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Three types of prime filters in strong quasi-ordered residuated system

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Abstract. The concept of quasi-ordered residuated system was introduced in 2018 by Bonzio and Chajda. The author introduced and analyzed the concept of filters as well as some types of filters in such an algebraic system. Additionally, the author also dealt with a strong quasi-ordered residuated system in which he determined prime and irreducible filters. In this paper, the author introduce three types of prime filters in a strong quasi-ordered residuated system and analyze their interconnections.

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1. Introduction

The concept of residuated relational systems ordered under a quasi-order relation, or quasi-ordered residuated systems (briefly, QRS), was introduced in 2018 by S. Bonzio and I. Chajda [2]. Previously, this concept was discussed in [1]. The author introduced and developed the concepts of ideals [15] and filters [9] in this algebraic structure as well as several types of filters such as implicative [11], weakly implicative [16], associated [10] and comparative filters [12]. In [12], it is shown that every comparative filter of a quasi-ordered residuated system $\mathfrak{A}$ is an implicative filter of $\mathfrak{A}$ and the reverse it need not be valid. The concept of a strong quasi-ordered residuated system was introduced and discussed in [13]. In such systems, comparative and implicative filters are coincide. The specificity of strong QRS's is that they allow us to determine the least upper bound for each their two elements.

In paper [14], as a continuation of the previous articles, the fundamental properties of the least upper bound in such systems are discussed. In addition, the concepts of prime and irreducible filters in strong quasi-ordered residuated systems have been introduced and some their important properties have been recognized. It is shown that each prime filter in a strong quasi-ordered residuated system is an irreducible filter.

For the purposes of this article, we recognize the concept of prime filter of a QRS as a 'prime filter of the first type' (Definition 3.1). In what follows in this article, we introduce two more types of prime filters in strong quasi-ordered relational systems. We introduce the concepts of 'prime filters of the second type' (Definition 3.2) and 'prime filters of the third type' (Definition 3.3) in a strong QRS. In addition to the

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above, the links between these three types of prime filters in a strong quasi-ordered residuated system are considered. Thus it is shown that each prime filter of the second type is a prime filter of the first type (Theorem 3.1) and a prime filter of the thitd type (Theorem 3.6) and that the reverse does not have to be (Example 3.5, Example 3.6 and Example 3.7).

We use the opportunity to draw the potential reader's attention to the fact that the results obtained in this and previous research on strong quasi-ordered residual systems differ from the results about ideals and filters of commutative residuated lattice ordered under a (quasi-)order (see, for example [5, 6]).

2. Preliminaries

2.1. Quasi-ordered residuated systems. In article [2], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

Definition 2.1 ([2], Definition 2.1)**.** A *residuated relational system* is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and R is a binary relation on A and satisfying the following properties:

(1) $(A, \cdot, 1)$ is a commutative monoid;
(2) $(\forall x \in A)((x, 1) \in R)$;
(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.

We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition *(3)* as residuation.

The basic properties for residuated relational systems are subsumed in the following

Theorem 2.1 ([2], Proposition 2.1)**.** *Let $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ be a residuated relational system. Then*

(4) $(\forall x, y \in A)(x \rightarrow y = 1 \Longrightarrow (x, y) \in R)$*;*
(5) $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$*;*
(6) $(\forall x \in A)((1, x \rightarrow 1) \in R)$*;*
(7) $(\forall x, y, z \in A)(x \rightarrow y = 1 \Longrightarrow (z \cdot x, y) \in R)$*;*
(8) $(\forall x, y \in A)((x, y \rightarrow 1) \in R)$.

Recall that a *quasi-order relation* $' \preccurlyeq '$ on a set A is a binary relation which is reflexive and transitive.

Definition 2.2 ([2], Definition 3.1)**.** A *quasi-ordered residuated system* is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, where $\preccurlyeq$ is a quasi-order relation in the monoid $(A, \cdot)$

Example 2.1. Let $A = \{1, a, b, c, d\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

and

$\rightarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$\preccurlyeq := \{(1,1),(a,1),(b,1),(c,1),(d,1),(b,b),(a,a),(c,c),(d,d),(c,a),(d,a),(d,b),(d,c)\}$.

Example 2.2. For a commutative monoid A, let $\mathfrak{P}(A)$ denote the powerset of A, ordered by set inclusion, and '$\cdot$' the usual multiplication of subsets of A. Then $\langle \mathfrak{P}(A), \cdot, \rightarrow, A, \subseteq\rangle$ is a quasi-ordered residuated system in which the residuum are given by

$$(\forall X, Y \in \mathfrak{P}(A))(Y \rightarrow X := \{z \in A : Yz \subseteq X\}).$$

Example 2.3. Let $\mathbb{R}$ be a field of real numbers. Define a binary operations '$\cdot$' and '$\rightarrow$' on $A = [0,1] \subset \mathbb{R}$ by

$$(\forall x, y \in [0,1])(x \cdot y := max\{0, x+y-1\}) \text{ and } x \rightarrow y := min\{1, 1-x+y\}).$$

Then, A is a commutative monoid with the identity 1 and $\langle A, \cdot, \rightarrow, \leqslant, 1\rangle$ is a quasi-ordered residuated system.

Example 2.4. Any commutative residuated lattice $\langle A, \cdot, \rightarrow, 0, 1, \wedge, \vee, R\rangle$ where R is a lattice quasi-order is a quasi-ordered residuated system.

Remark 2.1. Quasi-ordered residauted system, generally speaking, differs from the commutative residuated lattice $\langle A, \cdot, \rightarrow, 0, 1, \sqcap, \sqcup, R\rangle$ where R is a lattice quasi- order. First, our observed system does not have to be limited from below. Second, the observed system does not have to be a lattice. However, the difference between a quasi-ordered relational system and a CRPM (Example 2.4) is only in order relations since a quasi-order relation does not have to be antisymmetric. More about this last-mentioned algebraic structure can be found in [8].

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.2 ([2], Proposition 3.1)**.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then*

(9) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y))$;
(10) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (y \rightarrow z \preccurlyeq x \rightarrow z \wedge z \rightarrow x \preccurlyeq z \rightarrow y))$;
(11) $(\forall x, y \in A)(x \cdot y \preccurlyeq x \wedge x \cdot y \preccurlyeq y)$.

2.2. Filters in QRS. In this subsection we give some notions that will be used in this article.

Definition 2.3 ([9], Definition 3.1)**.** For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a *filter* of $\mathfrak{A}$ if it satisfies conditions

(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, and
(F3) $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \implies v \in F)$.

It is shown ([9], Proposition 3.4 and Proposition 3.2) that if a non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F-2), then it also satisfies the conditions

(F- 0): $1 \in F$ and
(F-1): $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$.

If $\mathfrak{F}(A)$ is the family of all filters in a QRS $\mathfrak{A}$, then $\mathfrak{F}(A)$ is a complete lattice ([9], Theorem 3.1).

Notions and notations that are used but not previously determined in this paper can be found in [2, 9, 10, 13].

Remark 2.2. In implicative algebras, the term 'implicative filter' is used instead of the term 'filter' we use (see, for example [3, 17]) because in the structure we study the concept of filter is determined more complexly than requirements (F3). It is obvious that our filter concept is also a filter in the sense of [3, 4, 17]. The term 'special implicative filter' is also used in the aforementioned sources if the implicative filter in the sense of [17] satisfies some additional condition.

2.3. Strong QRS. In this subsection we analyze the concept of strong quasi-ordered residuated systems. This concept was introduced and analyzed in [13]. Considering the fact that the quasi-order relation '$\preccurlyeq$', which appears in the determination of this algebraic system, does not have to be antisymmetric, the following definition gets a clearer meaning. It is generally known that a quasi-order relation $\preccurlyeq$ on a set A generates a equivalence relation $\equiv_{\preccurlyeq} := \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (9) and (10), this equivalence is compatible with the operations in $\mathfrak{A}$. Thus, the relation $\equiv_{\preccurlyeq}$ is a congruence on $\mathfrak{A}$.

Definition 2.4 ([13], Definition 6)**.** For a quasi-ordered residuated system $\mathfrak{A}$ it is said to be a *strong quasi-ordered residuated system* if the following holds
(14) $(\forall u, v \in A)((u \to v) \to v \equiv_{\preccurlyeq} (v \to u) \to u)$.

The following is an example of a non strong QRS:

Example 2.5. Let $A = \{1, a, b\}$ and the operations '$\cdot$' and '$\to$' be defined on A as follows:

$\cdot$	1	a	b
1	1	a	b
a	a	a	a
b	b	a	b

and

$\to$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	a

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1,1), (a,a), (b,b), (a,1), (b,1), (a,b)\}.$$

It can be easily checked that $\mathfrak{A}$ is a quasi-ordered residuated system. Since

$$(a \to b) \to b = 1 \to b = b \text{ and } (b \to a) \to a = a \to a = 1,$$

we have $(a \to b) \to b \preccurlyeq (b \to a) \to a$ but $\neg((b \to a) \to a \preccurlyeq (a \to b) \to b)$. Thus, $\mathfrak{A}$ is not a strong quasi-ordered residuated system.

Now, we give an example of strong quasi-ordered residuated system.

Example 2.6. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

and

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	c	1
b	1	c	1	c
c	1	b	b	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (a,a), (b,b), (c,c), (a,c)\}.$$

Direct verification it can prove that $\mathfrak{A}$ is a strong quasi-ordered residuated system.

In this paper, we shall investigate the structure of strong quasi-ordered residuated systems. In [13], Theorem 5, it is shown that comparative and implicative filters in such algebraic systems are coincide. Some of the propositions made in this article have already been shown in [13] such as Theorem 2.3 and Theorem 2.5. We are re-enclose them in this paper in order to achieve greater consistency of the material presented in it.

Theorem 2.3. *Let $\mathfrak{A}$ be a strong quasi-ordered residuated system. Then the following holds*

(15) $(\forall u, v \in A)(u \preccurlyeq v \implies v \equiv_{\preccurlyeq} (v \to u) \to u)$.

Corollary 2.4. *Let $\mathfrak{A}$ be a strong quasi-ordered residuated system. Then the following holds*

(16) $(\forall x, y \in A)(y \to x \equiv_{\preccurlyeq} ((y \to x) \to x) \to x)$ *and*
(17) $(\forall x, y \in A)((y \to x) \to x \equiv_{\preccurlyeq} (((y \to x) \to x) \to x) \to x)$.

The following theorem shows that in a strong quasi-ordered residuated system we can construct the least upper bound for each pair of elements.

Theorem 2.5. *Let $\mathfrak{A}$ be a strong quasi-ordered residuated system. For any $u, v \in A$, the element*

$$u \sqcup v := (v \to u) \to u \equiv_{\preccurlyeq} (u \to v) \to v$$

is the least upper bound of u and v.

Example 2.7. Let $A = \{1, a, b, c, d\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	a	a	a
b	b	a	b	a	a
c	c	a	a	c	d
d	d	a	a	d	d

and

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	c	1	d	1
c	1	b	d	1	1
d	1	d	d	d	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (d,1), (a,a), (b,b), (c,c), (d,d), (a,b), (a,c), (a,d), (b,d), (c,d)\}$. Direct verification it can prove that $\mathfrak{A}$ is a strong quasi-ordered residuated system. In this example, to illustrate, we see that for the incomparable elements b and c we have

$$b \sqcup c = (b \to c) \to c = (c \to b) \to b = d.$$

Theorem 2.6 ([13], theorem 7)**.** *Let $\mathfrak{A}$ be a strong quasi-ordered residuated system. Then $(\mathfrak{A}, \sqcup)$ is a distributive upper semi-lattice in the following sense*

$$(\forall x, y, z \in A)((x \sqcup y) \sqcup z \equiv_{\preccurlyeq} (x \sqcup z) \sqcup (y \sqcup z)).$$

Proposition 2.7 ([14], Proposition 2)**.** *Let $\mathfrak{A}$ b4 a strong quasi-ordered residuated system. Then*

(a) $(\forall u, v \in A)(u \sqcup 1 = 1 \sqcup u = 1$ *and* $u \sqcup v = v \sqcup u)$,
(b) $(\forall x, y, z \in A)((z \cdot x) \sqcup (z \cdot y) \preccurlyeq x \sqcup y)$,
(c) $(\forall x, y, z \in A)((x \sqcup y) \rightarrow z \preccurlyeq (x \rightarrow z) \sqcup (y \rightarrow z))$,
(d) $(\forall x, y, z \in A)((z \rightarrow x) \sqcup (z \rightarrow y) \preccurlyeq z \rightarrow (x \sqcup y))$,
(e) $(\forall, y \in A)(x \sqcup y \preccurlyeq (y \rightarrow x) \sqcup (x \rightarrow y))$,
(f) $(\forall x, y \in A)((x \sqcup y) \sqcup x \equiv_{\preccurlyeq} x \sqcup y)$.

3. Three types of prime filters in QRS

The following definition gives the concept of prime filters of the first type in QRS's.

Definition 3.1. ([14]) Let F be a filter of a strong quasi-ordered residuated system $\mathfrak{A}$. Then F is said to be a *prime filter of the first type* in $\mathfrak{A}$ if the following holds

(PF1) $(\forall u, v \in A)(u \sqcup v \in F \implies (u \in F \vee v \in F))$.

Example 3.1. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	a	a
c	c	b	a	a

and

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	1
c	1	b	c	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (a,a), (b,b), (c,c), (a,b), (a,c), (b,c)\}.$$

Direct verification it can prove that $\mathfrak{A}$ is a strong quasi-ordered residuated system. The only proper filter in this system is the subset $F := \{1\}$. It is easily concluded by directly checking that F is a prime filter of the first type.

Example 3.2. Let $\mathfrak{A}$ be as in the Example 2.6. Then the subsets $\{1\}$, $\{1, b\}$ and $\{1, c\}$ are filters in $\mathfrak{A}$. It can be checked that $F_1 := \{1, b\}$ and $F_2 := \{1, c\}$ are prime filters of the first type in $\mathfrak{A}$ while the filter $\{1\}$ is not prime of the first type because we have $b \sqcup c = 1 \in \{1\}$ but $b \notin \{1\}$ and $c \notin \{1\}$.

We first show one important feature of prime filters in strong QRSs.

Proposition 3.1. *Let F be a prime filter of the first type in a strong quasi-ordered residuated system $\mathfrak{A}$. Then*

$$(\forall x, y \in A)(x \sqcup y \in F \implies (x \rightarrow y \in F \vee y \rightarrow x \in F)).$$

Proof. The proof of this proposition follows directly from the claim (e) of Proposition 2.7, (F-2) and (PF1). □

The result of the previous proposition is the motive for introducing the notion of prime filters of the second type in a strong quasi-ordered residuated system.

Definition 3.2. A filter F of a strong quasi-ordered residuated system $\mathfrak{A}$ is a *prime filter of the second type* if the following holds

(PF2) $(\forall x, y \in A)(x \to y \in F \vee y \to x \in F)$.

Example 3.3. Let $\mathfrak{A}$ be as in the Example 2.6. Subsets $\{1, c\}$ is a prime UP-filter of the second type of $\mathfrak{A}$. The subset $F := \{1, b\}$ is a prime UP-filter of the first type but it is not a prime UP-filter of the second type because, for example, holds $a \to b = c \notin F$ and $b \to a = c \notin F$.

In the previous example it was shown that a filter of a quasi-ordered residuated system can be a prime filter of the first type but it does not have to be a prime filter of the second type. However, the following theorem shows that if F satisfies the condition (PF2), then it satisfies the condition (PF1) also, i.e. any prime filter of the second type of a strong quasi-ordered residuated system $\mathfrak{A}$ is a prime filter of the first type of $\mathfrak{A}$.

Theorem 3.2. *If F is s filter of the second type in a strong quasi-ordered residuated system $\mathfrak{A}$, then F is a prime filter of the first type in $\mathfrak{A}$.*

Proof. Let $\mathfrak{A}$ be a strong quasi-ordered residuated system and assume that a filter F in $\mathfrak{A}$ satisfies the condition (PF2). Let $x \sqcup y \in F$ be holds for elements $x, y \in A$, i.e. let $(x \to y) \to y \equiv_{\preccurlyeq} (y \to x) \to x \in F$ be holds. Then from $x \to y \in F$ and $(x \to y) \to y \in F$ it follows $y \in F$, and from $y \to x \in F$ and $(y \to x) \to x \in F$ it follows $x \in F$ according to (F-3). Therefore, F is a prime filter of the first type in $\mathfrak{A}$. □

Theorem 3.3 (Extension property for prime filters of the second type)**.** *Let $\mathfrak{A}$ be a strong quasi-ordered residuated system and let F and G be filter of $\mathfrak{A}$ such that $F \subseteq G$. If F is a prime filter of the second type, then G is a prime filter of the second kind also.*

Proof. Since F is a prime filter of the second type of $\mathfrak{A}$, i.e. since it satisfies the condition (PF2), it follows that the filter G also satisfies the condition (PF2). Therefore G is a prime filter of the second type of $\mathfrak{A}$. □

The following theorem gives one sufficient condition that a filter of first type in a strong quasi-ordered residuated system be a filter of the second type.

Theorem 3.4. *Let a strong quasi-ordered residuated system $\mathfrak{A}$ satisfies the condition*

(U) $(\forall x, y \in A)((x \to y) \sqcup (y \to x) = 1)$.

Then any prime filter of the first type in $\mathfrak{A}$ is a prime filter of the second type.

Proof. Let F be a prime filter of the first type in a strong quasi-ordered residuated system $\mathfrak{A}$. If $\mathfrak{A}$ satisfy the condition (U), then from $(x \to y) \sqcup (y \to x) = 1 \in F$ it follows $x \to y \in F$ or $y \to x \in F$ because F is a prime filter of the first type in $\mathfrak{A}$. So F is a prime filter of the second type in $\mathfrak{A}$. □

Our next theorem gives one important property of prime filters of the second type in a strong QRS.

Theorem 3.5. *If the relation $\preccurlyeq$ in a strong quasi-ordered residuated system $\mathfrak{A}$ is a linear relation in the following sense*

$$(\forall x, y \in A)(x \preccurlyeq y \lor y \preccurlyeq x),$$

then every filter in $\mathfrak{A}$ is a prime filter of the second type.

Proof. Let $x, y \in A$ be arbitrary elements. Then $x \preccurlyeq y$ or $y \preccurlyeq x$ by hypothesis. Thus $1 \preccurlyeq x \rightarrow y$ or $1 \preccurlyeq y \rightarrow x$ by (3). If F is a filter in $\mathfrak{A}$, then $x \rightarrow y \in F$ or $y \rightarrow x \in F$ by (F-0) and (F-2). Hence F is a prime filter of the second type in $\mathfrak{A}$. □

Example 3.4. Let $\mathfrak{A}$ be as in the Example 3.1. The relation $\preccurlyeq$ is linear and the subset $F := \{1\}$ is a prime filter in $\mathfrak{A}$.

One connection between the linearity of the relation $\preccurlyeq$ and the requirement that the filter $\{1\}$ be a prime filter (of the first type) in a strong quasi-ordered residuated system is given by the following theorem.

Theorem 3.6. *If $\{1\}$ is a prime filter of the first type in a strong quasi-ordered residuated system $\mathfrak{A}$, then holds*

$$(\forall x, y \in A)(x \sqcup y = 1 \implies (x \preccurlyeq y \lor y \preccurlyeq x)).$$

Proof. Let $\{1\}$ be a prime fin A and let $x, y \in A$ be such that $x \sqcup y \in \{1\}$. Then $x \in \{1\}$ or $y \in \{1\}$. Thus $y \rightarrow x \in \{1\}$ or $x \rightarrow y \in \{1\}$ by (F-2). Hence $x \preccurlyeq y$ or $y \preccurlyeq x$ by (4). □

Obviously, if F is a prime filter of the second type of a strong quasi-ordered residuated system $\mathfrak{A}$, then it holds

$$(\forall x, y \in A)((x \rightarrow y) \sqcup (y \rightarrow x) \in F).$$

Indeed, from $x \rightarrow y \in F$ or $y \rightarrow x \in F$ it follows $(x \rightarrow y) \sqcup (y \rightarrow x) \in F$ because $x \rightarrow y \preccurlyeq (x \rightarrow y) \sqcup (y \rightarrow x)$ or $y \rightarrow x \preccurlyeq (x \rightarrow y) \sqcup (y \rightarrow x)$ with respect to (F-2).

The procedure exposed in the previous analysis is the motive for introducing the concept of prime filter of the third type in a strong quasi-ordered residuated system.

Definition 3.3. A *prime filter F of the third type* of a strong quasi-ordered residated system $\mathfrak{A}$ is a filter of $\mathfrak{A}$ satisfying

(PF3) $(\forall x, y \in A)((x \rightarrow y) \sqcup (y \rightarrow x) \in F)$.

From the definitions it is also clear that a prime filters of the third type has the following two properties. Therefore, we will state the following two theorems without proof.

Theorem 3.7. *Any prime filter of the second type is a prime filter of the third type.*

Theorem 3.8 (Extension property for prime filters of the third type)**.** *Let $\mathfrak{A}$ be a strong quasi-ordered residuated system and let F and G be filter of $\mathfrak{A}$ such that $F \subseteq G$. If F is a prime filter of the third type, then G is a prime filter of the third type also.*

Corollary 3.9. *If $\{1\}$ is a prime filter of the third type of a strong quasi-ordered residuated system $\mathfrak{A}$, then every filter in $\mathfrak{A}$ is a prime filter of the third type in $\mathfrak{A}$.*

The following example shows that a prime filter of the first type does not have to be a prime filter of the third type.

Example 3.5. Let $\mathfrak{A}$ be as in Example 2.6. The subset $F_2 := \{1, c\}$ is a prime filter of the second type of $\mathfrak{A}$. Thus, F_2 is a prime filter of the third type of $\mathfrak{A}$ also, by Theorem 3.6. On the other side, $F_1 := \{1, b\}$ is a prime filter of the first type but it is not a prime filter of the second type (see Example 3.1). Direct verification it can show that F_1 is not a prime filter of the first third type, too. Indeed, for example, for $x = a$ and $y = b$, we have $a \to b = c$ and $b \to a = c$ but $(x \to y) \sqcup (y \to x) = c \sqcup c = c \notin F_1$.

The following example shows that a prime filter of the third type of a strong quasi-ordered residuated system does not have to be a prime filter of the second type. Also, this example shows that a prime filter of the third type of a strong quasi-ordered residuated system does not have to be a prime filter of the first type.

Example 3.6. Let $A = \{1, a, b, c, d\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	a	a	a
b	b	a	a	a	a
c	c	a	a	a	a
d	d	a	a	a	a

and

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	b	1	1	1
c	1	c	c	1	d
d	1	d	d	c	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $\preccurlyeq := \{(a, 1), (a, a), (a, b), (a, c), (a, d), (b, 1), (b, b), (b, c), (b, d), (c, 1), (c, c), (d, 1), (d, d)\}$. Direct verification it can prove that $\mathfrak{A}$ is a strong quasi-ordered residuated system. Here it is $(c \to d) \to d = d \to d = 1$, and $(d \to c) \to c = c \to c = 1$. Subset $F := \{1\}$ is a filter of $\mathfrak{A}$. So $c \sqcup d = 1$. Obviously, this filter is not a prime filter of the first type because $c \sqcup d = 1 \in F$ but $c \notin F$ and $d \notin F$. It can be shown by direct verification that F is a prime filter of the third type of $\mathfrak{A}$. Also, this filter is not a prime filter of the second type, because for $x = c$ and $y = d$ we have $x \to y = c \to d = d \notin F$ and $y \to x = d \to c = c \notin F$.

Of course, at the end of the section in the following example we show that there is a filter in a strong quasi-ordered residuated system that it is not a prime filter of any kind listed in this article.

Example 3.7. Let $\mathfrak{A}$ be as in Example 2.7. Subset $F := \{1\}$ is a filter in $\mathfrak{A}$. But:

(i) F is not a prime filter of the first type of $\mathfrak{A}$ because $c \sqcup d = 1 \in F$ but $c \notin F$ and $d \notin F$.

(ii) F is not a prime filter of the second type of $\mathfrak{A}$ because for example, we have $b \to c = d \notin F$ and $c \to b = d \notin F$.

(iii) The filter F is not a prime filter of the third type of $\mathfrak{A}$ because for example, we have $(b \to c) \sqcup (c \to b) = d \notin F$.

4. Conclusion

The concept of quasi-ordered residuated system was introduced in [2] by Bonzio and Chajda. The concept of filters in this algebraic structure as well as various types of filters were introduced by the author ([9, 10, 11, 12, 16]). The notion strong quasi-ordered residuated systems it is designed to form an environment in which implicative and comparative filters coincide ([13]). In such algebraic structure, the author designed the notions of prime and irreducible filters ([14]).

In this paper, as a continuation of previous research, the author has dealt with the possibility of establishing three different concepts of prime filters in a strong quasi-ordered residuared system. The situation with prime filters in commutative residuated lattice $(A, \cdot, \rightarrow, \wedge, \vee, 0, 1, \preccurlyeq)$, where $\preccurlyeq$ is a quasi-ordered on A is different from the situation presented here. To this end, in order to gain insight into the types of prime filters in commutative residuated lattice, the reader can look at articles [5, 7].

In further research of these algebraic structures, one could, among other things, pay attention to the conditions that would lead to some of the types of prime filters coincide. Judging by the results obtained in this and some of the previous research, more attention should be paid to strong quasi-ordered residuated systems in which $\preccurlyeq$ is a linear relation.

It is possible to design an algebraic structure $(A, \cdot, \rightarrow, 1, \sqcup, \preccurlyeq)$ which has the following properties

(a) $(A, \cdot, 1)$ is a commutative monoid;
(b) $(\forall x, y, z \in A)(x \preccurlyeq z \iff x \preccurlyeq y \rightarrow z)$;
(c) $(A, \sqcup, 1)$ is a distributive upper semi-lattice; and
(d) $(\forall x, y \in A)((x \rightarrow y) \sqcup (y \rightarrow x) = 1)$.

Thus, it is an algebraic structure in which the last lower bound for a pair of elements does not have to be determined and it does not have to be bounded from below. The algebraic structure designed in this way is reminiscent of the determination of MTL-algebra in which the constraint requirements on the underside are omitted and, and moreover, it does not have to be a lower semi-lattice. Thus, an algebraic structure designed in this way would be an incomplete MTL-algebra. This reasoning could be accepted as a justification for studying such algebraic structures. Of course, it would be a generalization of MTL-algebra. In such an algebraic structure, any filter would be a prime filter of the third type. At the same time, each a prime filter of the first type would be a prime filter of the second type.

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