Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 49(1), 2022, Pages 120–134, DOI:
ISSN: 1223-6934

Some results on statistically convergent triple sequences in an uncertainty space

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Abstract. The concept of statistical convergence plays a very prominent role in the study of sequence spaces. In this treatise, we extend the research on different types of statistical convergence viz., statistical convergence in mean, in measure, in distribution, in almost surely and with respect to uniformly almost surely of complex uncertain triple sequences in a given uncertainty space. We emphasize to focus on characterizing statistical convergence of complex uncertain triple sequence in some extent. Moreover, we initiate the notion of complex uncertain Cauchy triple sequence and establish the interconnection of it with a statistically convergent triple sequence.

2010 Mathematics Subject Classification. 60B10, 60B1, 60E05, 60F25, 40A05, 40A30, 40D25, 40F05.
Key words and phrases. uncertainty space, complex uncertain variable, triple sequence, statistical convergence, complex uncertain Cauchy triple sequence.

1. Introduction

The concept of statistical convergence of real and complex sequence was first introduced by Fast [10], Buck [1], Schohenberg [21] independently. But this research got momentum once the work of Salat [20] and Fridy [11] came into literature. Some more significant works in this area may be seen in [17], [22], [23]. The study of statistical convergence in double sequence has been initiated by Tripathy [24], Mursaleen and Edely [14], Moricz [13] independently. Sahiner et al. [19] studied the statistical convergence for triple sequence.

Let $K \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and $K(p,q,r) = |\{(i,j,k) \in K : i \le p, j \le q, k \le r\}|$, where the vertical bars stands for cardinality of the set. Then Sahiner et al. [19] defined the triple natural density as

$$\delta_3(K) = \lim_{p,q,r\to\infty} \frac{K(p,q,r)}{pqr} \text{ (Limit taken in Pringsheim's sense)}$$

and statistical Convergence of a triple sequence is as follows:
A real triple sequence $x = (x_{nkl})$ is said to be statistically convergent to the number L if for each $\varepsilon > 0$,

$$\delta_3(\{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |x_{nkl} - L| \ge \varepsilon\}) = 0.$$

After the introduction of uncertainty theory by Liu [12], many researchers investigated the nature of convergence of sequences in an uncertain environment. Four types of convergence (in mean, in measure, in distribution, with respect to almost

Received July 13, 2021. Accepted January 9, 2022.

surely) of a real uncertain sequence were studied by Liu [12] and then Chen et al. [2] extended this work in complex uncertain sequences. You [26] reported a new type of convergence called convergence with respect to uniformly almost surely of a complex uncertain sequence. These days, researchers are trying to explore the study of real sequence spaces in the environment of uncertainty. In this process different forms of convergence like almost convergence is studied by considering complex uncertain single, double and triple sequences by Saha et al. [18], Nath et al. [15] and Das et al. [3],[4], [9] respectively. Tripathy and Nath [25] introduced the notion of statistically convergent complex uncertain sequence and recently this work is extended to double sequences by Das et al. [5] and characterized this notion by establishing some of its properties. Very recently, Das et al. [6, 7, 8] made further progress in this direction and introduced the same concept by considering triple sequences of complex uncertain variable and established interrelationships among different type of statistical convergence to some extent.

In this current treatise, we study the concept of statistical convergence of a complex uncertain triple sequence via triple natural density operator and boundedness. Moreover, we initiate statistically Cauchy complex uncertain triple sequence and prove that a complex uncertain triple sequence is statistically convergent if and only if it is statistically Cauchy.

Before going to the main section we need some basic and preliminary ideas about the existing definitions and results which will play a major role in this study.

2. Preliminary

Definition 2.1. [12] An uncertain variable ζ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\zeta \in B\}=\{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ is an event.

Definition 2.2. [16] A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers, i.e., for any Borel set B of complex numbers, the set $\{\zeta \in B\}=\{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ is an event.

Definition 2.3. [6] The complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be statistically convergent in measure to ζ if

$$\lim_{n,m,l\to\infty} \tfrac{1}{nml}|\{(i,j,k) : i \leq n, j \leq m, k \leq l, \mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \varepsilon\} \geq \delta\}| = 0,$$

for every $\varepsilon, \delta > 0$.

The set of all complex uncertain triple sequences which are statistically convergent in measure is denoted by $st_3(\Gamma_{\mathcal{M}})$.

Definition 2.4. [6] A complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent to ζ in mean if

$$\lim_{n,m,l\to\infty} \tfrac{1}{nml}|\{(i,j,k) : i \leq n, j \leq m, k \leq l, E[||\zeta_{ijk} - \zeta||] \geq \varepsilon\}| = 0,$$

for every $\varepsilon > 0$.

The collection of all complex uncertain triple sequences which are statistically convergent in mean is denoted by $st_3(\Gamma_E)$.

Definition 2.5. [6] A triple sequence $\{\zeta_{nml}\}$ of complex uncertain variable is said to be statistically convergent to ζ with respect to almost surely if for any preassigned positive number ε there exists some event Λ having uncertain measure $\mathcal{M}(\Lambda) = 1$ such that

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{(i,j,k) : i \leq n, j \leq m, k \leq l, ||\zeta_{ijk}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}| = 0,$$

for every $\gamma \in \Lambda$.
The class of all such triple sequences is denoted by $st_3(\Gamma_{a.s})$.

Definition 2.6. [6] Let $\{\zeta_{nml}\}$ be a complex uncertain triple sequence and Φ, Φ_{nml} be the distribution functions for each complex uncertain variable $\zeta, \zeta_{nml}, n, m, l = 1, 2, 3, ...$ respectively. Then $\{\zeta_{nml}\}$ is said to be statistically convergent in distribution to ζ if

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{(i,j,k) : i \leq n, j \leq m, k \leq l, ||\Phi_{ijk}(z) - \Phi(z)|| \geq \varepsilon\}| = 0,$$

for all z at which Φ is continuous.
The family of all such complex uncertain triple sequences is denoted by $st_3(\Gamma_{\mathcal{D}})$.

3. Statistical convergence of triple sequence of complex uncertain variables

In this section, at first we introduce the fifth type of statistical convergence of complex uncertain triple sequence with repect to uniformly almost surely. This work is an extension of Das et al. [6] in a unique way. We characterize statistical convergence of a complex uncertain triple sequence to some extent via boundedness and triple density operator.

Definition 3.1. A complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be statistically convergent with respect to uniformly almost surely to ζ if for any preassigned $\varepsilon > 0$, there exist events E_t with $\mathcal{M}(E_t) \to 0$ such that

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{(i,j,k) : i \leq n, j \leq m, k \leq l, ||\zeta_{ijk}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\} = 0,$$

for all events $\gamma \in \Gamma - E_t$.
The set of all complex uncertain triple sequences which are statistically convergent with respect to uniformly almost surely is denoted by $st_3(\Gamma_{u.a.s})$.

Theorem 3.1. *A complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent in measure to ζ if and only if there exists $K \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_3(K) = 1$ and*

$$\lim_{i,j,k\to\infty} \mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \varepsilon\} = 0, \qquad (i,j,k) \in K.$$

Proof. Let $\{\zeta_{nml}\}$ be a statistically convergent complex uncertain triple sequence in measure to ζ.
Define two sets S_r and L_r as follows:

$$S_r = \{(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \delta\} \geq \tfrac{1}{r}\}$$

and

$$L_r = \{(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \delta\} < \tfrac{1}{r}\}.$$

Then

$$\delta_3(S_r) = 0,\ L_1 \supset L_2 \supset .... \supset L_i \supset L_{i+1} \supset .... \tag{1}$$

and

$$\delta_3(L_r) = 1,\ r = 1, 2, ... \tag{2}$$

We now show that for any $(i,j,k) \in L_r$, the complex uncertain triple sequence $\{\zeta_{ijk}\}$ is convergent in measure to ζ.
Let $\{\zeta_{ijk}\}$ is not convergent in measure to ζ.
Then, there exists $\varepsilon, \delta > 0$ such that

$$\mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \delta\} \geq \varepsilon.$$

Suppose $L_\varepsilon = \{(i,j,k) : \mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \delta\} < \varepsilon\}$ and $\varepsilon > \frac{1}{r}$ $\qquad (r = 1, 2, 3, ...)$.
Then

$$\delta_3(L_\varepsilon) = 0 \tag{3}$$

and by equation 1, $L_r \subset L_\varepsilon$.
Hence, $\delta_3(L_\varepsilon) = 0$, which contradicts the condition 2.
Thus $\{\zeta_{ijk}\}$ is convergent in measure to ζ.
Conversely, let there exists subset $K \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_3(K) = 1$ and

$$\lim_{i,j,k\to\infty} \mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \delta\} = 0, \qquad (i,j,k) \in K,$$

which implies there exists $n_0 \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$\mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \delta\} < \varepsilon, \qquad \forall\ i,j,k \geq n_0.$$

Now, $S_\varepsilon = \{(i,j,k) : \mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \delta\} \geq \varepsilon\}$
$\subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} - \{(i_{n_0+1}, j_{n_0+1}, k_{n_0+1}), (i_{n_0+2}, j_{n_0+2}, k_{n_0+2}), ...\}$,
that is, $\delta_3(S_\varepsilon) \leq 1 - 1 = 0$.
Hence, $\{\zeta_{ijk}\}$ is statistically convergent in measure to ζ. $\square$

Theorem 3.2. *The triple sequence $\{\zeta_{nml}\}$ is statistically convergent in distribution to ζ if and only if there exists $K \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with triple density 1 and*

$$\lim_{i,j,k\to\infty} \Phi_{ijk}(z) = \Phi(z), \qquad (i,j,k) \in K,$$

where Φ, Φ_{ijk} are the distribution functions of the complex uncertain variables ζ, ζ_{ijk} respectively and Φ is continuous at z.

Proof. Let $\{\zeta_{nml}\}$ be a statistically convergent complex uncertain triple sequence in distribution to ζ and S_r and L_r be two sets defined as follows:

$$S_r = \{(i,j,k) : ||\Phi_{ijk} - \Phi|| \geq \tfrac{1}{r}\}$$

and

$$L_r = \{(i,j,k) : ||\Phi_{ijk} - \Phi|| < \tfrac{1}{r}\},$$

uniformly for all $r \in \mathbb{N}$.
Then

$$L_1 \supset L_2 \supset .... \supset L_i \supset L_{i+1} \supset ....\ \textit{and}\ \delta_3(S_r) = 0. \tag{4}$$

The triple density of each of the sets $L_i (i = 1, 2, ...)$ being 1.
We now claim that $\{\zeta_{ijk}\}$ is convergent in distribution to ζ.
If possible let this assumption is wrong.
Therefore, there are $\varepsilon > 0$ such that

$$||\Phi_{ijk}(z) - \Phi(z)|| \geq \varepsilon, \text{ for infinitely many terms.}$$

Let $L_\varepsilon = \{(i,j,k) : ||\Phi_{ijk}(z) - \Phi(z)|| \geq \varepsilon\}$ and $\varepsilon > \frac{1}{r}, (r = 1, 2, 3, ...)$.
Then,

$$\delta_3(L_\varepsilon) = 0 \tag{5}$$

and by equation 4, $L_r \subset L_\varepsilon$.
Therefore, $\delta_3(L_\varepsilon) = 0$, which contradicts the hypothesis that the triple sequence is statistically convergent in distribution.
Hence $\{\zeta_{ijk}\}$ is convergent in distribution to ζ.
Conversely, let there exists $K \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_3(K) = 1$ and

$$\lim_{i,j,k\to\infty} \Phi_{ijk}(z) = \Phi(z), \qquad (i,j,k) \in K,$$

where Φ is continuous at the point z. This implies that there exists $n_0 \in \mathbb{N}$ such that for every $\varepsilon > 0$,

$$||\Phi_{ijk}(z) - \Phi(z)|| < \varepsilon, \qquad \forall\ i,j,k \geq n_0.$$

Now $T_\varepsilon = \{(i,j,k) : ||\Phi_{ijk}(z) - \Phi(z)|| \geq \varepsilon\}$
$\subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} - \{(i_{n_0+1}, j_{n_0+1}, k_{n_0+1}), (i_{n_0+2}, j_{n_0+2}, k_{n_0+2}), ...\}$,
that is, $\delta_3(T_\varepsilon) \leq 1 - 1 = 0$.
Hence, $\{\zeta_{ijk}\} \in st_3$-$(\Gamma_{\mathcal{D}})$ and it converges to ζ. □

Theorem 3.3. *A complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent in mean to ζ if and only if there exists a set $K \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with triple natural density unity such that*

$$\lim_{x,y,z\to\infty} E[||\zeta_{xyz} - \zeta||] = 0, \qquad (x,y,z) \in K.$$

Proof. This can be established by following the technique adopted to prove the theorem 3.1. In this case, we consider the expected value operator, in place of uncertain measure. □

Theorem 3.4. *The triple sequence $\{\zeta_{nml}\}$ of complex uncertain variable converges statistically to ζ with respect to almost surely if there exists a subset K of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with unit triple density and an event Λ with unit uncertain measure such that*

$$\lim_{x,y,z\to\infty} \zeta_{xyz}(\gamma) = \zeta(\gamma), \qquad \forall\ (x,y,z) \in K \text{ and } \gamma \in \Lambda.$$

Proof. Let $\{\zeta_{nml}\}$ be a statistically convergent complex uncertain triple sequence in almost surely to ζ in an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ and $\Lambda \subseteq \Gamma$ be a set of uncertain events with unit uncertain measure.
Now, construct the following two sets S_r and L_r as below:

$$S_r = \{(x,y,z) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \geq \tfrac{1}{r}\},\ \ \gamma \in \Lambda$$

and

$$L_r = \{(x,y,z) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| < \tfrac{1}{r}\},\ \ \gamma \in \Lambda,$$

where $r \in \mathbb{N}$.
Then

$$\delta_3(S_r) = 0,\ L_1 \supset L_2 \supset .... \supset L_i \supset L_{i+1} \supset .... \tag{6}$$

and

$$\delta_3(L_r) = 1,\ r = 1, 2, ... \tag{7}$$

The target is to show that for any $(x, y, z) \in L_r$, the complex uncertain triple sequence $\{\zeta_{xyz}\}$ is convergent in almost surely to ζ.
If possible let the complex uncertain triple sequence $\{\zeta_{xyz}\}$ is not convergent in almost surely to ζ.
Then there exists $\varepsilon > 0$ such that

$$||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \geq \varepsilon, \text{ for infinitely many points } (x, y, z) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$$

and $\forall\ \gamma \in \Lambda$.
Suppose $L_\varepsilon = \{(x, y, z) : ||\zeta_{xyz} - \zeta|| < \varepsilon\}$ and $\varepsilon > \frac{1}{r}$ $(r = 1, 2, 3, ...)$.
That is,

$$\delta_3(L_\varepsilon) = 0 \tag{8}$$

and by equation 6, $L_r \subset L_\varepsilon$.
Consequently, $\delta_3(L_\varepsilon) = 0$, which contradicts the condition 7.
Hence, the triple sequence $\{\zeta_{xyz}\}$ is convergent in almost surely to ζ.
Conversely, let there exists subset $K \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_3(K) = 1$ and

$$\lim_{x,y,z\to\infty} ||\zeta_{xyz} - \zeta|| = 0,\ (x, y, z) \in K.$$

This implies that there exists $n_0 \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$||\zeta_{xyz} - \zeta|| < \varepsilon, \qquad \forall\ x, y, z \geq n_0.$$

Now, $S_\varepsilon = \{(x, y, z) : ||\zeta_{xyz} - \zeta|| \geq \varepsilon\}$
$\subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} - \{(x_{n_0+1}, y_{n_0+1}, z_{n_0+1}), (x_{n_0+2}, y_{n_0+2}, z_{n_0+2}), ...\}$,
that is, $\delta_3(S_\varepsilon) \leq 1 - 1 = 0$.
Hence, $\{\zeta_{xyz}\}$ is statistically convergent in almost surely to ζ. □

Theorem 3.5. *The triple sequence $\{\zeta_{nml}\}$ of complex uncertain variable converges statistically to ζ with respect to uniformly almost surely if there exists a subset K of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with $\delta_3(K) = 1$ and a sequence of events $\{E_t\}$ having uncertain measure of each events approaching zero such that*

$$\lim_{x,y,z\to\infty} \zeta_{xyz}(\gamma) = \zeta(\gamma), \qquad \forall\ (x, y, z) \in K,\ \gamma \in \Gamma - E_t.$$

Proof. Replacing the subset Λ of Γ by the sub-collection $\Gamma - E_t$, where E_t are uncertain events with uncertain measure tending to zero, the theorem can be verified using similar technique as above theorem 3.4. □

Definition 3.2. A complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be bounded in measure if there exists a positive number δ such that

$$\mathcal{M}\{||\zeta_{nml}|| \geq \delta\} = 0.$$

The collection of all bounded complex uncertain triple sequences in measure is denoted by ${}_3\ell_\infty(\Gamma_\mathcal{M})$.

All other types of boundedness are defined as follows:

Definition 3.3. A complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be bounded in mean if there exists a positive number M such that

$$\sup_{n,m,l} E[||\zeta_{nml}||] < M.$$

The collection of such triple sequences is denoted by ${}_3\ell_\infty(\Gamma_E)$.

Definition 3.4. Let Φ_{nml} be the complex uncertainty distribution function for the complex uncertain variable ζ_{nml}. Then the triple sequence $\{\zeta_{nml}\}$ is said to be bounded in distribution if $\sup\limits_{n,m,l} ||\Phi_{nml}(z)||$ is finite, for any complex point z.
The collection of all triple sequences of such type is denoted by ${}_3\ell_\infty(\Gamma_D)$.

Definition 3.5. A complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be bounded in almost surely if for every $\varepsilon > 0$, there exists some event Λ with unit uncertain measure such that $\sup\limits_{n,m,l} ||\zeta_{nml}(\gamma)|| < \infty, \ \forall \ \gamma \in \Lambda$.
The class of all bounded complex uncertain triple sequences is denoted by ${}_3\ell_\infty(\Gamma_{a.s})$.

Definition 3.6. A complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be bounded with respect to uniformly almost surely if for any $\varepsilon > 0$, there exists events E_x with uncertain measure of each of the events tending to zero and $\sup\limits_{n,m,l} ||\zeta_{nml}(\gamma)|| < \infty$, for all $\gamma \in \Gamma - E_x$.
The set of all such types of triple sequences is denoted by ${}_3\ell_\infty(\Gamma_{u.a.s})$.

Theorem 3.6. *The class $st_3(\Gamma_{\mathcal{M}}) \cap \ {}_3\ell_\infty(\Gamma_{\mathcal{M}})$ is a closed linear subspace of the bounded complex uncertain sequence space ${}_3\ell_\infty(\Gamma_{\mathcal{M}})$.*

Proof. Let $\zeta^{nml} = \{\zeta^{nml}_{xyz}\} \in st_3(\Gamma_{\mathcal{M}}) \cap \ {}_3\ell_\infty(\Gamma_{\mathcal{M}})$ and $\zeta^{nml} \to \zeta \in \ {}_3\ell_\infty(\Gamma_{\mathcal{M}})$.
Since $\zeta^{nml} \in st_3(\Gamma_{\mathcal{M}}) \cap \ {}_3\ell_\infty(\Gamma_{\mathcal{M}})$, then there exists complex number s_{nml} such that

$$st_3(\Gamma_{\mathcal{M}})\text{-}\lim_{x,y,z\to\infty} \zeta^{nml}_{xyz} = s_{nml} \ (n,m,l = 1,2,.....).$$

As $\zeta^{nml} \to \zeta$, then for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that
$\mathcal{M}\{||\zeta^{pqr} - \zeta^{nml}|| \geq \delta\}$

$\leq \mathcal{M}\{||\zeta^{pqr} - \zeta|| \geq \delta'\} + \mathcal{M}\{||\zeta^{nml} - \zeta|| \geq \delta'\}$, for some positive $\delta' \leq \frac{\delta}{2}$

$< \frac{\varepsilon}{6} + \frac{\varepsilon}{6}$ (say) $= \frac{\varepsilon}{3}$, for all $p \geq n \geq n_0$, $q \geq m \geq n_0$ and $r \geq l \geq n_0$.

By 3.1, there exists $K_1, K_2 \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with triple natural density of both sets being 1 and
(1) $\lim\limits_{x,y,z\to\infty} \zeta^{nml}_{xyz} = s_{nml}, \qquad (x,y,z) \in K_1$
(2) $\lim\limits_{x,y,z\to\infty} \zeta^{pqr}_{xyz} = s_{pqr}, \qquad (x,y,z) \in K_2$
with $\delta_3(K_1 \cap K_2) = 1$.
Now, let $(k_1,k_2,k_3) \in K_1 \cap K_2$.
Then, from (1) and (2), we have

$$\mathcal{M}\left\{||\zeta^{nml}_{k_1k_2k_3} - s_{nml}|| \geq \delta\right\} \leq \frac{\varepsilon}{3} \tag{9}$$

and

$$\mathcal{M}\left\{||\zeta^{pqr}_{k_1k_2k_3} - s_{pqr}|| \geq \delta\right\} < \frac{\varepsilon}{3} \tag{10}$$

Then for each $p \geq n \geq n_0$, $q \geq m \geq n_0, r \geq l \geq n_0$ and $\delta > 0$, there exists a positive number $\delta'' \leq \frac{\delta}{3}$ such that
$\mathcal{M}\{||s_{pqr} - s_{nml}|| \geq \delta\}$

$\leq \mathcal{M}\{||s_{pqr} - \zeta^{pqr}_{k_1k_2k_3}|| \geq \delta''\} + \mathcal{M}\left\{||\zeta^{pqr}_{k_1k_2k_3} - \zeta^{nml}_{k_1k_2k_3}|| \geq \delta''\right\} + \mathcal{M}\left\{||\zeta^{nml}_{k_1k_2k_3} - s_{nml}|| \geq \delta''\right\}$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$
Therefore, the triple sequence $\{s_{nml}\}$ is a Cauchy triple sequence of complex numbers and hence convergent.
Let $\lim_{n,m,l\to\infty} s_{nml} = s \in \mathbb{C}$, and then there exists $n_1 \in \mathbb{N}$ such that

$$\mathcal{M}\{||s_{xyz} - s|| \geq \delta\} = 0, \ \forall\ x, y, z \geq n_1.$$

Now, we have to show that $\{\zeta_{xyz}\}$ is statistically convergent in measure to s.

Since $\{\zeta^{nml}_{xyz}\}$ is convergent in measure to ζ_{xyz}, then for each $\varepsilon > 0$ and $\delta > 0$ there exists $n_2 \in \mathbb{N}$ satisfying

$$\mathcal{M}\{||\zeta^{nml}_{xyz} - \zeta_{xyz}|| \geq \delta\} < \tfrac{\varepsilon}{2}, \ \forall\ x, y, z \geq n_2.$$

Also, $\{\zeta^{nml}_{xyz}\}$ is statistically convergent in measure to s_{nml}. Then there exists $K \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_3(k) = 1$ and for every $\varepsilon > 0, \delta > 0$ there exists $n_3 \in \mathbb{N}$ such that

$$\mathcal{M}\left\{||\zeta^{nml}_{xyz} - s_{xyz}|| \geq \delta\right\} < \tfrac{\varepsilon}{2}, \ \forall\ x, y, z \geq n_3.$$

Let $n_4 = \max\{n_1, n_2, n_3\}$. Then for any preassigned $\varepsilon > 0, \delta > 0$ and $(x, y, z) \in K$ with $x, y, z \geq n_4$, we have
$\mathcal{M}\{||\zeta_{xyz} - s_{xyz}|| > \delta\}$

$= \mathcal{M}\{||(\zeta_{xyz} - \zeta^{nml}_{xyz}) + (\zeta^{nml}_{xyz} - s_{xyz}) + (s_{xyz} - s)|| > \delta\}$

$\leq \mathcal{M}\{||\zeta_{xyz} - \zeta^{nml}_{xyz}|| > \delta_1\} + \mathcal{M}\{||\zeta^{nml}_{xyz} - s_{xyz}|| > \delta_1\} + \mathcal{M}\{||s_{xyz} - s|| > \delta_1\},$

for some $\delta_1 \leq \frac{\delta}{3}$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + 0 = \varepsilon,$.
Therefore, the triple sequence $\{\zeta_{xyz}\}$ is a convergent complex uncertain triple sequence in measure to s and consequently, it statistically converges in measure to $s \in st_3(\Gamma_{\mathcal{M}}) \cap\ _3\ell_\infty(\Gamma_{\mathcal{M}})$.
Hence, the space is a closed linear subspace. □

Theorem 3.7. *The set of all statistically convergent and bounded complex uncertain triple sequence in mean is a closed linear subspace of the bounded complex uncertain sequence space* $_3\ell_\infty(\Gamma_E)$.

Proof. The claim may be verified by the similar argument as of the theorem 3.6. By considering the expected value operator of complex uncertain variable, the result can be achieved easily. □

Theorem 3.8. *The space* $st_3(\Gamma_{u.a.s}) \cap\ _3\ell_\infty(\Gamma_{u.a.s})$ *of all statistically convergent and statistically bounded complex uncertain triple sequences is a closed linear subspace of* $_3\ell_\infty(\Gamma_{u.a.s})$.

Proof. Let $\zeta^{nml} = \{\zeta^{nml}_{xyz}\} \in st_3(\Gamma_{u.a.s}) \cap\ _3\ell_\infty(\Gamma_{u.a.s})$ and $\zeta^{nml} \to \zeta \in\ _3\ell_\infty(\Gamma_{u.a.s})$.
Since $\{\zeta^{nml}\}$ is statistically convergent with respect to uniformly almost surely, then there exists a sequence of events $\{E_t\}$ with $\mathcal{M}\{E_t\} \to 0$ and a complex number s_{nml} such that

$$st_3(\Gamma_{u.a.s})\text{-}\lim_{x,y,z\to\infty}\zeta^{nml}_{xyz}(\gamma) = s_{nml}, (n,m,l = 1,2,.....), \text{ where } \gamma \in \Gamma - E_t.$$

As $\zeta^{nml}(\gamma) \to \zeta(\gamma)$, for $\gamma \in \Gamma - E_t$, then for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $||\zeta^{pqr}(\gamma) - \zeta^{nml}(\gamma)||$

$\leq ||\zeta^{pqr}(\gamma) - \zeta(\gamma)|| + ||\zeta^{nml}(\gamma) - \zeta(\gamma)||$

$< \frac{\varepsilon}{6} + \frac{\varepsilon}{6}$ (say) $= \frac{\varepsilon}{3}$, for all $p \geq n \geq n_0$, $q \geq m \geq n_0$ and $r \geq l \geq n_0$.

Then there exists $K_1, K_2 \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with $\delta_3(K_1) = \delta_3(K_2) = 1$ and
(1) $\lim_{x,y,z\to\infty}\zeta^{nml}_{xyz}(\gamma) = s_{nml}, \qquad (x,y,z) \in K_1$
(2) $\lim_{x,y,z\to\infty}\zeta^{pqr}_{xyz}(\gamma) = s_{pqr}, \qquad (x,y,z) \in K_2$
where $\gamma \in \Gamma - E_t$ and $\delta_3(K_1 \cap K_2) = 1$.
Now let $(k_1, k_2, k_3) \in K_1 \cap K_2$.
Then
$$||\zeta^{nml}_{k_1k_2k_3}(\gamma) - s_{nml}|| \leq \frac{\varepsilon}{3},\ \gamma \in \Gamma - E_t \tag{11}$$
and
$$||\zeta^{pqr}_{k_1k_2k_3}(\gamma) - s_{pqr}|| < \frac{\varepsilon}{3},\ \gamma \in \Gamma - E_t \tag{12}$$
Thus, for each $p \geq n \geq n_0$, $q \geq m \geq m_0$ and $r \geq l \geq n_0$,
$||s_{pqr} - s_{nml}||$

$\leq ||s_{pqr} - \zeta^{pqr}_{k_1k_2k_3}(\gamma)|| + ||\zeta^{pqr}_{k_1k_2k_3}(\gamma) - \zeta^{nml}_{k_1k_2k_3}(\gamma)|| + ||\zeta^{nml}_{k_1k_2k_3}(\gamma) - s_{nml}||$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.
Therefore, the triple sequence $\{s_{nml}\}$ is a Cauchy triple sequence.
Let $\lim_{n,m,l\to\infty} s_{nml} = s$, that is for every $\varepsilon > 0$ there exists a $n_1(\varepsilon) \in \mathbb{N}$, such that $||s_{xyz} - s|| < \frac{\varepsilon}{3},\ \forall\ x,y,z \geq n_1$.
We are to show that $\{\zeta_{xyz}\}$ is statistically convergent with respect to uniformly almost surely to s.
Since, $\{\zeta^{nml}_{xyz}\}$ is convergent with respect to uniformly almost surely to ζ_{xyz} for each $\varepsilon > 0$, then there exist uncertain events E_u with uncertain measure of each events approaching zero and $n_2 \in \mathbb{N}$ so that

$$||\zeta^{nml}_{xyz}(\gamma) - \zeta_{xyz}(\gamma)|| < \tfrac{\varepsilon}{3},\ \forall\ x,y,z \geq n_2 \text{ where } \gamma \in \Gamma - E_u.$$

Also, $\{\zeta^{nml}_{xyz}\}$ is statistically convergent with respect to uniformly almost surely to s_{nml}. Then there exists $K = \{(x,y,z) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}\}$ such that $\delta_3(K) = 1$ and for every $\varepsilon > 0$ there exists $n_3 \in \mathbb{N}$ such that

$$||\zeta^{nml}_{xyz}(\gamma) - s_{nml}|| < \tfrac{\varepsilon}{3},\ \forall\ x,y,z \geq n_3.$$

Let $n_4 = \max\{n_1, n_2, n_3\}$. Then for any given positive number ε and $(x,y,z) \in K$ with $x,y,z \geq n_4$, we have
$||\zeta_{xyz}(\gamma) - z_{xyz}||$

$= ||(\zeta_{xyz}(\gamma) - \zeta^{nml}_{xyz}(\gamma)) + (\zeta^{nml}_{xyz}(\gamma) - s_{xyz}) + (s_{xyz} - s)||$

$\leq ||\zeta_{xyz}(\gamma) - \zeta_{xyz}^{nml}(\gamma)|| + ||\zeta_{xyz}^{nml}(\gamma) - s_{xyz}|| + ||s_{xyz} - s||,$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$

Thus, $\{\zeta_{xyz}\}$ statistically converges with respect to uniformly almost surely to s and hence the space $st_3(\Gamma_{u.a.s}) \cap\ {}_3\ell_\infty(\Gamma_{u.a.s})$ is a closed linear subspace. □

Theorem 3.9. *The space $st_3(\Gamma_{a.s}) \cap\ {}_3\ell_\infty(\Gamma_{a.s})$ is a closed linear subspace of the bounded complex uncertain sequence space ${}_3\ell_\infty(\Gamma_{a.s})$.*

Proof. The proof can be derived by following the above theorem 3.8. For this, we just need to consider such event Λ whose uncertain measure is 1, instead of the uncertain events excluding E_t mentioned in the theorem. □

Theorem 3.10. *The set of all statistically convergent and bounded complex uncertain triple sequence in distribution is a closed linear subspace of the bounded complex uncertain sequence space ${}_3\ell_\infty(\Gamma_{\mathcal{D}})$.*

Proof. By taking the complex uncertainty distribution functions of the complex uncertain variable instead of considering expected value operator in the theorem 3.7, this claim can be justified easily. □

4. Statistically complex uncertain Cauchy triple sequences

In this section, we present the notion of statistically complex uncertain Cauchy triple sequence and establish the interrelationships with statistically convergent complex uncertain triple sequence.

Definition 4.1. The complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be statistically Cauchy in measure if for every $\varepsilon, \delta > 0$, there exists $n_1, m_1, l_1 \in \mathbb{N}$ such that for all $i, p \geq n_1$, $j, q \geq m_1$ and $k, r \geq l_1$, we have

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{(i,j,k) : i \leq n, j \leq m, k \leq l : \mathcal{M}\{||\zeta_{ijk} - \zeta_{pqr}|| \geq \delta\} > \varepsilon\}| = 0.$$

Definition 4.2. The complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be statistically Cauchy in mean if for every $\varepsilon > 0$, there exists $n_1, m_1, l_1 \in \mathbb{N}$ such that for all $i, p \geq n_1$, $j, q \geq m_1$ and $k, r \geq l_1$,

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{(i,j,k) : i \leq n, j \leq m, k \leq l : E[||\zeta_{ijk} - \zeta_{pqr}||] \geq \varepsilon\}| = 0.$$

Definition 4.3. The complex uncertain triple sequence $\{\zeta_{nml}\}$ called statistically Cauchy in distribution if for every positive ε, there exists $n_1, m_1, l_1 \in \mathbb{N}$ such that for all $i, p \geq n_1$, $j, q \geq m_1$ and $k, r \geq l_1$,

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{(i,j,k) : i \leq n, j \leq m, k \leq l : ||\Phi_{ijk}(z) - \Phi_{pqr}(z)|| \geq \varepsilon\}| = 0,$$

where z is the point at which Φ is continuous and Φ, Φ_{ijk} are uncertain distribution function for ζ, ζ_{ijk}.

Definition 4.4. The complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be statistically Cauchy with respect to almost surely if for any positive $\varepsilon > 0$, there exist events Λ with unit uncertain measure and $n_1, m_1, l_1 \in \mathbb{N}$ such that

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{(i,j,k) : i \leq n, j \leq m, k \leq l : ||\zeta_{ijk}(\gamma) - \zeta_{pqr}(\gamma)|| \geq \varepsilon)| = 0,$$

for all $i, p \geq n_1, j, q \geq m_1, k, r \geq l_1$ and $\gamma \in \Lambda$.

Definition 4.5. A complex uncertain triple sequence $\{\zeta_{nml}\}$ is called statistically Cauchy with respect to uniformly almost surely if for any positive $\varepsilon > 0$, there exists sequence of events $\{E_t\}$ approaching to uncertain measure zero and natural numbers n_1, m_1, l_1 with $i, p \geq n_1, j, q \geq m_1, k, r \geq l_1$ such that

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{(i,j,k) : i \leq n, j \leq m, k \leq l : ||\zeta_{ijk}(\gamma) - \zeta_{pqr}(\gamma)|| \geq \varepsilon\}| = 0,$$

for all $\gamma \in \Gamma - E_t$.

Theorem 4.1. *A complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent in measure if and only if $\{\zeta_{nml}\}$ is statistically Cauchy in measure.*

Proof. Let $\{\zeta_{nml}\}$ be statistically convergent in measure to ζ. Then for each $\varepsilon, \delta > 0$, there exists $n_0, m_0, l_0 \in \mathbb{N}$ such that

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{(i,j,k) : i \leq n, j \leq m, k \leq l : \mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \delta\} \geq \varepsilon\}| = 0,$$

where $i, p \geq n_0, j, q \geq m_0, k, r \geq l_0$.
Let us choose two natural numbers n_1, n_2, n_3 such that

$$\mathcal{M}\{||\zeta_{n_1n_2n_3} - \zeta|| \geq \delta\} \geq \varepsilon.$$

Let us take three sets

$$A_\varepsilon = \{(i,j,k) : i \leq n, j \leq m, k \leq l : \mathcal{M}\{||\zeta_{ijk} - \zeta_{n_1n_2n_3}|| \geq \delta\} \geq \varepsilon\}$$

$$B_\varepsilon = \{(i,j,k) : i \leq n, j \leq m, k \leq l : \mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \delta\} \geq \varepsilon\}$$

$$C_\varepsilon = \{(i,j,k) : i = n_1 \leq n, j = n_2 \leq m, k = n_3 \leq l : \mathcal{M}\{||\zeta_{n_1n_2n_3} - \zeta|| \geq \delta\} \geq \varepsilon\}$$

Obviously, $A_\varepsilon \subseteq B_\varepsilon \cup C_\varepsilon$. Therefore $\delta_3(A_\varepsilon) \leq \delta_3(B_\varepsilon) + \delta_3(C_\varepsilon) = 0$, since $\{\zeta_{nml}\}$ is statistically convergent in measure to ζ.
Hence $\{\zeta_{nml}\}$ is statistically Cauchy in measure.
Conversely, let the complex uncertain triple sequence $\{\zeta_{nml}\}$ be statistically Cauchy in measure. Then $\delta_3(A_\varepsilon) = 0$. Hence for the set

$$E_\varepsilon = \{(i,j,k) : i \leq n, j \leq m, k \leq l : \mathcal{M}\{||\zeta_{ijk} - \zeta_{n_1n_2n_3}|| \geq \delta\} < \varepsilon\},$$

we have $\delta_3(E_\varepsilon) = 1$.
Now for each $\delta > 0$, there exists some $0 < \delta' \leq \frac{\delta}{2}$, such that

$$\mathcal{M}\{||\zeta_{ijk} - \zeta_{n_1n_2n_3}|| \geq \delta\} \leq 2\mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \delta'\} < \varepsilon. \tag{13}$$

Now if $\{\zeta_{nml}\}$ is not statistically convergent in measure, then $\delta_3(B_\varepsilon) = 1$.
Then for the set

$$F_\varepsilon = \{(i,j,k) : i \leq n, j \leq m, k \leq l : \mathcal{M}\{||\zeta_{ijk} - \zeta|| \geq \delta\} < \varepsilon\},$$

we have $\delta_3(F_\varepsilon) = 0$.
Thus from the equation 13, for the set

$$G_k = \{(i,j,k) : i \leq n, j \leq m, k \leq l : \mathcal{M}\{||\zeta_{ijk} - \zeta_{n_1n_2n_3}|| \geq \delta\} < \varepsilon\},$$

we have $\delta_3(G_k) = 0$, which implies that $\delta_3(A_\varepsilon) = 1$ and thus it arises a contradiction that $\{\zeta_{nml}\}$ is a statistically Cauchy sequence in measure.

Hence the complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent in measure to ζ. □

Theorem 4.2. *A complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent almost surely if and only if $\{\zeta_{nml}\}$ is statistically Cauchy with respect to almost sure.*

Proof. Let $\{\zeta_{nml}\}$ be a statistically convergent triple sequence with respect to almost surely to ζ. Therefore for every $\varepsilon > 0$ there exists $n_0, m_0, l_0 \in \mathbb{N}$ with $i, p \geq n_0, j, q \geq m_0, k, r \geq l_0$ and events Λ with unit uncertain measure such that

$$\lim_{n,m,l\to\infty} \tfrac{1}{nml} |\{(i,j,k) : i \leq n, j \leq m, k \leq l : ||\zeta_{ijk}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}| = 0,$$

for all $\gamma \in \Lambda$.

Take $n_1, n_2, n_3 \in \mathbb{N}$ such that $||\zeta_{n_1n_2n_3}(\gamma) - \zeta(\gamma)|| \geq \varepsilon,\ \gamma \in \Lambda$ and consider three sets

$$A_\varepsilon = \{(i,j,k) : i \leq n, j \leq m, k \leq l : ||\zeta_{ijk}(\gamma) - \zeta_{n_1n_2n_3}(\gamma)|| \geq \varepsilon\},$$

$$B_\varepsilon = \{(i,j,k) : i \leq n, j \leq m, k \leq l : ||\zeta_{ijk}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\},$$

$$C_\varepsilon = \{(i,j,k) : i = n_1 \leq n, j = n_2 \leq m, k = n_3 \geq l : ||\zeta_{n_1n_2n_3}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\},$$

where $\gamma \in \Lambda$.
Here $A_\varepsilon \subseteq B_\varepsilon \cup C_\varepsilon$ and hence $\delta_3(A_\varepsilon) \leq \delta_3(B_\varepsilon) + \delta_3(C_\varepsilon) = 0$, since $\{\zeta_{nml}\}$ is statistically convergent with respect to almost surely.

Hence the triple sequence $\{\zeta_{nml}\}$ is a statistically Cauchy sequence with respect to almost surely.

Conversely, let $\{\zeta_{nml}\}$ be statistically Cauchy with respect to almost surely. Then $\delta_3(A_\varepsilon) = 0$.

Therefore, for the set

$$E_\varepsilon = \{(i,j,k) : i \leq n, j \leq m, k \leq l : ||\zeta_{ijk}(\gamma) - \zeta_{n_1n_2n_3}(\gamma)|| < \varepsilon\},\ \gamma \in \Lambda$$

for some event Λ such that $\mathcal{M}(\Lambda) = 1$, we have $\delta_3(E_\varepsilon) = 1$.
In particular we can write

$$||\zeta_{ijk}(\gamma) - \zeta_{n_1n_2n_3}(\gamma)|| \leq 2||\zeta_{ijk}(\gamma) - \zeta(\gamma)|| < \varepsilon,\ if\ ||\zeta_{ijk}(\gamma) - \zeta(\gamma)|| < \frac{\varepsilon}{2}. \tag{14}$$

If possible let $\{\zeta_{nml}\}$ is not statistically convergent triple sequence with respect to almost surely. Then $\delta_3(B_\varepsilon) = 1$.
Hence for the set

$$F_\varepsilon = \{(i,j,k) : i \leq n, j \leq m, k \leq l : ||\zeta_{ijk}(\gamma) - \zeta(\gamma)|| < \varepsilon\},$$

we have $\delta_3(F_\varepsilon) = 0$.
Hence from equation 14, for the set

$$G_k = \{(i,j,k) : i \leq n, j \leq m, k \leq l : ||\zeta_{ijk}(\gamma) - \zeta_{n_1n_2n_3}(\gamma)|| < \varepsilon\},$$

we have $\delta_3(G_k) = 0$, which implies that $\delta_3(A_\varepsilon) = 1$ and so it is not a statistically Cauchy triple sequence with respect to almost surely. This is a contradiction to our assumption. Hence, $\{\zeta_{nml}\}$ is statistically convergent with respect to almost surely to ζ. □

The above results are true for mean, distribution and uniformly almost surely also and the proofs are also easy as they can be established by following the above techniques. So, we only put the statements only.

Theorem 4.3. *A complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent in mean if and only if $\{\zeta_{nml}\}$ is statistically Cauchy in mean.*

Theorem 4.4. *A complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent with respect to uniformly almost surely if and only if $\{\zeta_{nml}\}$ is statistically Cauchy with respect to uniformly almost surely.*

Theorem 4.5. *A complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent in distribution if and only if $\{\zeta_{nml}\}$ is statistically Cauchy in distribution.*

Finally, a necessary and sufficient condition for a complex uncertain triple sequence to be statistically Cauchy is obtained by observing the results established earlier (with respect to mean, measure, distribution, almost surely and uniformly almost surely).

Theorem 4.6. *Let $\{\zeta_{ijk}\}$ be a triple sequence of complex uncertain variables. Then $\{\zeta_{ijk}\}$ is statistically Cauchy in measure if and only if there exists a subsequence $\{\xi_{pqr}\}$ of $\{\zeta_{ijk}\}$ such that*

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{p \le n, q \le m, r \le l : \mathcal{M}\{||\xi_{pqr} - \zeta|| \ge \varepsilon\} \ge \delta\}| = 0,$$

for every $\varepsilon, \delta > 0$, where ζ is the limit to which the sequence $\{\zeta_{ijk}\}$ statistically converges in measure.

Proof. Combining the theorem 3.1 and theorem 4.1, the proof can be established. □

Theorem 4.7. *A complex uncertain triple sequence $\{\zeta_{ijk}\}$ is statistically Cauchy in mean if and only if there exists a subsequence $\{\xi_{pqr}\}$ of $\{\zeta_{ijk}\}$ converging statistically in mean to the same limit ζ that of $\{\zeta_{ijk}\}$, that is*

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{p \le n, q \le m, r \le l : E[||\xi_{pqr} - \zeta||] \ge \varepsilon\}| = 0, \text{ for every } \varepsilon > 0.$$

Proof. Straightforward from theorem 3.3 and theorem 4.3 and hence omitted. □

Theorem 4.8. *The triple sequence $\{\zeta_{ijk}\}$ of complex uncertain variables is statistically Cauchy in distribution if and only if there exists a subsequence $\{\xi_{pqr}\}$ of $\{\zeta_{ijk}\}$ such that*

$$\lim_{n,m,l\to\infty} \frac{1}{nml}|\{i \le n, j \le m, k \le l : ||\Phi_{pqr}(z) - \Phi(z)|| \ge \varepsilon\}| = 0,$$

where z is the point at which Φ is continuous.

Proof. The proof is straightforward from theorem 3.2 and theorem 4.5. □

Theorem 4.9. *A complex uncertain triple sequence $\{\zeta_{ijk}\}$ is statistically Cauchy with respect to almost surely if and only if there exists a subsequence $\{\xi_{pqr}\}$ of $\{\zeta_{ijk}\}$ converging statistically with respect to almost surely to the same limit as that of $\{\zeta_{ijk}\}$.*

Proof. Observing theorem 3.4 and theorem 4.2, one can prove the result easily. □

Theorem 4.10. *A complex uncertain triple sequence* $\{\zeta_{ijk}\}$ *is statistically Cauchy with respect to uniformly almost surely if and only if there exists a subsequence* $\{\xi_{pqr}\}$ *of* $\{\zeta_{ijk}\}$ *converging statistically with respect to uniformly almost surely to the same limit as that of* $\{\zeta_{ijk}\}$.

Proof. Obvious from theorem 3.5 and theorem 4.4. □

5. Conclusions

In this paper, we have characterized the concept of statistically convergent complex uncertain triple sequence through mean, measure, distribution, almost surely and uniformly almost surely. Several characterization of such sequences have been made. We also initiate statistical Cauchy triple sequence of complex uncertain variables and established interrelationship of it with statistically convergent complex uncertain triple sequence.

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