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A certain class of statistical convergence of martingale sequences and its applications to Korovkin-type approximation theorems

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Abstract. In this paper, we investigate and study the notions of statistical product convergence and statistical product summability via deferred Cesàro and deferred Nörlund product means for martingale sequences of random variables. We then establish an inclusion theorem concerning the relation between these two beautiful and definitively useful concepts. Also, based upon our proposed ideas, we demonstrate new thoughtful approximation of Korovkin-type theorems for a martingale sequence over a Banach space. Moreover, we establish that our theorems effectively extend and improve most (if not all) of the previously existing outcomes (in statistical and classical versions). Finally, by using the generalized Bernstein polynomials, we present an illustrative example of a martingale sequence in order to demonstrate that our established theorems are quite stronger than the traditional and statistical versions of different theorems existing in the literature.

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1. Introduction and motivation

Let (X_k) be a random variable defined over the probability measurable space $(\Omega, \mathfrak{F}, \mathbb{P})$. Suppose that $\mathfrak{F}_k \subseteq \mathfrak{F}$ $(k \in \mathbb{N})$ be a monotonically increasing sequence of σ-fields of measurable sets. Now, considering the random variable (X_k) with respect to measurable functions $(\mathfrak{F}_k)$, we adopt a stochastic sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$.

A given stochastic sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ is said to be a martingale sequence if

(i) $\mathbb{E}|X_k| < \infty$,

(ii) $\mathbb{E}(X_{k+1}|\mathfrak{F}_k) = X_k$ almost surely (a.s.) and

(iii) $(\mathfrak{F}_k)$ is a measurable sequence of functions,

where $\mathbb{E}$ is the mathematical expectation.

We now recall the definition of convergence of martingale sequences of random variables.

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Definition 1.1. A martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ with $\mathbb{E}|X_k|$ is bounded and $\text{Prob}(X_k) = 1$ (that is, with probability 1) is said to be convergent to a martingale $(X_0, \mathfrak{F}_0)$, if

$$\lim_{k\to\infty} (X_k, \mathfrak{F}_k) \longrightarrow (X_0, \mathfrak{F}_0) \qquad (\mathbb{E}|X_0| < \infty).$$

In the study of sequence space, the classical convergence of sequences and series has been achieved a high degree of development. Subsequently, a new concept, called the statistical convergence has been merged into this field, and it is more general than the ordinary convergence. Such a beautiful concept was introduced and studied independently by two eminent mathematicians, Fast [5] and Steinhaus [18]. Gradually, by using this valuable concept with different settings, various researchers developed many interesting and useful outcomes in several fields of mathematics such as Fourier series, Approximation theory, Probability theory, Machine Learning, Signal Processing, Measure theory, and so on. Moreover, the introduction of statistical probability convergence has enhanced the glory of this development. For some recent research works in this direction, see [2], [3], [4], [6], [7], [9], [11], [16] and [20].

Let $\mathfrak{Y} \subseteq \mathbb{N}$ and, let

$$\mathfrak{Y}_k = \{\vartheta : \vartheta \leqq k \quad \text{and} \quad \vartheta \in \mathfrak{Y}\} \qquad (k \in \mathbb{N}).$$

Then $\delta(\mathfrak{Y})$ is the natural density of $\mathfrak{Y}$, defined by

$$\delta(\mathfrak{Y}) = \lim_{k\to\infty} \frac{|\mathfrak{Y}_k|}{n} = \eta,$$

where η is a real finite number and $|\mathfrak{Y}_k|$ is the cardinality of $\mathfrak{Y}_k$.

Definition 1.2. (see [5] and [18]) A given sequence (u_k) is statistically convergent to u if, for each $\epsilon > 0$,

$$\mathfrak{Y}_\epsilon = \{\vartheta : \vartheta \leqq k \quad \text{and} \quad |u_\vartheta - u| \geqq \epsilon\} \qquad (k \in \mathbb{N})$$

has zero natural density. Thus, for each $\epsilon > 0$, we have

$$\delta(\mathfrak{Y}_\epsilon) = \lim_{k\to\infty} \frac{|\mathfrak{Y}_\epsilon|}{k} = 0.$$

We write

$$\text{stat} \lim_{k\to\infty} u_k = u.$$

We now introduce the definition of statistical convergence of martingale sequence of random variables.

Definition 1.3. A bounded martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ having its probability 1 is said to be statistically convergent to a martingale $(X_0, \mathfrak{F}_0)$ with $\mathbb{E}|X_0| < \infty$ if, for all $\epsilon > 0$,

$$\mathfrak{Y}_\epsilon = \{\vartheta : \vartheta \leqq k \quad \text{and} \quad |(X_\vartheta, \mathfrak{F}_\vartheta) - (X_0, \mathfrak{F}_0)| \geqq \epsilon\}$$

has zero natural density. This means that, for every $\epsilon > 0$, we have

$$\delta(\mathfrak{Y}_\epsilon) = \lim_{k\to\infty} \frac{|\mathfrak{Y}_\epsilon|}{k} = 0.$$

We write

$$\text{stat}_{\text{mart}} \lim_{k\to\infty} (X_k, \mathfrak{F}_k) = (X_0, \mathfrak{F}_0).$$

Example 1.1. Let $(\mathfrak{F}_k, k \in \mathbb{N})$ be a monotonically increasing sequence of 0-mean independent random variables over σ-fields. Also, let $(X_k) \in \mathfrak{F}_k$ be such that

$$X_k = \begin{cases} 1 & (k = 2^m;\ m \in \mathbb{N}) \\ \\ 0 & (\text{otherwise}). \end{cases}$$

It is easy to see that the martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ is statistically convergent to zero, but not simply martingale convergent.

Motivated essentially by the above-mentioned investigations, here we investigate and study the notions of statistical product convergence and statistical product summability via deferred Cesàro and deferred Nörlund product means for martingale sequences of random variables. We then establish an inclusion theorem concerning the relation between these two beautiful and definitively useful concepts. Also, based upon our proposed ideas, we demonstrate new thoughtful approximation of Korovkin-type theorems for a martingale sequence over a Banach space. Moreover, we establish that our theorems effectively extend and improve most (if not all) of the previously existing outcomes (in statistical and classical versions). Finally, by using the generalized Bernstein polynomials, we present an illustrative example of a martingale sequence in order to demonstrate that our established theorems are quite stronger than the traditional and statistical versions of different theorems existing in the literature.

2. A certain class of martingale sequences

Let (α_k) and (β_k) be sequences of non-negative integers such that $\alpha_k < \beta_k$ and

$$\lim_{k\to\infty} \beta_k = +\infty.$$

Then the deferred Cesàro mean for the martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ is defined by

$$\begin{aligned} \mathfrak{C}_k(X_k, \mathfrak{F}_k) &= \frac{(X_{\alpha_k+1}, \mathfrak{F}_{\alpha_k+1}) + (X_{\alpha_k+2}, \mathfrak{F}_{\alpha_k+2}) + \cdots + (X_{\beta_k}, \mathfrak{F}_{\beta_k})}{\beta_k - \alpha_k} \\ &= \frac{1}{\beta_k - \alpha_k} \sum_{i=\alpha_k+1}^{\beta_k} (X_i, \mathfrak{F}_i). \end{aligned}$$

Similarly, let (p_j) be a sequence of non-negative numbers such that

$$P_k = \sum_{j=\alpha_k+1}^{\beta_k} p_{\beta_k - j}.$$

Then the deferred Nörlund mean for the martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ of random variables is defined by

$$\mathfrak{N}_k(X_k, \mathfrak{F}_k) = \frac{1}{P_k} \sum_{j=\alpha_k+1}^{b_k} p_{\beta_k - j}(X_j, \mathfrak{F}_j).$$

We now define the product of deferred Cesàro and deferred Nörlund means for the martingale sequence as follows:

$$\Omega_k(X_k, \mathcal{F}_k) = (\mathfrak{C}\mathfrak{N})_k = \frac{1}{\beta_k - \alpha_k} \sum_{i=\alpha_k+1}^{\beta_k} (\mathfrak{N}_i)$$
$$= \frac{1}{\beta_k - \alpha_k} \sum_{i=\alpha_k+1}^{\beta_k} \frac{1}{P_i} \sum_{j=\alpha_k+1}^{\beta_k} p_{\beta_k - j}(X_j, \mathfrak{F}_j).$$

We now present the definitions of the statistical deferred Cesàro and deferred Nörlund product mean convergence (that is, DCN-mean convergence) and statistically deferred Cesàro and deferred Nörlund product mean summability (that is, DCN-mean summability) for martingale sequences of random variables.

Definition 2.1. Let (α_k) and (β_k) be sequences of non-negative integers, and let (p_ϑ) be a sequence of non-negative numbers. A bounded martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ having probability 1 is statistically deferred Cesàro and deferred Nörlund product convergent (DCN-mean convergent) to a martingale $(X_0, \mathfrak{F}_0)$ with $\mathbb{E}|X_0| < \infty$ if, for all $\epsilon > 0$,

$$\mathfrak{Y}_\epsilon = \{\vartheta : \vartheta \leqq (\beta_k - \alpha_k)P_k \quad \text{and} \quad p_{\beta_\vartheta - \vartheta}|(X_\vartheta, \mathfrak{F}_\vartheta) - (X_0, \mathfrak{F}_0)| \geqq \epsilon\}$$

has zero natural density. This means that, for every $\epsilon > 0$, we have

$$\lim_{k\to\infty} \frac{|\{\vartheta : \vartheta \leqq (\beta_k - \alpha_k)P_k \quad \text{and} \quad p_{\beta_\vartheta - \vartheta}|(X_\vartheta, \mathfrak{F}_\vartheta) - (X_0, \mathfrak{F}_0)| \geqq \epsilon\}|}{(\beta_k - \alpha_k)P_k} = 0.$$

We write

$$\Omega_{k\text{stat}} \lim_{k\to\infty} (X_k, \mathfrak{F}_k) = (X_0, \mathfrak{F}_0).$$

Definition 2.2. Let (α_k) and (β_k) be sequences of non-negative integers. A bounded martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ having probability 1 is statistically deferred Cesàro and deferred Nörlund product summable (DCN-mean summable) to a martingale $(X_0, \mathfrak{F}_0)$ with $\mathbb{E}|X_0| < \infty$ if, for all $\epsilon > 0$,

$$\mathfrak{Y}_\epsilon = \{\vartheta : \alpha_k < \vartheta \leqq \beta_k \quad \text{and} \quad |\Omega_\vartheta(X_k, \mathfrak{F}_k) - (X_0, \mathfrak{F}_0)| \geqq \epsilon\}$$

has zero natural density. This means that, for every $\epsilon > 0$, we have

$$\lim_{k\to\infty} \frac{|\{\vartheta : \alpha_k < \vartheta \leqq \beta_k \quad \text{and} \quad |\Omega_\vartheta(X_k, \mathfrak{F}_k) - (X_0, \mathfrak{F}_0)| \geqq \epsilon\}|}{\beta_k - \alpha_k} = 0.$$

We write

$$\text{stat}_{\Omega_k} \lim_{k\to\infty} \Omega_k(X_k, \mathfrak{F}_k) = (X_0, \mathfrak{F}_0).$$

We now establish an inclusion theorem concerning the above two new and interesting notions that, every statistical DCN-product mean convergent martingale sequence is statistically DCN-product mean summable, but the converse is not generally true.

Theorem 2.1. *If a given martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ is statistical DCN-mean convergent to a martingale $(X_0, \mathfrak{F}_0)$ with $\mathbb{E}|X_0| < \infty$, then it is statistically DCN- mean summable to the same martingale, but not conversely.*

Proof. Suppose the given martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ is statistically DCN-mean convergent to a martingale $(X_0, \mathfrak{F}_0)$ with $\mathbb{E}|X_0| < \infty$. Then, by Definition 2.1, we have

$$\lim_{k\to\infty} \frac{|\{\vartheta : \vartheta \leqq (\beta_k - \alpha_k)P_k \quad \text{and} \quad p_{\beta_\vartheta - \vartheta}|(X_\vartheta, \mathfrak{F}_\vartheta) - (X_0, \mathfrak{F}_0)| \geqq \epsilon\}|}{(\beta_k - \alpha_k)P_k} = 0.$$

Now, for the following two sets:

$$\mathcal{H}_\epsilon = \{\vartheta : \vartheta \leqq (\beta_k - \alpha_k)P_k \quad \text{and} \quad p_{\beta_\vartheta - \vartheta}|(X_\vartheta, \mathfrak{F}_\vartheta) - (X_0, \mathfrak{F}_0)| \geqq \epsilon\}$$

and

$$\mathcal{H}_\epsilon^c = \{\vartheta : \vartheta \leqq (\beta_k - \alpha_k)P_k \quad \text{and} \quad p_{\beta_\vartheta - \vartheta}|(X_\vartheta, \mathfrak{F}_\vartheta) - (X_0, \mathfrak{F}_0)| < \epsilon\},$$

we find that

$$\begin{aligned}
|\Omega_k(X_k, \mathfrak{F}_k) - (X_0, \mathfrak{F}_0)| &= \left| \frac{1}{\beta_k - \alpha_k} \sum_{i=\alpha_k+1}^{\beta_k} \frac{1}{P_i} \sum_{j=\alpha_k+1}^{\beta_k} p_{\beta_\vartheta - \vartheta}(X_j, \mathfrak{F}_j) - (X_0, \mathfrak{F}_0) \right| \\
&\leqq \left| \frac{1}{\beta_k - \alpha_k} \sum_{i=\alpha_k+1}^{\beta_k} \left[\frac{1}{P_i} \sum_{j=\alpha_k+1}^{\beta_k} p_{\beta_\vartheta - \vartheta}(X_j, \mathfrak{F}_j) - (X_0, \mathfrak{F}_0) \right] \right| \\
&\quad + \left| \frac{1}{\beta_k - \alpha_k} \sum_{i=\alpha_k+1}^{\beta_k} (X_0, \mathfrak{F}_0) - (X_0, \mathfrak{F}_0) \right| \\
&\leqq \frac{1}{(\beta_k - \alpha_k)P_k} \sum_{\substack{i=\alpha_k+1 \\ (\vartheta \in \mathcal{H}_\epsilon)}}^{\beta_k} |(X_k, \mathfrak{F}_k) - (X_0, \mathfrak{F}_0)| \\
&\quad + \frac{1}{(\beta_k - \alpha_k)P_k} \sum_{\substack{i=\alpha_k+1 \\ (\vartheta \in \mathcal{H}_\epsilon^c)}}^{\beta_k} |(X_k, \mathfrak{F}_k) - (X_0, \mathfrak{F}_0)| \\
&\quad + |(X_0, \mathfrak{F}_0)| \left| \frac{1}{\beta_k - \alpha_k} \sum_{\lambda=\alpha_k+1}^{\beta_k} -1 \right| \\
&\leqq \frac{1}{(\beta_k - \alpha_k)P_k} |\mathcal{H}_\epsilon| + \frac{1}{(\beta_k - \alpha_k)P_k} |\mathcal{H}_\epsilon^c| = 0.
\end{aligned}$$

Thus, clearly, we obtain

$$|\Omega_k(X_k, \mathfrak{F}_k) - (X_0, \mathfrak{F}_0)| < \epsilon.$$

Therefore, the martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ is statistically DCN-mean summable to the martingale $(X_0, \mathfrak{F}_0)$ with $\mathbb{E}|X_0| < \infty$.

Next, in support of the non-validity of the converse statement, we present here an example demonstrating that a statistically DCN- mean summable of martingale sequence is not necessarily statistically DCN- mean convergent.

Example 2.1. Let us set

$$\alpha_k = 2k \quad \beta_k = 4k \quad \text{and} \quad p_k = k \quad (k \in \mathbb{N}).$$

Also, let $(\mathfrak{F}_k, k \in \mathbb{N})$ be a monotonically increasing sequence of 0-mean independent random variables of σ-fields with $(X_k) \in \mathfrak{F}_k$ such that for k is even

$$X_k = \begin{cases} 1 & (k = 2^m;\ m \in \mathbb{N}) \\ \\ 0 & (\text{otherwise}) \end{cases}$$

and for k is odd

$$X_k = \begin{cases} -1 & (k = 2^m;\ m \in \mathbb{N}) \\ \\ 0 & (\text{otherwise}). \end{cases}$$

It is easy to see that, the martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ is neither ordinarily DCN- mean convergent nor statistically DCN- mean convergent; however, it is statistically DCN- mean summable to 0.

□

3. Korovkin-type theorems for martingale sequence

Recently, a number of researchers worked toward extending (or generalizing) the approximation aspects of the Korovkin-type theorems in different fields of mathematics such as (for example) sequence spaces, Probability space, Measurable space, and so on. This concept is extremely valuable in Real Analysis, Functional Analysis, Harmonic Analysis, and other related areas. Here, in this connection, we choose to refer the interested readers to the recent works [12], [14] and [15].

We establish here the statistical versions of new approximation of Korovin-type theorems for martingale sequences of positive linear operators via DCN product (that is, deferred Cesàro and deferred Nörlund) mean.

Let $\mathcal{C}([0,1])$ be the space of all real-valued continuous functions defined on $[0,1]$ under the norm $\|\cdot\|_\infty$. Also, let $\mathcal{C}[0,1]$ be a complete norm linear space. Then, for $g \in \mathcal{C}[0,1]$, the norm of g denoted by $\|g\|$ is given by

$$\|g\|_\infty = \sup\{|g(x)| : x \in [0,1]\}.$$

We say that the operator $\mathfrak{Z}$ is a martingale sequence of positive linear operators, provided that

$$\mathfrak{Z}(g;x) \geqq 0 \quad \text{whenever} \quad g \geqq 0 \text{ with } \mathfrak{Z}(g;x) < \infty \text{ and } \mathrm{Prob}\big(\mathfrak{Z}(g;x)\big) = 1.$$

Theorem 3.1. *Let*

$$\mathfrak{Z}_i : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$$

be a martingale sequence of positive linear operators. Then, for all $g \in \mathcal{C}[0,1]$,

$$\Omega_{k\text{stat}} \lim_{i\to\infty} \|\mathfrak{Z}_i(g;x) - g(x)\|_\infty = 0 \tag{1}$$

if and only if

$$\Omega_{k\text{stat}} \lim_{i\to\infty} \|\mathfrak{Z}_i(1;x) - 1\|_\infty = 0, \tag{2}$$

$$\Omega_{k\text{stat}} \lim_{i\to\infty} \|\mathfrak{Z}_i(2x;x) - 2x\|_\infty = 0 \tag{3}$$

and

$$\Omega_{k\text{stat}} \lim_{i\to\infty} \|\mathfrak{Z}_i(3x^2;x) - 3x^2\|_\infty = 0. \tag{4}$$

Proof. Since each of the following functions:

$$g_0(x) = 1, \quad g_1(x) = 2x \quad \text{and} \quad g_2(x) = 3x^2$$

belong to $\mathcal{C}[0,1]$ and are continuous, the implication given by (1) implies that the conditions (2) to (4) is obvious.

In order to complete the proof of the Theorem 3.1, we first assume that the conditions (2) to (4) hold true. If $f \in \mathcal{C}[0,1]$, then there exists a constant $\mathcal{V} > 0$ such that

$$|g(x)| \leqq \mathcal{V} \quad (\forall\ x \in [0,1]).$$

We thus find that

$$|g(t) - g(x)| \leqq 2\mathcal{V} \qquad (t, x \in [0,1]). \tag{5}$$

Clearly, for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|g(t) - g(x)| < \epsilon \tag{6}$$

whenever

$$|t - x| < \delta \quad \text{for all} \quad t, x \in [0,1].$$

Let us choose

$$\varphi_1 = \varphi_1(t,x) = 4(t-x)^2.$$

If $|t - x| \geqq \delta$, then we find that

$$|g(t) - g(x)| < \frac{2\mathcal{V}}{\delta^2}\varphi_1(t,x). \tag{7}$$

Thus, from the equations (6) and (7), we get

$$|g(t) - g(x)| < \epsilon + \frac{2\mathcal{V}}{\delta^2}\varphi_1(t,x),$$

which implies that

$$-\epsilon - \frac{2\mathcal{V}}{\delta^2}\varphi_1(t,x) \leqq g(t) - g(x) \leqq \epsilon + \frac{2\mathcal{V}}{\delta^2}\varphi_1(t,x). \tag{8}$$

Now, since $\mathfrak{Z}_i(1;x)$ is monotone and linear, by applying the operator $\mathfrak{Z}_i(1;x)$ to this inequality, we have

$$\begin{aligned}\mathfrak{Z}_i(1;x)\left(-\epsilon - \frac{2\mathcal{V}}{\delta^2}\varphi_1(t,x)\right) &\leqq \mathfrak{Z}_i(1;x)(g(t) - g(x)) \\ &\leqq \mathfrak{Z}_i(1;x)\left(\epsilon + \frac{2\mathcal{V}}{\delta^2}\varphi_1(t,x)\right).\end{aligned}$$

We note that x is fixed and so $h(x)$ is a constant number. Therefore, we have

$$\begin{aligned}-\epsilon\mathfrak{Z}_i(1;x) - \frac{2\mathcal{V}}{\delta^2}\mathfrak{Z}_i(\varphi_1;x) &\leqq \mathfrak{Z}_i(g;x) - g(x)\mathfrak{Z}_i(1;x) \\ &\leqq \epsilon\mathfrak{Z}_i(1;x) + \frac{2\mathcal{V}}{\delta^2}\mathfrak{Z}_i(\varphi_1;x).\end{aligned} \tag{9}$$

We also know that

$$\mathfrak{Z}_i(g;x) - g(x) = [\mathfrak{Z}_i(g;x) - g(x)\mathfrak{Z}_i(1;x)] + g(x)[\mathfrak{Z}_i(1;x) - 1]. \tag{10}$$

So, by using (9) and (10), we have

$$\mathfrak{Z}_i(g;x) - g(x) < \epsilon\mathfrak{Z}_i(1;x) + \frac{2\mathcal{V}}{\delta^2}\mathfrak{Z}_i(\varphi_1;x) + g(x)[\mathfrak{Z}_i(1;x) - 1]. \tag{11}$$

We now estimate $\mathfrak{Z}_i(\varphi_1;x)$ as follows:

$$\begin{aligned}\mathfrak{Z}_i(\varphi_1;x) &= \mathfrak{Z}_i((2t-2x)^2;x) = \mathfrak{Z}_i(2t^2 - 8xt + 4x^2;x)\\ &= \mathfrak{Z}_i(4t^2;x) - 8x\mathfrak{Z}_i(t;x) + 4x^2\mathfrak{Z}_i(1;x)\\ &= 4[\mathfrak{Z}_i(t^2;x) - x^2] - 8x[\mathfrak{Z}_i(t;x) - x] + 4x^2[\mathfrak{Z}_i(1;x) - 1].\end{aligned}$$

Thus, by using (11), we obtain

$$\begin{aligned}\mathfrak{Z}_i(g;x) - g(x) &< \epsilon\mathfrak{Z}_i(1;x) + \frac{2\mathcal{V}}{\delta^2}\{4[\mathfrak{Z}_i(t^2;x) - x^2]\\ &\qquad - 8x[\mathfrak{Z}_i(t;x) - x] + 4x^2[\mathfrak{Z}_i(1;x) - 1]\} + f(x)[\mathfrak{Z}_i(1;x) - 1].\\ &= \epsilon[\mathfrak{Z}_i(1;x) - 1] + \epsilon + \frac{2\mathcal{V}}{\delta^2}\{4[\mathfrak{Z}_i(t^2;x) - x^2]\\ &\qquad - 8x[\mathfrak{Z}_i(t;x) - x] + 4x^2[\mathfrak{Z}_i(1;x) - 1]\} + h(x)[\mathfrak{Z}_i(1;x) - 1].\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we can write

$$\begin{aligned}|\mathfrak{Z}_i(g;x) - g(x)| &\leqq \epsilon + \left(\epsilon + \frac{8\mathcal{V}}{\delta^2} + \mathcal{V}\right)|\mathfrak{Z}_i(1;x) - 1|\\ &\qquad + \frac{16\mathcal{V}}{\delta^2}|\mathfrak{Z}_i(t;x) - x| + \frac{8\mathcal{V}}{\delta^2}|\mathfrak{Z}_i(t^2;x) - x^2|\\ &\leqq \mathcal{E}(|\mathfrak{Z}_i(1;x) - 1| + |\mathfrak{Z}_i(t;x) - x| + |\mathfrak{Z}_i(t^2;x) - x^2|),\end{aligned} \tag{12}$$

where

$$\mathcal{E} = \max\left(\epsilon + \frac{8\mathcal{V}}{\delta^2} + \mathcal{V}, \frac{16\mathcal{V}}{\delta^2}, \frac{8\mathcal{V}}{\delta^2}\right).$$

Now, for a given $r > 0$, there exists $\epsilon > 0$ $(\epsilon < r)$, we get

$$\mathfrak{A}_i(x;r) = \{i : i \leqq (\beta_k - \alpha_k)P_k \quad \text{and} \quad p_{\beta_\vartheta - \vartheta}\,|\mathfrak{Z}_i(g;x) - g(x)| \geqq r\}.$$

Furthermore, for $j = 0, 1, 2$, we have

$$\mathfrak{A}_{i,j}(x;r) = \left\{i : i \leqq (\beta_k - \alpha_k)P_k \quad \text{and} \quad p_{\beta_\vartheta - \vartheta}\,|\mathfrak{Z}_m(g;x) - g_j(x)| \geqq \frac{r-\epsilon}{3\mathcal{E}}\right\},$$

so that

$$\mathfrak{A}_i(x;r) \leqq \sum_{j=0}^{2}\mathfrak{A}_{i,j}(x;r).$$

Clearly, we obtain

$$\frac{\|\mathfrak{A}_i(x;r)\|_{\mathcal{C}[0,1]}}{(\beta_k - \alpha_k)P_k} \leqq \sum_{j=0}^{2}\frac{\|\mathfrak{A}_{i,j}(x;r)\|_{\mathcal{C}[0,1]}}{(\beta_k - \alpha_k)P_k}. \tag{13}$$

Now, using the above assumption about the implications in (2) to (4) and, by Definition 2.1, the right-hand side of (13) tends to 0 as $n \to \infty$. Consequently, we get

$$\lim_{k\to\infty}\frac{\|\mathfrak{A}_i(x;r)\|_{\mathcal{C}[0,1]}}{(\beta_k - \alpha_k)P_k} = 0 \ (\delta,\ r > 0).$$

Therefore, the implication (1) holds true. This completes the proof of Theorem 3.1. □

Next, by using Definition 2.2, we present the following theorem.

Theorem 3.2. *Let $\mathfrak{Z}_i : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ be a martingale sequence of positive linear operators. Also, let $g \in \mathcal{C}[0,1]$. Then*

$$\mathrm{stat}_{\Omega_k} \lim_{i\to\infty} \|\mathfrak{Z}_i(g;x) - g(x)\|_\infty = 0 \tag{14}$$

if and only if

$$\mathrm{stat}_{\Omega_k} \lim_{i\to\infty} \|\mathfrak{Z}_i(1;x) - 1\|_\infty = 0, \tag{15}$$

$$\mathrm{stat}_{\Omega_k} \lim_{i\to\infty} \|\mathfrak{Z}_i(2x;x) - 2x\|_\infty = 0 \tag{16}$$

and

$$\mathrm{stat}_{\Omega_k} \lim_{i\to\infty} \|\mathfrak{Z}_i(3x^2;x) - 3x^2\|_\infty = 0. \tag{17}$$

Proof. The proof of the Theorem 3.2 is similar to the proof of Theorem 3.1. We, therefore, choose to skip the details involved. □

We present below an illustrative example for the martingale sequence of positive linear operators that does not satisfy the conditions of the DCN- product mean of statistical convergence versions of Korovkin-type approximation Theorem 3.1, and also the results of Jena and Paikray *et al.* ([7], [8]), but it satisfies the conditions of statistical DCN- product mean summability versions of our Korovkin-type approximation Theorem 3.2. Thus, clearly, our Theorem 3.2 is quite stronger than the results asserted by Theorem 3.1 and also the results of Jena and Paikray *et al.* ([7], [8]).

We now recall the operator

$$\tau(1+\tau D) \qquad \left(D = \frac{d}{d\tau}\right),$$

which was used by Al-Salam [1] and, more recently, by Viskov and Srivastava [19] (see [13] and[17]). Here, in our Example 3.1 below, we use this operator in conjunction with the Bernstein polynomials.

Example 3.1. Let us consider the *Bernstein polynomials* $\mathcal{B}_i(g;\tau)$ on $C[0,1]$ given by

$$\mathcal{B}_i(g;\tau) = \sum_{i=0}^{k} g\left(\frac{i}{k}\right)\binom{k}{i}\tau^i(1-\tau)^{k-i} \quad (\tau \in [0,1]). \tag{18}$$

Next, we present the martingale sequences of positive linear operators on $C[0,1]$ defined as follows:

$$\mathfrak{Z}_i(g;\tau) = [1 + (X_k, \mathfrak{F}_k)]\tau(1+\tau D)\mathcal{B}_i(g;\tau) \quad (\forall\ g \in C[0,1]) \tag{19}$$

with $(X_k, \mathfrak{F}_k)$ as already mentioned in above Example 2.1.

Now, by using our proposed operators (19), we calculate the values of the functions 1, 2τ and $3\tau^2$ as follows:

$$\mathfrak{Z}_i(1;\tau) = [1 + (X_i, \mathfrak{F}_i)]\tau(1+\tau D)1 = [1 + (X_i, \mathfrak{F}_i)]\tau,$$

$$\mathfrak{Z}_i(2\tau;\tau) = [1 + (X_i, \mathfrak{F}_i)]\tau(1 + \tau D)2\tau = [1 + (X_i, \mathfrak{F}_i)]\tau(1 + 2\tau),$$

and

$$\begin{aligned}\mathfrak{Z}_i(3\tau^2;\tau) &= [1 + (X_i, \mathfrak{F}_i)]\tau(1 + \tau D)3\left\{\tau^2 + \frac{\tau(1-\tau)}{i}\right\} \\ &= [1 + (X_i, \mathfrak{F}_i)]\left\{\tau^2\left(6 - \frac{9\tau}{i}\right)\right\},\end{aligned}$$

so that, we have

$$\mathrm{stat}_{\Omega_k} \lim_{i\to\infty} \|\mathfrak{Z}_i(1;\tau) - 1\|_\infty = 0,$$
$$\mathrm{stat}_{\Omega_k} \lim_{i\to\infty} \|\mathfrak{Z}_i(2\tau;\tau) - 2\tau\|_\infty = 0$$

and

$$\mathrm{stat}_{\Omega_k} \lim_{i\to\infty} \|\mathfrak{Z}_i(3\tau^2;\tau) - 3\tau^2\|_\infty = 0.$$

Consequently, the sequence $\mathfrak{Z}_i(g;\tau)$ satisfies the conditions (15) to (17). Therefore, by Theorem 3.2, we have

$$\mathrm{stat}_{\Omega_k} \lim_{i\to\infty} \|\mathfrak{Z}_i(g;\tau) - g\|_\infty = 0.$$

Here clearly, the given martingale sequence $(X_i, \mathfrak{F}_i)$ of functions in Example 2.1 is statistically DCN- product mean summable but not DCN- product mean statistically convergent. Thus, the martingale operators defined by (19) satisfy Theorem 3.2. However, these operators do not satisfy Theorem 3.1.

4. Concluding remarks and observations

In this concluding section of our investigation, we present several further remarks and observations concerning the various results which we have proved in this article.

Remark 4.1. Let $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ be a martingale sequence given in Example 2.1. Then, since

$$\mathrm{stat}_{\Omega_k} \lim_{i\to\infty} X_i = \frac{1}{2} \text{ on } [0,1],$$

we have

$$\mathrm{stat}_{\Omega_k} \lim_{i\to\infty} \|\mathfrak{Z}_i(g_j;x) - g_j(x)\|_\infty = 0 \qquad (j = 0,1,2). \tag{20}$$

Thus, by Theorem 3.2, we can write

$$\mathrm{stat}_{\Omega_k} \lim_{i\to\infty} \|\mathfrak{Z}_i(g;x) - g(x)\|_\infty = 0, \tag{21}$$

where

$$g_0(x) = 1, \quad g_1(x) = 2x \quad \text{and} \quad g_2(x) = 3x^2.$$

Here the martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ is neither statistically convergent nor it converges uniformly in the ordinary sense; thus, clearly, the classical and statistical versions of Korovkin-type theorems do not work here for the operators defined by (19). However, our Theorem 3.2 still works. Hence, this application indicates that our Theorem 3.2 is a non-trivial generalization of the classical as well as statistical versions of Korovkin-type theorems (see [5] and [10]).

Remark 4.2. Let $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ be a martingale sequence given in Example 2.1. Then, since

$$\operatorname{stat}_{\Omega_k} \lim_{i\to\infty} X_i = \frac{1}{2} \text{ on } [0,1],$$

so (20) holds true. Now, by applying (20) and Theorem 3.2, the condition (21) also holds true. However, since the martingale sequence $(X_k, \mathfrak{F}_k; k \in \mathbb{N})$ is not statistically DCN- product mean convergent, but it is statistically DCN- product mean summable. Thus, Theorem 3.2 is certainly a non-trivial extension of Theorem 3.1. Therefore, Theorem 3.2 is stronger than Theorem 3.1.

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