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Stochastic differential inclusions with Hilfer fractional derivative

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Abstract. In this paper, we study the existence of mild solutions of Hilfer fractional stochastic differential inclusions driven by sub fractional Brownian motion in the cases when the multivalued map is convex and non convex. The results are obtained by using fixed point theorem. Finally an example is given to illustrate the obtained results.

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1. Introduction

A differential inclusion is a generalization of the notion of an ordinary differential equation, which is often used to deal with differential equations with a discontinuous right-hand side or an inaccurately known right-hand side [19], [2]. Differential equation and inclusion with fractional order arise in many mathematics models you can see in [18], [36], [28], [22]. Basic theory of differential equation involving Caputo and Riemann-Liouville fractional derivatives can be found in [31], [30], [17], [13], [15], [16], [38]. Hilfer proposed a general operator for fractional derivative called "Hilfer fractional derivative". The two parameter family of Hilfer fractional derivative $D_{a+}^{\alpha,\beta}$ of order α and type β permits to combine between the Caputo and Riemann derivatives. the two parameters gives extra degree of feedom on the initial conditions and produces more type of stationary states. Models with Hilfer fractional derivatives are discussed in [6], [13]. Sandev et al. [35] derived the existence results of fractional diffusion equation with Hilfer fractional derivative which attained in terms of Mittag Leffer functions. Mahmudov and Mc Kibben [27] studied the controllability of fractional dynamical equation with generalized Riemann-Liouville fractional derivative by using Schauder fixed point theorem and fractional calculus. Recently, Gu and Trujillo [10] reported the existence results of fractional differential equations with Hilfer derivative based on non compact measure method.

the deterministic models often fluctuate due to noise. Naturally, the extension of such models is necessary to consider stochastic models, where the related parameters are considered as appropriate Brownian motion and stochastic process. The modeling of most problems in real situations is described by stochastic differential equations

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rather than deterministic equations. Thus, it is of great importance to design stochastic effects in the study of fractional order dynamical systems.

Chen and Li [29] reported the existence results of fractional stochastic integrodifferential equations with nonlocal initial conditions in Hilbert space. Wang [13] investigated the existence results of fractional stochastic differential equation by using Picard type approximation. Lu and Liu [25] studied the controllability of fractional stochastic hemivariational inequalities based on multivalued maps and fixed point theorem.

To the best of our knowledge, there is no work reported on stochastic differential inclusion driven by sub-fractional Brownian motion with Hilfer fractional derivative. Inspired by the previously mentioned works, in this article, we aim to study this interesting problem. We prove the existence of $\mathcal{PC}_\gamma$-mild solutions for stochastic differential inclusion driven by sub-fractional Brownian motion with Hilfer fractional derivative of the form

$$\begin{cases} D_{0^+}^{\alpha,\beta} x(t) \in Ax(t) + F(t, x_t) + g(t)\frac{dS_Q^H}{dt}, t \in J = [0, b], \\ (I_0^{1-\gamma} x)(t)|_{t=0} = \varphi \in \mathcal{B}. \end{cases} \tag{1}$$

Where $D_{0^+}^{\alpha,\beta}$ is the generalized Hilfer fractional derivative of orders $\alpha \in (0, 1)$ and type $\beta \in [0, 1]$. A is the infinitesimal generator of strongly continuous semigroup of bounded linear operator $\{T(t)\}_{t\geq 0}$.
Assume that $F : J \times \mathcal{H} \to \mathcal{P}(\mathcal{H})$ is a bounded, closed and convex multivalued map, $g : J \to L_Q^0(\mathcal{K}, \mathcal{H})$, $\mathcal{K}$ is another real separable Hilbert space with product $\langle ., . \rangle_{\mathcal{K}}$. Here $L_Q^0(\mathcal{K}, \mathcal{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{H}$ and S_Q^H is an Q-sub-fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$.
$I_0^{1-\gamma}$ is the fractional integral of orders $1 - \gamma$ $(\gamma = \alpha + \beta - \alpha\beta)$.

The plan of this paper is as follows. In section 2 we introduce some notations, definitions and preliminary facts about sub-fractional Brownian motion and fractional calculus which are useful throughout the paper. In section 3 we prove the existence of $\mathcal{PC}_\gamma$-mild solutions for problem 1 under both convexity and nonconvexity conditions of multivalued right-hand side. Finally an example is given to illustrate our result in section 4.

2. Preliminaries

In this section, we give some basic definitions, notations, lemmas and some basic facts about sub-fractional Brownian motion and fractional calculus.
Let $(\mathcal{H}, \| \,.\, \|_{\mathcal{H}}, (.,.)_{\mathcal{H}})$ and $(\mathcal{K}, \| \,.\, \|_{\mathcal{K}}, (.,.)_{\mathcal{K}})$ be the separable Hilbert spaces. The notation $\mathcal{C}(J, \mathcal{H})$ stand for the Banach space of continuous functions from J to $\mathcal{H}$ with supremum norm i.e., $\| x \|_J = \sup_{t\in J} \| x(t) \|$ and $L^1(J, \mathcal{H})$ denotes the Banach space of function $x : J \to \mathcal{H}$ which are Bochner integrable normed by $\| x \|_{L^1} = \int_0^b \| x(t) \| \, dt$, for all $x \in L^1(J, \mathcal{H})$. A measurable function $x : J \to \mathcal{H}$ is Bochner integrable if and only if $\| x \|$ is Lebesgue integrable.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets).

Definition 2.1. The sub-fractional Brownian motion (sub-fBm in short) with Hurst parameter $H \in (0,1)$ is a mean zero Gaussian process $S^H = \left\{S_t^H : t \geq 0\right\}$ with $S_H^0 = 0$ and the covariance

$$C_H(t,s) = \mathbb{E}\left[S_t^H S_s^H\right] = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + \mid t-s \mid^{2H}\right], \tag{2}$$

for all, $s, t \geq 0$.

For $H = \frac{1}{2}$, S^H coincides with the standard Brownian motion B. S^H is neither semimartingale nor a Markov process when $H \neq \frac{1}{2}$. The sub-fBm S^H has properties analogous to those of fBm (self-similarity, long-range dependence, Holder paths), but it does not have stationary increments. More works for sub-fBm can be found in Bojdecki et al. [33], [34], Tudor [4], Shen et al. [8].

The sub-fractional Brownian motion satisfies the following estimates:

$$\left[(2-2^{2H-1}) \wedge 1\right] \mid t-s \mid^{2H} \mathbb{E} \mid S^H(t) - S^H(s) \mid^2 \leqslant \left[(2-2^{2H-1}) \wedge 1\right] \mid t-s \mid^{2H}. \tag{3}$$

Thus, Kolmogorov's continuity criterion implies that sub-fBm is holder continuous of order γ for any $\gamma < H$ on any finite interval. Therefore, if y is a stochastic process with Holder continuous trajectories of order $\beta > 1-H$ then the pathwise Riemann-Stieltjes integral $\int_0^b y_t(\omega) dS^H(t)(\omega)$ exists for all $b \geq 0$. In particular, if $H > \frac{1}{2}$, the pathwise integral $\int\limits_0^b f^{'}(S_t^H) dS_t^H$ exists for all $f \in \mathcal{C}^2(\mathbb{R})$, and

$$f(S_b^H) - f(0) = \int\limits_0^b f^{'}(S_t^H) dS_t^H. \tag{4}$$

However, when $H < \frac{1}{2}$ the pathwise Riemann-Stieltjes integral $\int_0^b f^{'}(S_t^H) dS_t^H(\omega)$ does not exist. For more details, we refer the reader to [8], [23], [24].

Now we aim at introducing the Wiener integral with respect to one dimensional sub-fBm S^H. Fix a time interval $[0,b]$. We denote by Λ the linear space of $\mathbb{R}$-valued step functions on $[0,b]$, that is, $y \in \Lambda$ if

$$y(t) = \sum_{i=1}^{n-1} x_i 1_{[t_i, t_{i+1}](t)},$$

Where $t \in [0,b]$, $x_i \in \mathbb{R}$ and $0 = t_1 < t_2 < ... < t_n = b$. For $y \in \Lambda$ we define its Wiener integral with respect to S^H as

$$\int\limits_0^b y(s) dS_Q^H(s) = \sum_{i=1}^{n-1} x_i (S_{t_i+1}^H - S_{t_i}^H).$$

Let $\mathcal{H}_{\mathcal{S}^H}$ be the canonical Hilbert space associated to the sub-fBm S^H. That is $\mathcal{H}_{\mathcal{S}^H}$ is the cloture of the linear span Λ with respect to the scalar product

$$\left(1_{[0,t]}, 1_{[0,s]}\right)_{\mathcal{H}_{\mathcal{S}^H}} = C_H(t,s).$$

We know that the covariance of sub-fBm can be written as

$$\mathbb{E}\left[S_t^H S_s^H\right] = \int_0^t \int_0^s \eta_H(u,v) du dv = C_H(t,s), \tag{5}$$

where $\eta_H(u,v) = H(2H-1)\left(| u-v |^{2H-2} - (u+v)^{2H-2}\right)$.
Equation (5) implies that

$$(y,z)_{\mathcal{H}_{\mathcal{S}^H}} = \int_0^t \int_0^s y_u z_v \eta_H(u,v) du dv, \tag{6}$$

for any pair step functions y and z on $[0,b]$. Consider the kernel

$$K_H(t,s) = \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H-\frac{1}{2})} s^{3/2-H} \left(\int_0^t (x^2-s^2)^{H-3/2} ds\right) 1_{[0,t]}(s), \tag{7}$$

By Dzhaparidze and Van Zanten [20], we have

$$C_H(t,s) = c_H^2 \int_0^{t\wedge s} K_H(t,u) K_H(s,u) du \tag{8}$$

where

$$c_H^2 = \frac{\Gamma(1+2H) sin(\pi H)}{\pi}.$$

Then, (8) implies that $C_H(s,t)$ is non-negative definite. Consider the linear operator $K_H^* : \Lambda \to L^2([0,b])$ defined by

$$\left(K_H^* y\right)(s) = c_H \int_s^r y_r \frac{\partial K_H}{\partial r}(r,s) dr.$$

Using (6) (8) we have

$$\begin{aligned}
(K_H^* y, K_H^* z)_{L^2([0,b])} &= c_H^2 \int_0^b \left(\int_s^b y_r \frac{\partial K_H}{\partial r}(r,s)dr\right)\left(\int_s^b z_u \frac{\partial K_H}{\partial u}(u,s)du\right) ds \\
&= c_H^2 \int_0^b \int_0^b \left(\int_0^{r\wedge u} \frac{\partial K_H}{\partial r}(r,s) \frac{\partial K_H}{\partial u}(u,s) ds\right) y_r z_u dr du \\
&= c_H^2 \int_0^b \int_0^b \frac{\partial^2 K_H}{\partial r \partial u}(u,s) y_r z_u dr du \\
&= H(2H-1) \int_0^b \int_0^b \left(| u-r |^{2H-2} - (u+r)^{2H-2}\right) y_r z_u dr du \\
&= (y,z)_{\mathcal{H}_{\mathcal{S}^H}}.
\end{aligned} \tag{9}$$

As a consequence, the operator K_H^* provides an isometry between the Hilbert space $\mathcal{H}_{\mathcal{S}^H}$ and $L^2([0,b])$. Hence, the process W defined by $W(t) := S^H((K_H^*)^{-1}(1_{[0,t]}))$ is a Wiener process, and S^H has the following Wiener integral representation:

$$S^H(t) = c_H \int_0^t K_H(t,s) dW(s)$$

because $(K_H^*)(1_{[0,t]})(s) = c_H K_H(t,s)$. By [20], we have

$$W(t) = \int_0^t Z_H(t,s) dS^H(s),$$

where

$$Z_H(t,s) = \frac{s^{H-\frac{1}{2}}}{\Gamma(\frac{3}{2}-H)} \left[t^{H-\frac{3}{2}}(t^2-s^2)^{\frac{1}{2}-H} - \left(H-\frac{3}{2}\right)\int_s^t (x^2-s^2)^{\frac{1}{2}-H} x^{H-\frac{3}{2}} dx\right] (1_{[0,t]})(s).$$

In addition, for any $y \in \mathcal{H}_{\mathcal{S}^H}$,

$$\int_0^b y(s)dS^H(s) = \int_0^b (K_H^* y)(t)dW(t)$$

if and only if $K_H^* y \in L^2([0,b])$.
Also, denoting $L^2_{\mathcal{H}_{\mathcal{S}^H}}([0,b]) = \left\{y \in \mathcal{H}_{\mathcal{S}^H}, K_H^* y \in L^2([0,b])\right\}$. Since $H > \frac{1}{2}$, we have by (9) and lemma 2.1 of [11],

$$L^2([0,b]) \subset L^{\frac{1}{H}}([0,b]) \subset L^2_{\mathcal{H}_{\mathcal{S}^H}([0,b])}. \tag{10}$$

Lemma 2.1. [5] *For $y \in L^{\frac{1}{H}}([0,b])$,*

$$H(2H-1)\int_0^b\int_0^b | y_r || y_u || u-r |^{2H-2} drdu \leq C_H \| y \|_{L^{\frac{1}{H}}([0,b])},$$

where $C_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{1/2}$, with β denoting the beta function.

Let $L(\mathcal{K},\mathcal{H})$ denote the space of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ with the usual norm $\|.\|_{L(\mathcal{K},\mathcal{H})}$. Let $Q \in L(\mathcal{K},\mathcal{H})$ be a non-negative self-adjoint operator. Denote by $L^0_Q(\mathcal{K},\mathcal{H})$ the space of all $\xi \in L(\mathcal{K},\mathcal{H})$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$\|\xi\|^2_{L^0_Q(\mathcal{K},\mathcal{H})} = \|\xi Q^{\frac{1}{2}}\|^2_{HS} = tr(\xi Q \xi^*).$$

Then ξ is called a Q-Hilbert-Schmidt operator from $\mathcal{K}$ to $\mathcal{H}$. Let $\{S^H_n(t)\}_{n\in\mathbb{N}}$ be a sequence of one-dimensionnal standard sub-fractional Brownian motions mutually independent over $(\Omega, \mathcal{F}, \mathbb{P})$.
Set

$$S^H_Q(t) = \sum_{n=1}^{\infty} S^H_n(t) Q^{\frac{1}{2}} e_n, \qquad t \geq 0,$$

where $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in $\mathcal{K}$.
If Q is a non-negative self-adjoint trace class operator, then the above $\mathcal{K}$-valued stochastic process $S^H_Q(t)$ is called Q-cylindrical sub-fractional Brownian motion with covariance operator Q.

Lemma 2.2. [5] *For any $y : [0,b] \longrightarrow L^0_Q(\mathcal{K},\mathcal{H})$ such that $\sum\limits_{n=1}^{\infty} \|yQ^{\frac{1}{2}}e_n\|_{L^{\frac{1}{H}}([0,b],\mathcal{H})} < \infty$ holds, and for any $u, v \in [0,b]$ with $u > v$,*

$$\mathbb{E}\|\int_v^u y(s)dS^H_Q(s)\|^2_{\mathcal{H}} \leq C_H(u-v)^{2H-1}\sum_{n=1}^{\infty}\int_v^u \|y(s)Q^{\frac{1}{2}}e_n\|^2_{\mathcal{H}}ds.$$

If, in addition,

$$\sum_{n=1}^{\infty} \|y(s)Q^{\frac{1}{2}}e_n\|^2_{\mathcal{H}} \quad \textit{is uniformly convergent for} \;\; t \in [0,b],$$

then

$$\mathbb{E}\|\int_v^u y(s)dS^H_Q(s)\|^2_{\mathcal{H}} \leq C_H(u-v)^{2H-1}\int_v^u \|y(s)\|^2_{L^0_Q(\mathcal{K},\mathcal{H})}ds.$$

We suppose that $\mathcal{F}_t = \sigma\{S_Q^H; 0 \leq s \leq t\}$ is the σ-algebra generated by the $\mathcal{K}$-valued Q-cylindrical sub-fractional Brownian motion, $\mathcal{F}_b = \mathcal{F}$.

Definition 2.2. [1] The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(.)$ is the gamma function.

Definition 2.3. The Riemann-Liouville fractional derivative of order $\alpha > 0$ $n-1 < \alpha < n, n \in \mathbb{N}$, is defined as

$${}^{(R-L)}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-1-\alpha} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.4. [32] The Hilfer fractional derivative of order $0 \leq \alpha \leq 1$ and $0 < \beta < 1$ for a function f is defined by

$$D_{0+}^{\alpha,\beta} f(t) = I_{0+}^{\alpha(1-\beta)} \frac{d}{dt} I_{0+}^{(1-\alpha)(1-\beta)} f(t).$$

Remark 2.1. When $\alpha = 0$, $0 < \beta < 1$, the Hilfer fractional derivative coincides with classical Riemann-Liouville farctional derivative

$$D_{0+}^{0,\beta} f(t) = \frac{d}{dt} I_{0+}^{1-\beta} f(t) =^L D_{0+}^{\beta} f(t).$$

When $\alpha = 1$, $0 < \beta < 1$, the Hilfer fractional derivative coincides with classical Caputo fractional derivative

$$D_{0+}^{1,\beta} f(t) = I_{0+}^{1-\beta} \frac{d}{dt} f(t) =^c D_{0+}^{\beta} f(t).$$

We suppose that the phase space $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a seminormed linear space of $\mathcal{F}_0$-measurable function mapping $(-\infty, 0]$ into $\mathcal{H}$, and satisfying the following fundamental axioms due to Hale and Kato [9].

i. If $x : (-\infty, b) \to \mathcal{H}$, $b > 0$, is continuous on $(0, b]$ and x_0 in $\mathcal{B}$, then for every $t \in [0, b)$ the following conditions hold:
(a) x_t is in $\mathcal{B}$;

(b) $\| x(t) \|_{\beta} \leq \tilde{H} \| x_t \|_{\mathcal{B}}$;

(c)$\| x_t \|_{\mathcal{B}} \leq K(t) \sup\{\| x(s) \|_{\beta}: 0 \leq s \leq t\} + M(t) \| x_0 \|_{\mathcal{B}}$, where $\tilde{H} \geq 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous, M is locally bounded, and $\tilde{H}$, K, M are independent of $x(.)$.

ii. For the function $x(.)$ in i., x_t is a $\mathcal{B}$-valued function $[0, a)$.

iii. The space $\mathcal{B}$ is complete.

The following result is a consequence of the phase space axioms.

Lemma 2.3. [39] *Let $x : (-\infty, b] \to \mathcal{H}$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi(t) \in L_2^0(\Omega, \mathcal{B})$ and the restriction $x : J \to L_2^{\mathcal{F}}(\Omega, \mathcal{B})$ is continuous, then*

$$\| x_s \|_{\mathcal{B}} \leq M_b \mathbb{E} \| \varphi \|_{\mathcal{B}} + K_b \sup_{0\leq s\leq b} \mathbb{E} \| x(s) \|_{\mathcal{B}},$$

where $K_b = \sup\{K(t) : t \in J\}$ and $M_b = \sup\{M(t) : t \in J\}$.

We denote

$$\mathcal{P}_{cl}(\mathcal{H}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{H}) : \mathcal{Y} \quad \text{is closed}\}, \quad \mathcal{P}_{bd}(\mathcal{H}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{H}) : \mathcal{Y} \quad \text{is bounded}\},$$

$$\mathcal{P}_{cv}(\mathcal{H}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{H}) : \mathcal{Y} \quad \text{is convex}\}, \quad \mathcal{P}_{cp}(\mathcal{H}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{H}) : \mathcal{Y} \quad \text{is compact}\}.$$

A multi-valued map $\mathcal{G} : \mathcal{H} \to \mathcal{P}(\mathcal{H})$ is convex (closed) valued if $\mathcal{G}(\mathcal{H})$ is convex (closed) for all $x \in \mathcal{H}$. $\mathcal{G}$ is bounded on bounded sets if $\mathcal{G}(B) = \bigcup_{x\in B} \mathcal{G}(x)$ is bounded in $\mathcal{H}$ for any bounded set B of $\mathcal{H}$, that is, $\sup_{x\in B}\{\sup \|y\|_{\mathcal{H}} : y \in \mathcal{G}(x)\} < \infty$.

$\mathcal{G}$ is called upper semi continuous (u.s.c) on $\mathcal{H}$ if, for each $x \in \mathcal{H}$, the set $\mathcal{G}(x)$ is nonempty closed subset of $\mathcal{H}$ and if, for each open set V of $\mathcal{H}$ containing $\mathcal{G}(x)$, there exists an open neighborhood N of x such that $\mathcal{G}(N) \subseteq V$.
$\mathcal{G}$ is said to be completely continuous if $\mathcal{G}(B)$ is relatively compact, for every bounded subset B of $\mathcal{H}$. If the multi-valued map $\mathcal{G}$ is completely continuous with nonempty compact values, then $\mathcal{G}$ is u.s.c, if and only if $\mathcal{G}$ has closed graph i.e. $x_n \to x_*, y_n \to y_*, y_n \in \mathcal{G}(x_n)$ imply $y_* \in \mathcal{G}(x_*)$.

Definition 2.5. The multivalued map $F : J \times \mathcal{H} \longrightarrow \mathcal{P}(\mathcal{H})$ is said to be L^2-Carathéodory if

i) $t \longrightarrow F(t, v)$ is measurable for each $v \in \mathcal{H}$,
ii) $t \longrightarrow F(t, v)$ is u.s.c for almost all $t \in J$,
iii) for each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R}^+)$ such that $\|F(t, v)\|^2 = \sup_{f\in F(t,v)} E\|f\|^2 \leq h_q(t)$, for all $\|v\|^2_{\mathcal{H}} \leq q$ and for a.e. $t \in J$.

Lemma 2.4. *Let I be a compact interval and $\mathcal{H}$ be a Hilbert space. Let F be an L^2-Carathéodory multivalued map with $S_{F,x} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^2(J, \mathcal{H})$ to $C(J, \mathcal{H})$. Then the operator $F \circ S_F : C(J, \mathcal{H}) \longrightarrow \mathcal{P}_{cp,cv}(\mathcal{H}), x \longrightarrow (\Gamma \circ S_F)(x) = \Gamma(S_{F,x})$ is a closed graph operator in $C(J, \mathcal{H}) \times C(J, \mathcal{H})$, where $S_{F,x}$ is known as the selectors set from F and given by*

$$f \in S_{F,x} = \{f \in L^2([0, t], \mathcal{H}) : f(t) \in F(t, x) \ \textit{ for a.e. } \ t \in [0, T]\}.$$

Now we introduce the space $\mathcal{PC}$ formed by all $\mathcal{F}_t$-adapted measurable square integrable $\mathcal{H}$-valued stochastic process $\{x(t) : t \in [0, b]\}$ with norm $\|x\|^2_{\mathcal{PC}} = \sup_{t\in[0,b]} E\|x(t)\|^2$, then $(\mathcal{PC}, \|.\|_{\mathcal{PC}})$ is a Banach space.

We define $\mathcal{PC}_\gamma = \{x : (-\infty, b] \longrightarrow \mathcal{H} : t^{1-\gamma}x(t) \in \mathcal{PC}\}$ with norm $\|.\|_{\mathcal{PC}_\gamma}$ defined by

$$\|.\|^2_{\mathcal{PC}_\gamma} = \sup_{t\in[0,b]} E\|t^{1-\gamma}x(t)\|^2.$$

Obviously, $\mathcal{PC}_\gamma$ is a Banach space.
Let us define the operators $\{S_{\alpha,\beta}(t) : t \geq 0\}$ and $\{P_\beta(t) : t \geq 0\}$ by

$$S_{\alpha,\beta}(t) = I_{0+}^{\alpha(1-\beta)} P_\beta(t),$$
$$P_\beta(t) = t^{\beta-1} T_\beta(t),$$
$$T_\beta(t) = \int_0^\infty \beta\theta\Psi_\beta(\theta) T(t^\beta\theta) d\theta;$$

where

$$\Psi_\beta(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)\Gamma(1-n\beta)}, 0 < \beta < 1, \theta \in (0,\infty)$$

is a function of wright type which satisfies

$$\int_0^\infty \theta^\xi \Psi_\beta(\theta) d\theta = \frac{\Gamma(1+\xi)}{\Gamma(1+\beta\xi)}, \quad \xi \in (-1,\infty).$$

Lemma 2.5. [10] *The operator $S_{\alpha,\beta}$ and P_β have the following properties*

i) *For any fixed $t \geq 0$, $S_{\alpha,\beta}(t)$ and P_β are bounded linear operators, and*

$$\|P_\beta(t)x\|^2 \leq M \frac{t^{2(\beta-1)}}{(\Gamma(\beta))^2} \|x\|^2 \quad \textit{and}$$

$$\|S_{\alpha,\beta}(t)x\|^2 \leq M \frac{t^{2(\alpha-1)(1-\beta)}}{(\Gamma(\alpha(1-\beta)+\beta))^2} \|x\|^2.$$

ii) *$\{P_\beta(t) : t \geq 0\}$ is compact if $\{T(t) : t \geq 0\}$ is compact.*

Remark 2.2. $D_{0+}^{\alpha(1-\beta)} S_{\alpha,\beta}(t) = P_\beta(t)$.

Definition 2.6. An $\mathcal{H}$-valued stochastic process $\{x(t)\}$ is said to be mild solution of system 1 if the process x satisfies the following equation:

$$x(t) = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)F(s,x(s))ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s), \quad t \in J.$$

3. Existence of mild solution

3.1. The convex case. In this section, we will show the existence results of mild solutions for convex case of system 1.
So we impose the following assumptions to show the main results:
(H1) The operator A i the infinitesimal generator of a strongly continuous of bounded linear operators $\{S(t)\}_{t\geq 0}$ which is compact for $t > 0$ in $\mathcal{H}$ such that $\| S(t) \|^2 \leq M$ for each $t \in [0,b]$.
(H2)The maps $F : J \times \mathcal{H} \longrightarrow \mathcal{P}_{cp,cv}(\mathcal{H})$ is an L^2-Caratheodory function and for any $t \in [0,b]$ the multifunction $t \longrightarrow F(t,x(t))$ is measurable.
(H3) There exists a function $h_q \in L^2(J,\mathcal{H})$ such that

$$\| F(t,x) \|^2 \leq h_q(t).$$

(H4) There exist a constant $k \geq 0$ such that

$$\| F(t,x_2(t)) - F(t,x_1(t)) \|^2 \leq K \| x_2 - x_1 \|^2 .$$

(H5) There exist a constant $p > \frac{1}{2\beta-1}$ such that $g : J \longrightarrow L_2^0(J, \mathcal{H})$ satisfies $\int\limits_0^b \|g(s)\|_{L_2^0}^{2p} ds < \infty$.

Theorem 3.1. *If the assumptions (H1)-(H4) are satisfied then system 1 has a unique mild solution on $\mathcal{PC}_\gamma$ provided that*

$$\frac{\tilde{M} b^{2(\beta-\gamma)+1}}{(\Gamma(\beta))^2(2\beta-1)} < 1.$$

Proof. For an arbitrary x, we define the operator Φ on $\mathcal{PC}_\gamma$ as follows

$$(\Phi x)(t) = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)F(s,x(s))ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s).$$

We will prove that Φ has a fixed point on $\mathcal{PC}_\gamma$, the proof will be given in serval steps.
Step1: We show that Φ maps $\mathcal{PC}_\gamma$ into itself.
We divide the proof into two claims
Claim1: from lemma 2.5, Holder's inequality and hypotheses (H1)-(H4), we have

$$\begin{aligned}
&E\left\|t^{1-\gamma}x(t)\right\|^2\\
&= E\left\|t^{1-\gamma}S_{\alpha,\beta}(t)\varphi + t^{1-\gamma}\int_0^t P_\beta(t-s)F(s,x(s))ds + t^{1-\gamma}\int_0^t P_\beta(t-s)g(s)dS_Q^H(s)\right\|^2\\
&\leq 3E\left\|t^{1-\gamma}S_{\alpha,\beta}(t)\varphi\right\|^2 + 3E\left\|t^{1-\gamma}\int_0^t P_\beta(t-s)F(s,x(s))ds\right\|^2\\
&\quad + 3E\left\|t^{1-\gamma}\int_0^t P_\beta(t-s)g(s)dS_Q^H(s)\right\|^2\\
&\leq I_1 + I_2 + I_3.
\end{aligned}$$

$$\begin{aligned}
I_1 &:= 3E\left\|t^{1-\gamma}S_{\alpha,\beta}(t)\varphi\right\|^2\\
&\leq 3t^{2(1-\gamma)}M\frac{t^{2(\gamma-1)}}{(\Gamma(\gamma))^2}E\|\varphi\|^2 \leq 3\frac{M}{(\Gamma(\gamma))^2}E\|\varphi\|^2.
\end{aligned}$$

$$\begin{aligned}
I_2 &:= 3E\left\|t^{1-\gamma}\int_0^t P_\beta(t-s)F(s,x(s))ds\right\|^2\\
&\leq 3b^{2(1-\gamma)}E\left(\int_0^t \|P_\beta(t-s)F(s,x(s))\|ds\right)^2\\
&\leq 3b^{2(1-\gamma)}\frac{M}{(\Gamma(\beta))^2}E\left(\int_0^t (t-s)^{(\beta-1)}\|F(s,x(s))\|ds\right)^2\\
&\leq \frac{3Mb^{2\alpha(\beta-1)}}{(\Gamma(\beta))^2(2\beta-1)}E\int_0^t \|F(s,x(s))\|^2 ds\\
&\leq \frac{3Mb^{2\alpha(\beta-1)}}{(\Gamma(\beta))^2(2\beta-1)}E\int_0^t h_q(s)ds.
\end{aligned}$$

$$
\begin{aligned}
I_3 := 3E\left\| t^{1-\gamma}\int_0^t P_\beta(t-s)g(s)dS_Q^H(s)\right\|^2 \\
&\le 3t^{2(1-\gamma)}c_H(-t)^{2H-1}\int_0^t \|P_\beta(t-s)g(s)\|^2_{L_Q^0(\mathcal{K},\mathcal{H})}ds \\
&\le 3b^{2(1-\gamma)}c_H(-b)^{2H-1}\frac{M}{(\Gamma(\beta))^2}\int_0^t (t-s)^{2(\beta-1)}\|g(s)\|^2_{L_Q^0(\mathcal{K},\mathcal{H})}ds \\
&\le 3b^{2(1-\gamma)}c_H(-b)^{2H-1}\frac{M}{(\Gamma(\beta))^2}\left(\int_0^t (t-s)^{\frac{2p(\beta-1)}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_0^t \|g(s)\|^{2p}_{L_Q^0(\mathcal{K},\mathcal{H})}ds\right)^{\frac{1}{p}} \\
&\le 3b^{1-2\gamma+2H}c_H\frac{M}{(\Gamma(\beta))^2}\left(\int_0^t (t-s)^{\frac{2p(\beta-1)}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_0^t \|g(s)\|^{2p}_{L_Q^0(\mathcal{K},\mathcal{H})}ds\right)^{\frac{1}{p}}.
\end{aligned}
$$

Therefore Φ maps $\mathcal{PC}_\gamma$ into itself.
Claim2: We prove that $(\Phi x)(t)$ is continuous on J for any $x \in \mathcal{PC}_\gamma$.
Let $\varepsilon > 0$ and $t \in J$, then

$$
\begin{aligned}
\| (\Phi x)(t+\varepsilon) - (\Phi x)(t) \|^2_{\mathcal{PC}_\gamma} &= \sup_{0\le t\le b} E \| t^{(1-\gamma)}((\Phi x)(t+\varepsilon) - (\Phi x)(t)) \|^2 \\
&= \sup_{0\le t\le b} t^{2(1-\gamma)} E \| (\Phi x)(t+\varepsilon) - (\Phi x)(t) \|^2 \\
&\le \sup_{0\le t\le b} t^{2(1-\gamma)} E \| S_{\alpha,\beta}(t+\varepsilon)\varphi + \int_0^{t+\varepsilon} P_\beta(t+\varepsilon-s)F(s,x(s))ds \\
&\quad + \int_0^{t+\varepsilon} P_\beta(t+\varepsilon-s)g(s)dS_Q^H(s) - S_{\alpha,\beta}(t)\varphi - \int_0^t P_\beta(t-s)F(s,x(s))ds \\
&\quad - \int_0^t P_\beta(t-s)g(s)dS_Q^H(s) \|^2 \\
&\le 3\sup_{0\le t\le b} t^{2(1-\gamma)} E \| S_{\alpha,\beta}(t+\varepsilon)\varphi - S_{\alpha,\beta}(t)\varphi \|^2 \\
&\quad + 3\sup_{0\le t\le b} t^{2(1-\gamma)} E \| \int_0^{t+\varepsilon} P_\beta(t+\varepsilon-s)F(s,x(s))ds - \int_0^t P_\beta(t-s)F(s,x(s))ds \|^2 \\
&\quad + 3\sup_{0\le t\le b} t^{2(1-\gamma)} E \| \int_0^{t+\varepsilon} P_\beta(t+\varepsilon-s)g(s)dS_Q^H(s) - \int_0^t P_\beta(t-s)g(s)dS_Q^H(s) \|^2 .
\end{aligned}
$$

By Lemma 2.5 and hypothesis (H1)-(H4), we deduce that the right hand side of the above inequality tends to zero as $\varepsilon \longrightarrow 0$, then $(\Phi x)(t)$ is continuous.
Step2: (Φx) is convex for each $x \in \mathcal{PC}_\gamma$.
If $\rho_1, \rho_2 \in \Phi(x)$, then we have

$$
\rho_i = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)F(s,x_i(s))ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s).
$$

Let $0 \leq \delta \leq 1$, then for each $t \in [0, b]$ we have

$$(\delta\rho_1 + (1-\delta)\rho_2)(t) = \delta S_{\alpha,\beta}(t)\varphi + \delta \int_0^t P_\beta(t-s)F(s, x_1(s))ds + \delta \int_0^t P_\beta(t-s)g(s)dS_Q^H(s)$$
$$+ (1-\delta)S_{\alpha,\beta}(t)\varphi + (1-\delta) \int_0^t P_\beta(t-s)F(s, x_2(s))ds + (1-\delta) \int_0^t P_\beta(t-s)g(s)dS_Q^H(s)$$
$$= S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)(\delta F(s, x_1(s)) + (1-\delta)F(s, x_2(s))ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s)$$

$F(t, x)$ has a convex values, then $\delta\rho_1 + (1-\delta)\rho_2 \in \Phi(x)$.
Step3: Φ is a contraction.
For any x_1 and $x_2 \in \mathcal{PC}_\gamma$, we have

$$(\Phi x_1)(t) = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)F(s, x_1(s))ds - \int_0^t P_\beta(t-s)g(s)dS_Q^H(s).$$

$$\begin{aligned}
\|(\Phi x_2)(t) - (\Phi x_1)(t)\|^2_{\mathcal{PC}_\gamma} &= \sup_{0\leq t\leq b} E\left\|t^{1-\gamma}\left((\Phi x_2)(t) - (\Phi x_1)(t)\right)\right\|^2\\
&\leq \sup_{0\leq t\leq b} t^{2(1-\gamma)} E\left\|((\Phi x_2)(t) - (\Phi x_1)(t))\right\|^2\\
&\leq \sup_{0\leq t\leq b} t^{2(1-\gamma)} E \parallel \int_0^t P_\beta(t-s)(F(s, x_2(s)) - F(s, x_1(s))ds \parallel^2\\
&\leq \sup_{0\leq t\leq b} t^{2(1-\gamma)} E \int_0^t \parallel P_\beta(t-s)(F(s, x_2(s)) - F(s, x_1(s)) \parallel^2 ds\\
&\leq b^{2(1-\gamma)} \frac{M}{(\Gamma(\beta))^2} \parallel F(s, x_2(s)) - F(s, x_1(s)) \parallel^2 \int_0^t (t-s)^{2(\beta-1)} ds\\
&\leq \frac{\tilde{M}}{(\Gamma(\beta))^2(2\beta-1)} b^{2(\beta-\gamma)+1} \parallel x_2 - x_1 \parallel^2 .
\end{aligned}$$

Step4: $\Phi(x)$ is closed for each $x \in \mathcal{PC}_\gamma$.
Let $\{h_n\}_{n\geq 0} \in \Phi(x)$ such that $h_n \longrightarrow h$ in $\mathcal{PC}_\gamma$. Then $h \in \mathcal{PC}_\gamma$ and there exist $\{v_n\} \in S_{F,x}$ such that for each $t \in J$

$$h_n(t) = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)v_n(s)ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s).$$

Due to the fact that F has compact values, we may pass to a subsequence if necessary to get that v_n converges to v in $L^2(J, \mathcal{H})$ and hence $v \in S_{F,x}$. Then for each $t \in J$

$$h_n(t) \longrightarrow h(t) = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)v(s)ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s).$$

Thus, $h \in \Phi(x)$.

□

3.2. The non convex case. In this section, we give a non convex version of system (1).
Let $\mathcal{A}$ be a subset of $J \times \mathcal{B}$. $\mathcal{A}$ is $\mathcal{L} \otimes D$ measurable if $\mathcal{A}$ belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{B}$, where $\mathcal{J}$ is Lebesgue measurable in J and $\mathcal{B}$ is

Borel measurable in $\mathcal{B}$. A subset $\mathcal{A}$ of $L^2(J,\mathcal{H})$ is decomposable if for all $w, v \in \mathcal{A}$ and $\mathcal{J} \in J$ measurable, $w\mathcal{X}_{\mathcal{J}} + v\mathcal{X}_{J-\mathcal{J}} \in A$, where $\mathcal{X}$ denotes the characteristic function. Let $F : J \times \mathcal{H} \longrightarrow \mathcal{P}_{cp}(\mathcal{H})$. Assign to F the multivalued operator

$$\mathcal{F} : C(J,\mathcal{H}) \longrightarrow \mathcal{P}(L^2(J,\mathcal{H})),$$

Let $\mathcal{F}(x) = S_{F,x} = \{f \in L^2(J,\mathcal{H}) : f(t) \in F(t, x(t)) \text{ for a.e } t \in J\}$. The operator $\mathcal{F}$ is called the Niemytzki operator associated to F.

Definition 3.1. [26] Let Y be a separable metric space and let $N : Y \longrightarrow \mathcal{P}(L^2(J,\mathcal{H}))$ be a multivalued operator. We say that N has property (BC) if
1) N is lower semi continuous;
2) N has nonempty closed and decomposable values .

Definition 3.2. [26] $F : J \times \mathcal{H} \longrightarrow \mathcal{P}_{cp}(\mathcal{H})$ be a multivalued function with nonempty compact values. We say that F is lower semi continuous type (l.s.c type) if its associated Niemytski operator $\mathcal{F}$ is l.s.c and has nonempty closed and decomposable values.

Consider $H_d : \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \longrightarrow \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$.
Now, we give a selection theorem due to Bressan and Colombo [3].

Theorem 3.2. *Let Y be a separable metric space and let $N : Y \longrightarrow \mathcal{P}(L^2(J,\mathcal{H}))$ be a multivalued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $\tilde{g} : Y \longrightarrow L^2(J,\mathcal{H})$ such that $\tilde{g}(y) \in N(y)$ for every $y \in Y$.*

Lemma 3.3. *Let (X, d) be a complete metric space. If the multivalued operator $G : X \longrightarrow \mathcal{P}_{cl}(X)$ is a contraction then G has at least one fixed point.*

Now, we introduce the following hypothesis
(H6) $F : J \times \mathcal{H} \longrightarrow \mathcal{P}(\mathcal{H})$ is nonempty compact valued multifunction map such that
a) $(t, y) \longrightarrow F(t, y)$ is $\mathcal{L} \times \mathcal{D}$ measurable and for every $t \in J$, the multifunction $t \longrightarrow F(t, y_t)$ is measurable,
b) $(t, y) \longrightarrow F(t, y)$ is lower semi continuous for a.e.$t \in J$.

Theorem 3.4. *Under assumption (H1)-(H6), the problem 1 has at least one $\mathcal{PC}_\gamma$-mild solution.*

Proof. the proof is given in serval steps.
Consider the problem (1) on $[0, b]$

$$\begin{cases} D_{0^+}^{\alpha,\beta} x(t) \in Ax(t) + F(t, x_t) + g(t)\frac{dS_Q^H}{dt}, t \in J = [0, b], \\ (I_0^{1-\gamma} x)(t)|_{t=0} = \varphi \in \mathcal{B}. \end{cases} \tag{11}$$

Let $\mathcal{PC}_\gamma = \{x : (-\infty, b] \longrightarrow \mathcal{H} : t^{1-\gamma}x(t) \in \mathcal{PC}\}$, with $\| x \|_{\mathcal{PC}_\gamma} = (\sup_{t\in J} E \| t^{1-\gamma}x(t) \|^2)^{\frac{1}{2}}$. Thus $(\mathcal{PC}_\gamma, \| . \|_{\mathcal{PC}_\gamma})$ is a Banach space.
Let $\mathcal{D} = \mathcal{B} \cap \mathcal{PC}_\gamma$.

We transform the problem into fixed point theorem. Consider the multivalued operator $\Phi : \mathcal{D} \longrightarrow \mathcal{P}(\mathcal{D})$ defined by

$$\Phi(x) = \{\rho \in \mathcal{D} : \rho(t) = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)F(s,x(s))ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s)\}.$$

Let $\hat{\phi} : [0,b] \longrightarrow \mathcal{H}$ be a function defined by $\hat{\phi}(t) = S_{\alpha,\beta}(t)\varphi$. Then $\hat{\phi}(t)$ is an element of $\mathcal{D}$. Let $x(t) = z(t) + \hat{\phi}(t)$ for $t \in [0,b]$, with $z(t) = \int_0^t P_\beta(t-s)f(s)ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s)$, where
$f(s) \in F(t, z_t + \hat{\phi}_t)$ for a.e. $t \in [0,b]$.
Let consider the operator $\hat{\Phi} : \mathcal{PC}_\gamma \longrightarrow \mathcal{P}(\mathcal{PC}_\gamma)$ defined by

$$\hat{\Phi}(z) = \{\hat{\rho} \in \mathcal{PC}_\gamma : \hat{\rho}(t) = \int_0^t P_\beta(t-s)f(s)ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s)\}.$$

Now we show that $\hat{\Phi}$ satisfies the assumption of Lemma 3.3.
Step1: $\hat{\Phi}(t) \in \mathcal{P}(\mathcal{PC}_\gamma)$ for each $z \in \mathcal{PC}_\gamma$.
Let $z_n \in \hat{\Phi}(z)$ and $\| z_n - z \|^2_{\mathcal{PC}_\gamma} \longrightarrow 0$ for $z \in \mathcal{PC}_\gamma$ and there exist $f_n \in S_{F,z+\hat{\phi}}$ such that

$$z_n(t) = \int_0^t P_\beta(t-s)f_n(s)ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s).$$

Since $F(t, z(t) + \hat{\phi}(t))$ is compact values and from hypothesis (H6), we pass to a subsequence if necessary to get that f_n converges to f in $L^2(J, \mathcal{H})$.
Then for each $t \in [0,b]$,

$$E \| z_n(t) - \int_0^t P_\beta(t-s)f(s)ds - \int_0^t P_\beta(t-s)g(s)dS_Q^H(s) \| \longrightarrow 0 \ \text{as} \ n \longrightarrow 0,$$

so there exist a $f(.) \in S_{F,z_t+\hat{\phi}}$ such that $z(t) = \int_0^t P_\beta(t-s)f(s)ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s)$.
Step2: There exist $\delta < 1$ such that $EH_d^2(\hat{\Phi}(z_1), \hat{\Phi}(z_2)) \leq \delta \| z_1 - z_2 \|_{\mathcal{PC}_\gamma}$ for any $z_1, z_2 \in \mathcal{PC}_\gamma$.
Since for all $h_1 \in \hat{\Phi}(z_1)$, there exist $f_1(.) \in S_{F,z_1+\hat{\phi}}$ such that

$$h_1(t) = \int_0^t P_\beta(t-s)f_1(s)ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s).$$

We have $H_d(F(t, z_1(t)) + \hat{\phi}(t), F(t, z_2(t)) + \hat{\phi}(t)) \leq l(t) \| z_1 - z_2 \|$, so there exist $h_2(t) = \int_0^t P_\beta(t-s)f_2(s)ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H$.

We have

$$\begin{aligned}
\|h_2(t)-h_1(t)\|^2_{\mathcal{PC}_\gamma} &= \| \int_0^t P_\beta(t-s)(f_2(s)-f_1(s))ds \|^2_{\mathcal{PC}_\gamma} \\
&\leq \sup_{0\leq t\leq b} t^{2(1-\gamma)} E\int_0^t \| P_\beta(t-s)(f_2(s)-f_1(s)) \|^2 ds \\
&\leq b^{2(1-\gamma)} \frac{M}{(\Gamma(\beta))^2} l_f(t) \| z_2(t)-z_1(t) \|^2 \int_0^t (t-s)^{2(\beta-1)}ds \\
&\leq \frac{Ml_f(t)}{(\Gamma(\beta))^2(2\beta-1)} b^{2(\beta-\gamma)+1} \| z_2-z_1 \|^2 \\
&\leq \tilde{l}(t) \| z_1-z_2 \|^2 .
\end{aligned}$$

with $\tilde{l}(t) = \frac{b^{2\beta-2\gamma+1}}{(2\beta-1)(\Gamma(\beta))^2} Ml_f(t)$.

$EH_d^2(\hat{\Phi}(z_1)-\hat{\Phi}(z_2)) \leq \tilde{l}(t) \| z_2-z_1 \|^2$. So we conclude that $\hat{\Phi}$ is a contraction, and thus by Lemma 3.3, $\hat{\Phi}$ has a fixed point so the problem admit at least one mild solution. □

4. An example

Consider the following stochastic differential inclusion

$$\begin{cases}
D_{0^+}^{\frac{1}{2},\frac{1}{4}} y(t,\xi) \in \frac{\partial^2 y(t,\xi)}{\partial\xi^2} + F(t,x_t) + g(t)\frac{dS_Q^H}{dt}, t\in J=[0,b], \xi\in[0,\pi], \\
(I_0^{1-\gamma}y)(0) = y_0, \\
y(t,0)=y(t,\pi)=0.
\end{cases}$$

Where $D_{0^+}^{\frac{1}{2},\frac{1}{4}}$ denotes the Hilfer fractional derivative.
Let $\mathcal{H} = L^2([0,\pi],\mathbb{R})$, $F:[0,b]\times\mathcal{H}\longrightarrow\mathcal{P}(\mathcal{H})$ is bounded, closed and convex multivalued map and satisfies the condition (H1)-(H3).
The operator $A: D(A)\subset\mathcal{H}\longrightarrow\mathcal{H}$ is defined by

$$D(A) = \{y\in\mathcal{H}/y, y^{'} \text{are obsolutely continuous}, x^{''}\in\mathcal{H}|y(0)=y(\pi)=0\}.$$

S_Q^H is Q-sub fractional Brownian motion with Hurst parameter $H\in(\frac{1}{2},1)$.
$I_0^{1-\gamma}$ is the fractional integral of orders $1-\gamma$.
$Ay=y^{''}$ then $Ay = \sum_{n=1}^{\infty} n^2 < y, y_n > y_n$. where $y_n(t)=\sqrt{\frac{2}{n}}\sin(nt)\ \ n=1,2,...$
We see that A generates a compact analytic semi group $\{T(t)\}_{t>0}$ in $\mathcal{H}$.
We assume that $f_i:[0,b]\times\mathcal{H}\longrightarrow\mathcal{H}, i=1,2$ such that

i) f_1 and f_2 are u.s.c.
ii) $f_1<f_2$.
iii) For every $s>0$ there exists a function $h_q\in L^2([0,b]\times\mathcal{H})$ such that $f_i(t,x)\leq h_q(t)$.

Let $g:J\longrightarrow L_2^0([0,4],\mathcal{H})$ such that $\int_0^4 \frac{\sin(t)}{t^{\frac{1}{3}}}ds<\infty,\ \ p>-\frac{1}{2}$.

We take $F(t,x)=[f_1(t,x),f_2(t,x)]$.
All the assumptions in theorem 3.1 are verified thus this inclusion has a mild solution.

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