Boolean $BL$- algebra of fractions

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Abstract. In [4] we have introduced the notions of $BL$- algebra of fractions and maximal $BL$- algebra of quotients. The scope of this paper is to prove that these algebras are Boolean algebras (see Proposition 4.3, Corollary 4.1 and Remark 5.1) and to define the notions of $BL$ - algebra of fractions and maximal $BL$- algebra of quotients for a $BL$ - algebra $A$ relative to a Boolean subalgebra $B$ of $A (B \subseteq B(A))$.

In the last part of this paper, for a $BL$- algebra $A$ and Boolean subalgebra $B \subseteq A$, is proved the existence of a maximal $BL$ - algebra of quotients for $A$ relative to $B$ (which is a Boolean algebra, by Corollary 4.1) and we give explicit descriptions of this $BL$-algebra for some classes of $BL$-algebras and particular Boolean subalgebras $B$ of $A$. For $B = B(A)$ we obtain the results of [4]. If $BL$- algebra $A$ is an $MV$- algebra we obtain the results of [5], [6] (for $MV$- algebras).

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1. Introduction

The concept of maximal lattice of quotients for a distributive lattice was defined by J. Schmid in [17], [18] taking as a guide-line the construction of complete ring of quotients by partial morphisms introduced by G. Findlay and J. Lambek (see [15], p.36). For the case of Hilbert algebras and $MV$-algebras see [2] and [5]. The central role in this constructions is played by the concept of multiplier (defined for a distributive lattice by W. H. Cornish in [11], [12]).

For some informal explanations of notion of fraction see [15], p. 37.

The paper is organized as follows.

In Section 2 we recall the basic definitions and put in evidence many rules of calculus in $BL$-algebras which we need in the rest of paper.

In Section 3 we present the $MV$- center of a $BL$- algebra (defined by Turunen and Sessa in [20]). This is a very important construction, which associates an $MV$-algebra with every $BL$-algebra. In this way, many properties can be transferred from $MV$-algebras to $BL$-algebras and backwards.

In Section 4 we define the notion of $B$-multiplier for a $BL$-algebra $A$ relative to a Boolean subalgebra $B$ of $B(A)$; also we put in evidence many results which we need in the rest of the paper (especially in Section 5).

In Section 5 we define the notions of $BL$-algebra of fractions relative to $B$ and maximal $BL$-algebra of quotients relative to $B$ for a $BL$-algebra $A$ and a Boolean subalgebra $B$ of $B(A)$.

In the last part of this paper for a $BL$- algebra $A$ is proved the existence of the maximal $BL$-algebra of quotients of $A$ relative to a Boolean subalgebra $B \subseteq B(A)$.
for a $BL$–algebra (Theorem 5.1) and we give explicit descriptions of this $BL$–algebra for some classes of $BL$–algebras $A$ ($MV$– algebras, local $BL$–algebras, $BL$–chains, and Boolean algebras) and particular Boolean subalgebras $B$ of $A$. For $B = B(A)$ we obtain the results of [4]. In particular $BL$– algebra $A$ is an $MV$– algebras we obtain the results from [5] and [6].

### 2. Definitions and first properties

**Definition 2.1.** ([19]) An algebra $(L, \land, \lor, \circ, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ is called a residuated lattice if $(L, \land, \lor, 0, 1)$ is a distributive lattice with 0 and 1, the operation $\circ$ is an isotone, associative and commutative binary operation on $L$, and for every $x, y, z \in L$, $x \circ y \leq z$ iff $x \leq y \rightarrow z$.

**Definition 2.2.** A $BL$–algebra ([13], [19]) is an algebra

$$A = (A, \land, \lor, \circ, \rightarrow, 0, 1)$$

of type $(2,2,2,2,0,0)$ satisfying the following:

1. $(A, \land, \lor, 0, 1)$ is a bounded lattice,
2. $(A, \circ, 1)$ is a commutative monoid,
3. $\circ$ and $\rightarrow$ form an adjoint pair, i.e. $c \leq a \rightarrow b$ iff $a \circ c \leq b$ for all $a, b, c \in A$,
4. $a \land b = a \circ (a \rightarrow b)$,
5. $(a \rightarrow b) \lor (b \rightarrow a) = 1$, for all $a, b \in A$.

The origin of $BL$–algebras is in Mathematical Logic; they where invented by Hájek in [13] in order to study the „Basic Logic“ ($BL$, for short) arising from the continuous triangular norms, familiar in the framework of fuzzy set theory. They play the role of Lindenbaum algebras from classical Propositional calculus. Apart from their logical interest, $BL$–algebras have important algebraic properties (see [13], [14], [19]).

**Remark 2.1.** $BL$–algebras are exactly the commutative residuated lattices satisfying $a_4, a_5$ (see Definition 2.1).

In order to simplify the notation, a $BL$–algebra $A = (A, \land, \lor, \circ, \rightarrow, 0, 1)$ will be referred by its support set, $A$. So, in the rest of this paper by $A$ we denote a $BL$–algebra.

A $BL$–algebra is nontrivial if $0 \neq 1$. For any $BL$–algebra $A$, the reduct $L(A) = (A, \land, \lor, 0, 1)$ is a bounded distributive lattice. A $BL$–chain is a totally ordered $BL$–algebra, i.e. a $BL$–algebra such that its lattice order is total.

For any $a \in A$, we define $a^* = a \rightarrow 0$ and denote $(a^*)^*$ by $a^{**}$. Clearly, $0^* = 1$.

We define $a^0 = 1$ and $a^n = a^{n-1} \circ a$ for $n \geq 1$. The order of $a \in A, a \neq 1$, in symbols ord$(a)$ is the smallest $n \in \omega$ such that $a^n = 0$; if no such $n$ exists, then ord$(a) = \infty$.

A $BL$–algebra is called locally finite if all non unit elements in it have finite order.

**Example 2.1.** Define on the real unit interval $I = [0, 1]$ the binary operations $\circ$ and $\rightarrow$ by

$$x \circ y = \max\{0, x + y - 1\}$$

$$x \rightarrow y = \min\{1, 1 - x + y\}.$$

Then $(I, \leq, \circ, \rightarrow, 0, 1)$ is a $BL$–algebra (called Lukasiewicz structure).
Example 2.2. Define on the real unit interval $I = [0, 1]$

$$x \circ y = \min\{x, y\}$$

and $x \to y = 1$ iff $x \leq y$ and $y$ otherwise.

Then $(I, \leq, \circ, \to, 0, 1)$ is a BL-algebra (called G"odel structure).

Example 2.3. Let $\circ$ be the usual multiplication of real numbers on the unit interval $I = [0, 1]$ and $x \to y = 1$ iff $x \leq y$ and $y/x$ otherwise. Then $(I, \leq, \circ, \to, 0, 1)$ is a BL-algebra (called Product structure or Gaines structure).

Remark 2.2. Not every residuated lattice, however, is a BL-algebra (see [19], p.16).

Consider, for example a residuated lattice defined on the unit interval, for all $x, y, z \in I$, such that

$$x \circ y = 0, \text{ iff } x + y \leq \frac{1}{2} \text{ and } x \land y \text{ elsewhere}$$

$$x \to y = 1 \text{ if } x \leq y \text{ and } \max\{\frac{1}{2} - x, y\} \text{ elsewhere.}$$

Let $0 < y < x$, $x + y < \frac{1}{2}$. Then $y < \frac{1}{2} - x$ and $0 \neq y = x \land y$, but $x \circ (x \to y) = x \circ (\frac{1}{2} - x) = 0$. Therefore $a_4$ does not hold.

Example 2.4. If $(A, \wedge, \vee, 0, 1)$ is a Boolean algebra, then $(A, \land, \lor, \circ, \to, 0, 1)$ is a BL-algebra where the operation $\circ$ coincide with $\land$ and $x \to y = |x \lor y|$, for all $x, y \in A$.

Example 2.5. If $(A, \land, \lor, \to, 0, 1)$ is a relative Stone lattice (see [1], p.176), then $(A, \land, \lor, \circ, \to, 0, 1)$ is a BL-algebra where the operation $\circ$ coincide with $\land$.

Example 2.6. If $(A, \circ, \ast, 0)$ is an MV-algebra (see [10]), then $(A, \land, \lor, \circ, \to, 0, 1)$ is a BL-algebra, where for $x, y \in A$:

$$x \circ y = (x^\ast \circ y^\ast)^\ast,$$

$$x \to y = x^\ast \circ y, 1 = 0^\ast,$$

$$x \lor y = (x \to y) \to y = (y \to x) \to x \text{ and } x \land y = (x^\ast \lor y^\ast)^\ast.$$
(A_2) \((\phi \land \psi) \rightarrow \phi, \)
(A_3) \((\phi \land \psi) \rightarrow (\psi \land \phi), \)
(A_4) \((\phi \land (\phi \rightarrow \psi)) \rightarrow (\psi \land (\psi \rightarrow \phi)), \)
(A_5) \((\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\phi \land \psi) \rightarrow \chi), \)
(A_6) \(((\phi \land \psi) \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi)), \)
(A_7) \(((\phi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \phi) \rightarrow \chi) \rightarrow \chi), \)
(A_8) \(0 \rightarrow \phi. \)

The deduction rule is modus ponens: if \(\phi\) and \(\phi \rightarrow \psi\) then \(\psi\). We say that \(\phi\) is a theorem and we denote by \(\vdash \phi\) if there is a proof of \(\phi\) from \(A_1 - A_8\) using modus ponens. The completeness theorem for BL says that \(\vdash \phi\) if and only if \(\phi\) is a tautology in every standard BL-algebra.

On the set \(\text{Fmla}\) of all formulas we define the equivalence relation \(\equiv\) by:

\(\phi \equiv \psi\) iff \(\vdash \phi \leftrightarrow \psi.\)

Let us denote by \([\phi]\) the equivalence class of the formula \(\phi\), and \(L_{BL}\) the set of all equivalence classes. We define

\[0 := [\emptyset],\]
\[1 := [T],\]
\([\phi] \land [\psi] := [\phi \land \psi],\]
\([\phi] \lor [\psi] := [\phi \lor \psi],\]
\([\phi] \circ [\psi] := [\phi \& \psi],\]
\([\phi] \rightarrow [\psi] := [\phi \rightarrow \psi].\)

Then \((L_{BL}, \land, \lor, \circ, \rightarrow, 0, 1)\) is a BL-algebra.

**Example 2.8.** A product algebra (or \(P\)-algebra) \([13]\) is a BL-algebra \(A\) satisfying:

\((P_1)\) \(c^{**} \leq (a \circ c \rightarrow b \circ c) \rightarrow (a \rightarrow b),\)
\((P_2)\) \(a \land a^{*} = 0.\)

Product algebras are the algebraic counterparts of propositional Product Logic \([13]\). The standard product algebra is the Product structure.

**Example 2.9.** A G-algebra \([13], \text{Definition 4.2.12}\) is a BL-algebra \(A\) satisfying:

\((G)\) \(a \circ a = a, \text{ for all } a \in A.\)

G-algebras are the algebraic counterpart of Gödel Logic. The standard G-algebra is the Gödel structure.

**Example 2.10.** If \((A, \land, \lor, \circ, \rightarrow, 0, 1)\) is a BL-algebra and \(X\) is a nonempty set, then the set \(A^X\) becomes a BL-algebra \((A^X, \land, \lor, \circ, \rightarrow, 0, 1)\) with the operations defined pointwise. If \(f, g \in A^X\), then

\[(f \land g)(x) = f(x) \land g(x),\]
\[(f \lor g)(x) = f(x) \lor g(x),\]
\[(f \circ g)(x) = f(x) \circ g(x),\]
\[(f \rightarrow g)(x) = f(x) \rightarrow g(x)\]

for all \(x, y \in X\) and \(0, 1 : X \rightarrow A\) are the constant functions associated with 0, 1 \(\in A.\)

**Example 2.11.** \([14], [16]\)

We give an example of a finite BL-algebra which is not an MV-algebra. Let \(A = \{0, a, b, c, 1\}.\)
Define on \( A \) the following operations:

\[
\begin{array}{cccccc}
\rightarrow & 0 & c & a & b & 1 \\
0 & 1 & 1 & 1 & 1 & 1 \\
c & 0 & 1 & 1 & 1 & 1 \\
a & 0 & b & 1 & b & 1 \\
b & 0 & a & a & 1 & 1 \\
1 & 0 & c & a & b & 1 \\
\end{array}
\quad
\begin{array}{cccccc}
\odot & 0 & c & a & b & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & c & c & c & c \\
a & 0 & c & a & c & a \\
b & 0 & c & c & b & b \\
1 & 0 & c & a & b & 1 \\
\end{array}
\]

We have, \( 0 \leq c \leq a, b \leq 1 \), but \( a, b \) are incomparable, hence \( A \) is not a \( BL\)-chain.

We remark that \( x \odot y = x \land y \) for all \( x, y \in A \), so \( \text{ord}(x) = \infty \) for all \( x \in A, x \neq 0 \). It follows also that \( x \odot x = x \land x = x \) for all \( x \in A \), so \( A \) is a \( G \)-algebra. It is easy to see that \( 0^* = 1 \) and \( x^* = 0 \) for all \( x \in A, x \neq 0 \), so \( 0^{**} = 0 \) and \( x^{**} = 1 \) for all \( x \in A, x \neq 0 \). Thus, \( A \) is not an \( MV\)-algebra.

**Example 2.12.** ([14], [16])

We give an example of a finite \( MV\)-algebra which is not an \( MV\)-chain. The set

\[
L_{3 \times 2} = \{0, a, b, c, d, 1\} \approx L_A \times L_2 = \{0, 1, 2\} \times \{0, 1\} = \\
= \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}
\]

organized as lattice as in figure

\[
\begin{array}{cccc}
1 & & & \\
& c & d & \\
& a & b & \\
0 & & & \\
\end{array}
\]

and as \( BL\)-algebra with the operation \( \rightarrow \) and

\[
x \odot y = \min\{z : x \leq y \rightarrow z\} = (x \rightarrow y^*)^*, x^* = x \rightarrow 0
\]

as in the following tables, is a non-linearly ordered \( MV\)-algebra.
It is easy to see that $0^* = 1, a^* = d, b^* = c, c^* = b, d^* = a, 1^* = 0$ and $x^{**} = x$, for all $x \in A$, hence $L_{3 \times 2}$ is an MV-algebra which is not chain.

In [3], [7], [13], [19] it is proved that if $A$ is a BL-algebra and $a, a', a_1, ..., a_n, b, b', c, b_i \in A, \ (i \in I)$ then we have the following rules of calculus:

$(c_1)$ $a \odot b \leq a, b$, hence $a \odot b \leq a \land b$ and $a \odot 0 = 0$,

$(c_2)$ $a \leq b$ implies $a \odot c \leq b \odot c$,

$(c_3)$ $a \leq b$ iff $a \rightarrow b = 1$,

$(c_4)$ $1 \rightarrow a = a, a \rightarrow a = 1, a \leq b \rightarrow a, a \rightarrow 1 = 1$,

$(c_5)$ $a \odot a^* = 0$,

$(c_6)$ $a \odot b = 0$ iff $a \leq b^*$,

$(c_7)$ $a \odot b = 1$ implies $a \odot b = a \land b$,

$(c_8)$ $a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow c \rightarrow (a \rightarrow c)$,

$(c_9)$ $(a \rightarrow b) \rightarrow (a \rightarrow c) = (a \land b) \rightarrow c$,

$(c_{10}$) $a \rightarrow (b \rightarrow c) \geq (a \rightarrow b) \rightarrow (a \rightarrow c)$,

$(c_{11})$ $a \leq b$ implies $c \rightarrow a \leq b \rightarrow a \rightarrow b \rightarrow c \leq a \rightarrow c$ and $b^* \leq a^*$,

$(c_{12})$ $a \leq (a \rightarrow b)$ implies $(a \rightarrow b) \rightarrow b = a \rightarrow b$,

$(c_{13})$ $a \odot (b \lor c) = (a \odot b) \lor (a \odot c)$,

$(c_{14})$ $a \odot (b \land c) = (a \odot b) \land (a \odot c)$,

$(c_{15})$ $a \lor b = ((a \rightarrow b) \rightarrow b) \land ((b \rightarrow a) \rightarrow a)$,

$(c_{16})$ $(a \land b)^n = a^n \land b^n, (a \lor b)^n = a^n \lor b^n$, hence $a \lor b = 1$ implies $a^n \lor b^n = 1$ for any $n \geq 0$,

$(c_{17})$ $a \rightarrow (b \land c) = (a \rightarrow b) \land (a \rightarrow c)$,

$(c_{18})$ $(b \lor c) \rightarrow a = (b \lor a) \lor (c \lor a)$,

$(c_{19})$ $(a \lor b) \rightarrow c = (a \lor c) \land (b \lor c)$,

$(c_{20})$ $a \rightarrow b \leq (b \rightarrow c) \rightarrow (a \rightarrow c)$,

$(c_{21})$ $a \rightarrow b \leq (c \rightarrow a) \rightarrow (c \rightarrow b)$,

$(c_{22})$ $a \rightarrow b \leq (a \odot c) \rightarrow (b \odot c)$,

$(c_{23})$ $a \odot (b \rightarrow c) \leq b \rightarrow (a \odot c)$,

$(c_{24})$ $(b \rightarrow c) \odot (a \rightarrow b) \leq a \rightarrow c$,

$(c_{25})$ $(a_1 \rightarrow a_2) \odot (a_2 \rightarrow a_3) \odot \ldots \odot (a_{n-1} \rightarrow a_n) \leq a_1 \rightarrow a_n$,

$(c_{26})$ $a \leq b$ and $c \rightarrow a = b \rightarrow c$ implies $a = b$,

$(c_{27})$ $a \lor (b \odot c) \geq (a \lor b) \lor (a \lor c)$, hence $a^m \lor b^n \geq (a \lor b)^mn$, for any $m, n \geq 0$. 

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We have in $L_{3 \times 2}$ the following operations:
Proposition 2.2. If \( \text{Remark 2.4.} \) then Boolean algebra of all complemented elements in \( \text{Lemma 2.1.} \) by \( \text{Proposition 2.2.} \) if \( \text{Remark 2.4.} \) then \( \text{Proposition 2.2.} \) by \( \text{Lemma 2.1.} \). In the rest of this paper by \( A \) we denote a \( BL \)-algebra; by \( B(A) \) we denote the Boolean algebra of all complemented elements in \( L(A) \) (hence \( B(A) = B(L(A)) \)) and by \( B \subseteq B(A) \) we denote a Boolean subalgebra of \( A \).

**Proposition 2.1.** ([13], [19]) For \( e \in A \), the following are equivalent:

(i) \( e \in B(A) \),
(ii) \( e \circ e = e \) and \( e \circ e = e^{**} \),
(iii) \( e \circ e = e \) and \( e^{**} = e \),
(iv) \( e \circ e^{**} = e \).

**Remark 2.4.** If \( a \in A \) and \( e \in B \), then \( e \circ a = e \wedge a \), \( a \rightarrow e = (a \circ e^{**})^{*} = a^{*} \circ e \); if \( e \leq a \vee a^{*} \), then \( e \circ a \in B \).

**Proposition 2.2.** ([7]) For \( e \in A \), the following are equivalent:

(i) \( e \in B(A) \),
(ii) \( e \circ x \rightarrow e = e \), for every \( x \in A \).

**Lemma 2.1.** If \( e, f \in B \) and \( x, y \in A \), then:

\( c_{40} \) \( e \circ (x \circ y) = (e \circ x) \circ (e \circ y) \),
\( c_{41} \) \( e \wedge (x \circ y) = (e \wedge x) \circ (e \wedge y) \),
\( c_{42} \) \( e \circ (x \rightarrow y) = e \circ [(e \circ x) \rightarrow (e \circ y)] \),
\( c_{43} \) \( x \circ (e \rightarrow f) = x \circ [(x \circ e) \rightarrow (x \circ f)] \),
\( c_{44} \) \( e \rightarrow (x \rightarrow y) = (e \rightarrow x) \rightarrow (e \rightarrow y) \).

**Proof.** \( c_{40} \). We have

\[
(e \circ x) \circ (e \circ y) \equiv [(e \circ x) \circ e] \circ [(e \circ y) \circ e] \equiv [(e \circ x) \circ e] \circ [(e \circ y) \circ (x \circ y)]
\]

\[= [(e \circ x) \wedge e] \circ [(e \circ y) \circ (x \circ y)] = e \circ (e \circ y) \circ (x \circ y) = e \circ (x \circ y).
\]
Lemma 2.2. If $e \leq (e \circ x) \circ (x \circ y) = e \circ (x \circ y) = e \wedge (x \circ y)$. 

(c41). We have $(e \wedge x) \circ (e \wedge y) = (e \circ x) \circ (e \circ y) = (e \circ e) \circ (x \circ y) = e \circ (x \circ y) = e \wedge (x \circ y)$. 

(c42). By c22 we have $x \rightarrow y \leq (e \circ x) \rightarrow (e \circ y)$, hence $e \circ (x \rightarrow y) \leq e \circ [(e \circ x) \rightarrow (e \circ y)]$. Conversely, $e \circ [(e \circ x) \rightarrow (e \circ y)] \leq e$ and $(e \circ x) \circ [(e \circ y) \rightarrow (e \circ y)] \leq e \circ y \leq y$ so $e \circ [(e \circ x) \rightarrow (e \circ y)] \leq x \rightarrow y$. Hence $e \circ [(e \circ x) \rightarrow (e \circ y)] \leq e \circ (x \rightarrow y)$.

(c43). We have $x \circ [(x \circ e) \rightarrow (x \circ f)] = x \circ [(x \circ e) \rightarrow (x \wedge f)] \leq x \circ [(x \circ e) \rightarrow x] \wedge ((x \circ e) \rightarrow f)] = x \circ [1 \wedge ((x \circ e) \rightarrow f)] = x \circ ((x \circ e) \rightarrow f)$.

Definition 2.3. ([13], [19]) Let $A$ and $B$ be $BL$-algebras. A function $f : A \rightarrow B$ is a morphism of $BL$-algebras iff it satisfies the following conditions, for every $x, y \in A$:

(a6) $f(0) = 0$,

(a7) $f(x \circ y) = f(x) \circ f(y)$,

(a8) $f(x \rightarrow y) = f(x) \rightarrow f(y)$.

Remark 2.5. ([13], [19]) It follows that:

$$f(1) = 1,$$

$$f(x^*) = [f(x)]^*$$

$$f(x \vee y) = f(x) \vee f(y),$$

$$f(x \wedge y) = f(x) \wedge f(y),$$

for every $x, y \in A$.

If $f$ is bijective then the morphism $f$ is called an isomorphism of $BL$-algebras; in this case we write $A \cong B$.

Lemma 2.2. If $a, b, x$ are elements of $A$ and $a, b \leq x$ then

(c45) $a \circ (x \rightarrow b) = b \circ (x \rightarrow a)$.

Proof. We have

$$a \circ (x \rightarrow b) = (x \wedge a) \circ (x \rightarrow b) = [x \circ (x \rightarrow a)] \circ (x \rightarrow b)$$

$$= [x \circ (x \rightarrow b)] \circ (x \rightarrow a) = (x \wedge b) \circ (x \rightarrow a) = b \circ (x \rightarrow a).$$

3. MV-center of a BL-algebra

In this section we present the $MV$-center of a $BL$-algebra, defined by Turunen and Sessa in [20]. This is a very important construction, which associates an $MV$-algebra with every $BL$-algebra. In this way, many properties can be transferred from $MV$-algebras to $BL$-algebras and backwards. We shall use more times this construction in our paper.

As we saw in Example 2.6, $MV$-algebras are $BL$-algebras, and more, a $BL$-algebra $A$ is an $MV$-algebra iff $a^* = a$ for every $a \in A$.

The $MV$-center of a $A$, denoted by $MV(A)$ is defined as

$$MV(A) = \{a \in A : a^* = a\} = \{a^* : a \in A\}.$$ 

Hence, a $BL$-algebras $A$ is an $MV$-algebra iff $A = MV(A)$.

By Proposition 2.1 follow that $B(A) \subseteq MV(A)$. 


**Example 3.1.** ([20]) If $A$ is a product algebra or a $G$-algebra, then $MV(A)$ is a Boolean $BL$-algebra; if $A$ is the Product structure or the G"odel structure, then $MV(A) = \{0, 1\}$; if $A$ is the 5-element $BL$-algebra from Example 2.11, $MV(A) = \{0, 1\}$.

**Proposition 3.1.** ([20]) If $A$ be a $BL$-algebra and let us define for all $a, b \in A$,
\[ a^* \oplus b^* = (a \odot b)^*. \]

Then
(i) $(MV(A), \oplus^*, 0)$ is an $MV$-algebra,
(ii) the order $\leq$ of $A$ agrees with the one of $MV(A)$, defined by
\[ a \leq_{MV} b \iff a^* \oplus b = 0, \text{ for all } a, b \in MV(A), \]
(iii) the residuum $\rightarrow$ of $A$ coincides with the residuum $\rightarrow_{MV}$ in $MV(A)$, defined by
\[ a \rightarrow_{MV} b = a^* \oplus b, \text{ for all } a, b \in MV(A), \]
(iv) the product $\circ_{MV}$ on $MV(A)$ is such that
\[ a \odot_{MV} b = (a \odot b)^{**} = a \odot b, \text{ for all } a, b \in MV(A), \]
(v) $MV(A)$ is the largest $MV$-subalgebra of $A$.

**Proposition 3.2.** ([8]) If $A$ be a $BL-$ algebra, then $B(A) = B(MV(A))$.

4. B-Multipliers on a $BL$-algebra

**Definition 4.1.** Let $(P, \leq)$ an ordered set and $I \subseteq P$. $I$ is an order ideal (alternative terms include down-set or decreasing set) if, whenever $x \in I$, $y \in P$ and $y \leq x$, we have $y \in I$. We denote by $I(P)$ the set of all order ideals of $P$; clearly, $I(P)$ is closed under arbitrary intersections. For a nonempty set $M \subseteq P$ we denote by $< M >_P$ the order ideal of $P$ generated by $M$.

**Remark 4.1.** Is easy to prove that for a nonempty set $M \subseteq P$,
\[ < M >_P = \{x \in P : \text{there exists } a \in M \text{ such that } x \leq a\}. \]

We denote by $Id(A)$ the set of all ideals of the lattice $L(A)$ and by $I(A)$ the set of all order ideals of $A$, that is:
\[ I(A) = \{I \subseteq A : \text{if } x, y \in A, x \leq y \text{ and } y \in I, \text{then } x \in I\}. \]

**Remark 4.2.** Clearly, $Id(A) \subseteq I(A)$ and if $I_1, I_2 \in I(A)$, then $I_1 \cap I_2 \in I(A)$. Also, if $I \in I(A)$, then $0 \in I$.

By $B \subseteq B(A)$ we denote a Boolean subalgebra of $A$.

**Definition 4.2.** By $B-$ partial multiplier on $A$ we mean a map $f : I \to A$, where $I \in I(A)$, which verifies the next conditions:
\begin{enumerate}
  \item[(a9)] $f(e \odot x) = e \odot f(x)$, for every $e \in B$ and $x \in I$,
  \item[(a10)] $f(x) \leq x$, for every $x \in I$,
  \item[(a11)] If $e \in I \cap B$, then $f(e) \in B$,
  \item[(a12)] $x \land f(e) = e \land f(x)$, for every $e \in I \cap B$ and $x \in I$.
\end{enumerate}

**Remark 4.3.** For every $e \in I \cap B$ and $x \in I$, $x \land f(e) = e \land f(x) \Leftrightarrow x \odot f(e) = e \odot f(x)$. 


By \( \text{dom}(f) \in I(A) \) we denote the domain of \( f \); if \( \text{dom}(f) = A \), we called \( f \) total.

To simplify the language, we will use multiplier instead \( B \) - partial multiplier using total to indicate that the domain of a certain multiplier is \( A \).

**Examples**

1. The map \( 0 : A \to A \) defined by \( 0(x) = 0 \), for every \( x \in A \) is a total multiplier on \( A \); indeed if \( x \in A \) and \( e \in B \), then \( 0(e \circ x) = 0 = e \circ 0 = e \circ 0(x) \) and \( 0(x) \leq x \).

   Clearly, if \( e \in A \cap B = B \), then \( 0(e) = 0 \in B \) and for \( x \in A \), \( x \land 0(e) = e \land 0(x) = 0 \).

2. The map \( 1 : A \to A \) defined by \( 1(x) = x \), for every \( x \in A \) is also a total multiplier on \( A \); indeed if \( x \in A \) and \( e \in B \), then \( 1(e \circ x) = e \circ x = e \circ 1(x) \) and \( 1(x) = x \leq x \).

   The conditions \( a_{11} - a_{12} \) are obviously verified.

3. For \( a \in B \) and \( I \in I(A) \), the map \( f_a : I \to A \) defined by \( f_a(x) = a \land x \), for every \( x \in I \) is a multiplier on \( A \) (called principal). Indeed, for \( x \in I \) and \( e \in B \), we have \( f_a(e \circ x) = e \land (e \circ x) = e \land (e \land x) = e \circ (a \land x) = e \circ f_a(x) \) and clearly \( f_a(x) \leq x \).

   Also, if \( e \in I \cap B \), \( f_a(e) = e \land a \in B \) and \( x \land (a \land e) = e \land (a \land x) \), for every \( x \in I \).

**Remark 4.4.** The condition \( a_{12} \) is not a consequence of \( a_9 - a_{11} \). As example, \( f : I \to A \), \( f(x) = x 
\land x^* \) for every \( x \in I \), verify \( a_9 - a_{11} \), but if \( e \in I \cap B \) and \( x \in I \), then

\[
x \land f(e) = x \land 0 \neq e \land (x \land x^*) = e \land f(x).
\]

**Remark 4.5.** In general, if consider \( a \in A \), then \( f_a : I \to A \) verifies only \( a_9, a_{10} \) and \( a_{12} \) but does not verify \( a_{11} \).

If \( \text{dom}(f_a) = A \), we denote \( f_a \) by \( \overline{f_a} \); clearly, \( \overline{0} = 0 \).

For \( I \in I(A) \), we denote

\[
M(I, A) = \{ f : I \to A \mid f \text{ is a multiplier on } A \}
\]

and

\[
M(A) = \bigcup_{I \in I(A)} M(I, A).
\]

If necessary, we denote \( M(I, A) \) by \( M_{SG}(I, A) \) to indicate that we work in \( BL \)-algebras; for the case of \( MV \)-algebras we denote \( M(I, A) \) by \( M_{MV}(I, A) \).

**Remark 4.6.** From Propositions 3.1 and 3.2 we deduce that for every \( I \in I(A) \) the algebra of multipliers \( M_{SG}(I, A) \) for a \( BL \)-algebra is in fact a generalization of the algebra of multipliers \( M_{MV}(I, A) \) for \( MV \)-algebras (see [5], [6]). Also, we deduce that if \( A \) is an \( MV \)-algebra (that is \( A = MV(A) \), then \( M_{SG}(I, A) = M_{MV}(I, A) \) for every \( I \in I(A) \).

**Definition 4.3.** If \( I_1, I_2 \in I(A) \) and \( f_i \in M(I_i, A) \), \( i = 1, 2 \), we define \( f_1 \land f_2, f_1 \lor f_2, f_1 \land f_2, f_1 \lor f_2 : I_1 \cap I_2 \to A \) by

\[
(f_1 \land f_2)(x) = f_1(x) \land f_2(x),
\]

\[
(f_1 \lor f_2)(x) = f_1(x) \lor f_2(x),
\]

\[
(f_1 \land f_2)(x) = f_1(x) \circ [x \to f_2(x)] \land \exists f_2(x) \circ [x \to f_1(x)],
\]

\[
(f_1 \lor f_2)(x) = x \circ [f_1(x) \to f_2(x)].
\]

for every \( x \in I_1 \cap I_2 \).

**Lemma 4.1.** \( f_1 \land f_2 \in M(I_1 \cap I_2, A) \).
Lemma 4.3. \( f_1 \circ f_2(x) = f_1(e \circ x) \circ (e \circ f_2(x)) = (e \circ f_1(x)) \circ (e \circ f_2(x)) = e \circ f_1(x) \circ e \circ f_2(x) = e \circ (f_1 \circ f_2)(x) \).

Since \( f_1 \in M(I, A), i = 1, 2 \), we have \( f_1 \circ f_2(x) = f_1(x) \circ f_2(x) \leq x \circ x = x, \) for every \( x \in I \cap I_2, \) and if \( e \in I_1 \cap I_2 \cap B, \) then
\[
(f_1 \circ f_2)(e) = f_1(e) \circ f_2(e) \in B.
\]

Proof. If \( x \in I_1 \cap I_2 \) and \( e \in B, \) then \( (f_1 \wedge f_2)(e \circ x) = f_1(e \circ x) \wedge f_2(e \circ x) = (e \circ f_1(x)) \wedge (e \circ f_2(x)) = (e \wedge f_1(x)) \wedge (e \wedge f_2(x)) = e \wedge (f_1 \wedge f_2)(x). \)

Lemma 4.4. \( f_1 \circ f_2 \in M(I \cap I_2, A). \)

Proof. If \( x \in I_1 \cap I_2 \) and \( e \in B, \) then \( (f_1 \wedge f_2)(e \circ x) = f_1(e \circ x) \wedge f_2(e \circ x) = (e \circ f_1(x)) \wedge (e \circ f_2(x)) = e \wedge (f_1 \wedge f_2)(x). \)

Since \( f_1 \in M(I, A), i = 1, 2 \), we have \( f_1 \wedge f_2(x) = f_1(x) \wedge f_2(x) \leq x \wedge x = x, \) for every \( x \in I \cap I_2, \) and if \( e \in I_1 \cap I_2 \cap B, \) then
\[
(f_1 \circ f_2)(e) = f_1(e) \circ f_2(e) \in B.
\]

Lemma 4.3. \( f_1 \circ f_2 \in M(I \cap I_2, A). \)

Proof. If \( x \in I_1 \cap I_2 \) and \( e \in B, \) then
\[
(f_1 \circ f_2)(e \circ x) = f_1(e \circ x) \circ (e \circ f_2(x)) = [e \circ f_1(x)] \circ [(e \circ x) \rightarrow (e \circ f_2(x))] = f_1(x) \circ [(e \circ x) \rightarrow (e \circ f_2(x))] \circ [e \circ (x \rightarrow f_2(x))].
\]

Clearly, \( (f_1 \circ f_2)(x) = f_1(x) \circ [x \rightarrow f_2(x)] \leq f_1(x) \leq x, \) for every \( x \in I \cap I_2, \) and if \( e \in I_1 \cap I_2 \cap B, \) then by Remark 2.4 we have
\[
(f_1 \circ f_2)(e) = f_1(e) \circ [e \rightarrow f_2(e)] = f_1(e) \circ (e^* \circ f_2(e)) \in B.
\]

Lemma 4.4. \( f_1 \rightarrow f_2 \in M(I \cap I_2, A). \)
Proof. If \( x \in I_1 \cap I_2 \) and \( e \in B \), then
\[
(f_1 \rightarrow f_2)(e \circ x) = (e \circ x) \circ [f_1(e \circ x) \rightarrow f_2(e \circ x)] = (e \circ x) \circ [(e \circ f_1(x)) \rightarrow (e \circ f_2(x))] = x \circ [(e \circ f_1(x)) \rightarrow (e \circ f_2(x))] = x \circ [e \circ ((e \circ f_1(x)) \rightarrow (e \circ f_2(x)))]
\]
which is an algebra for every algebra \( A \). To prove that
\[
\longrightarrow \quad (f_1 \rightarrow f_2)(e) = e \circ (f_1 \rightarrow f_2)(x) = e \circ [(f_1(e))^* \vee f_2(e)] \in B.
\]
for \( e \in I_1 \cap I_2 \cap B \), then by Remark 2.4 we have
\[
(f_1 \rightarrow f_2)(e) = e \circ (f_1(e) \rightarrow f_2(e)) = e \circ [(f_1(e))^* \vee f_2(e)] \in B.
\]
For \( e \in I_1 \cap I_2 \cap B \) and \( x \in I_1 \cap I_2 \) we have:
\[
e \wedge (f_1 \rightarrow f_2)(x) = e \wedge [x \circ (f_1(x) \rightarrow f_2(x))] =
\]
\[
= e \circ [x \circ (f_1(x) \rightarrow f_2(x))] = e \circ [x \circ (f_1(x) \rightarrow f_2(x))]
\]
which is an algebra for every algebra \( A \). To prove that
\[
\longrightarrow \quad x \wedge (f_1 \rightarrow f_2)(x) = e \wedge (f_1 \rightarrow f_2)(x),
\]
that is \( f_1 \rightarrow f_2 \in M(I_1 \cap I_2, A) \).

Proposition 4.1. \((M(A), \wedge, \vee, \sqcup, \rightarrow, 0, 1)\) is a BL-algebra.

Proof. See [4], Proposition 13.

Remark 4.7. To prove that \((M(A), \wedge, \vee, \sqcup, \rightarrow, 0, 1)\) is a BL-algebra it is sufficient to ask for multipliers to verify only the axioms \( a_9 \) and \( a_{10} \).

Proposition 4.2. If \( BL^{-} \) algebra \((A, \wedge, \vee, \circ, \rightarrow, 0, 1)\) is an \( MV^{-} \) algebra \((A, \circ^*, \rightarrow)\) (i.e. \( x^{**} = x \), for all \( x \in A \)), then \( BL^{-} \) algebra \((M(A), \land, \lor, \rightarrow, 0, 1)\) is an \( MV^{-} \) algebra \((M(A), \land^*, \rightarrow, 0, 1)\), see [6]. If \( I_1, I_2 \in \mathcal{I}(A) \) and \( f_i \in M(I_i, A), i = 1, 2 \), we have \( f_1 \overline{\sqcup} f_2 : I_1 \cap I_2 \rightarrow A \),
\[
(f_1 \overline{\sqcup} f_2)(x) = (f_1(x) \circ f_2(x)) \wedge x,
\]
for every \( x \in I_1 \cap I_2 \); for \( I \in \mathcal{I}(A) \) and \( f \in M(I, A) \) we have \( f^* : I \rightarrow A \)
\[
f^*(x) = (f \rightarrow 0)(x) = x \circ (f(x) \rightarrow 0(x)) = x \circ (f(x) \rightarrow 0) = x \circ (f(x))^*,
\]
for every \( x \in I \).

Proof. To prove that \( BL^{-} \) algebra \((M(A)) \) is an \( MV^{-} \) algebra let \( f \in M(I, A) \) with \( I \in \mathcal{I}(A) \).

Then
\[
f^{**}(x) = [(f \rightarrow 0) \rightarrow 0]((f \rightarrow 0)(x)) = x \circ [(f \rightarrow 0)(x)]^* = x \circ [x \circ (f(x))^*]^*
\]
which is an algebra for every algebra \( A \). To prove that
\[
\longrightarrow \quad x \circ [x \circ (f(x))^*]^* \rightarrow 0 \quad \text{is an algebra for every algebra \( A \).}
\]
\[
(f_1 \overline{\sqcup} f_2)(x) = x \circ (f_1(x))^* \circ (x \rightarrow f_2(x))^* = x \circ [(f_1(x))^* \circ x \circ (x \rightarrow (f_2(x))^*)] = x \circ [(f_1(x))^* \circ x \circ (f_2(x))^*]
\]
for every \( x \in I \).

So, \( f^{**} = f \) and \( BL^{-} \) algebra \((M(A)) \) is an \( MV^{-} \) algebra.

We have \( f_1 \overline{\sqcup} f_2 = (f_1^* \sqcup f_2^*)^* \) and \( f^* = f \rightarrow 0 \).

Clearly,
\[
(f_1 \overline{\sqcup} f_2)(x) = x \circ (f_1(x))^* \circ (x \rightarrow f_2(x))^* = x \circ (f_1(x))^* \circ (x \rightarrow x \circ (f_2(x))^*) = x \circ (f_1(x))^* \circ x \circ (f_2(x))^*
\]
that
Remark 4.8. The condition $I \in R(A)$ is equivalent with the condition: for every $x, y \in A$, if $f_{x\cap I} = f_{y\cap I}$, then $x = y$.

Lemma 4.6. If $I_1, I_2 \in I(A) \cap R(A)$, then $I_1 \cap I_2 \in I(A) \cap R(A)$.

Proof. See [4], Lemma 15. ■

Remark 4.9. By Lemma 4.6, we deduce that

$$M_r(A) = \{ f \in M(A) : \text{dom}(f) \in I(A) \cap R(A) \}$$

is a BL-subalgebra of $M(A)$.

Proposition 4.3. $M_r(A)$ is a Boolean subalgebra of $M(A)$. 

\[
x \circ [x \circ (f_1(x))^* \circ (f_2(x))^*]^* \cong x \circ [x \rightarrow ((f_1(x))^* \circ (f_2(x))^*)^*] \\
\cong x \land (f_1(x) \oplus f_2(x)),
\]

for all $x \in I_1 \cap I_2$. Then $(M(A), \oplus^*,0)$ is an MV-algebra. ■

Lemma 4.5. The map $v_A : B \rightarrow M(A)$ defined by $v_A(a) = \overline{a}$ for every $a \in B$, is a monomorphism of BL-algebras.

Proof. Clearly, $v_A(0) = \overline{0} = 0$. Let $a, b \in B$ and $x \in A$. We have:

$$(v_A(a) \square v_A(b))(x) = v_A(a)(x) \circ (x \rightarrow v_A(b)(x)) = (a \land x) \circ (x \rightarrow (b \land x))$$

$$= (a \circ x) \circ (x \rightarrow (b \land x)) = a \circ [x \circ (x \rightarrow (b \land x))] = a \circ [x \land (b \land x)]$$

$$= a \land [x \land (b \land x)] = a \land (b \land x) = (a \land b) \land x = (v_A(a \land b))(x) = (v_A(a \land b))(x),$$

hence

$$v_A(a \land b) = v_A(a) \square v_A(b).$$

Also,

$$(v_A(a) \rightarrow v_A(b))(x) = x \circ [v_A(a)(x) \rightarrow v_A(b)(x)] = x \circ [(a \land x) \rightarrow (b \land x)]$$

$$= x \circ [(x \circ a) \rightarrow (x \circ b)] \cong x \circ (a \rightarrow b) = x \land (a \rightarrow b)$$

(since $a \rightarrow b \in B$)

$$= v_A(a \rightarrow b)(x),$$

hence

$$v_A(a) \rightarrow v_A(b) = v_A(a \rightarrow b),$$

that is $v_A$ is a morphism of BL-algebras.

To prove the injectivity of $v_A$ let $a, b \in B$ such that $v_A(a) = v_A(b)$. Then $a \land x = b \land x$, for every $x \in A$, hence for $x = 1$ we obtain that $a \land 1 = b \land 1 \Rightarrow a = b$. ■

Definition 4.4. A nonempty set $I \subseteq A$ is called regular if for every $x, y \in A$ such that $x \land e = y \land e$ for every $e \in I \cap B$, then $x = y$.

For example $A$ is a regular subset of $A$ (since if $x, y \in A$ and $x \land e = y \land e$ for every $e \in A \cap B = B$, then for $e = 1$ we obtain $x \land 1 = y \land 1 \Leftrightarrow x = y$).

More generally, every subset of $A$ which contains 1 is regular.

We denote

$$R(A) = \{ I \subseteq A : I \text{ is a regular subset of } A \}.$$
Lemma 4.10. Every multiplier \( f \in M_r(A) \) can be extended to a maximal multiplier.

Proof. See [4], Lemma 17. ■

Lemma 4.11. The axioms \( a_{11} \) and \( a_{12} \) are necessary in the proof of Proposition 4.3.

Definition 4.5. Given two multipliers \( f_1, f_2 \) on \( A \), we say that \( f_2 \) extends \( f_1 \) if \( \text{dom}(f_1) \subseteq \text{dom}(f_2) \) and \( f_2|_{\text{dom}(f_1)} = f_1 \); we write \( f_1 \leq f_2 \) if \( f_2 \) extends \( f_1 \). A multiplier \( f \) is called maximal if \( f \) can not be extended to a strictly larger domain.

Lemma 4.7. If \( f_1, f_2 \in M(A), f \in M_r(A) \) and \( f \leq f_1, f \leq f_2 \), then \( f_1 \) and \( f_2 \) agree on the \( \text{dom}(f_1) \cap \text{dom}(f_2) \).

Proof. See [4], Lemma 17. ■

Lemma 4.9. Each principal multiplier \( f_a \) with \( a \in B \) and \( \text{dom}(f_a) \in I(A) \cap R(A) \) can be uniquely extended to the total multiplier \( \overline{f_a} \) and each non-principal multiplier can be extended to a maximal non-principal one.

Proof. See [4], Lemma 17. ■

On the Boolean algebra \( M_r(A) \) we consider the relation \( \rho_A \) defined by
\[
(f_1, f_2) \in \rho_A \iff f_1 \text{ and } f_2 \text{ agree on the intersection of their domains).
\]

Lemma 4.10. \( \rho_A \) is a congruence on Boolean algebra \( M_r(A) \).

Proof. The same proof as in the case of BL-algebras (see [4], Lemma 18). ■

Definition 4.6. For \( f \in M_r(A) \) with \( I = \text{dom}(f) \in I(A) \cap R(A) \), we denote by \([f, I]\) the congruence class of \( f \) modulo \( \rho_A \) and \( A_B = M_r(A)/\rho_A \).

Corollary 4.1. By Proposition 4.3 and Lemma 4.10 we deduce that \( A_B \) is a Boolean algebra.

Remark 4.11. If we denote by \( \mathcal{F} = I(A) \cap R(A) \) and consider the partially ordered systems \( \{ \delta_{I, J} \}_{I, J \in \mathcal{F}, I \subseteq J} \) (where for \( I, J \in \mathcal{F}, I \subseteq J, \delta_{I, J} : M(J, A) \rightarrow M(I, A) \) is defined by \( \delta_{I, J}(f) = f|_I \), then by above construction of \( A_B \) we deduce that \( A_B \) is the inductive limit
\[
A_B = \lim_{I \in \mathcal{F}} M(I, A).
\]

Lemma 4.11. Let the map \( \overline{\tau}_A : B \rightarrow A_B \) defined by \( \overline{\tau}_A(a) = [\overline{f_a}, A] \) for every \( a \in B \). Then
(i) \( \overline{\mathcal{V}}_A \) is an injective morphism of Boolean algebras,
(ii) For every \( a \in B, [\overline{\mathcal{V}}_A, A] \in B(A_B) \),
(iii) \( \overline{\mathcal{V}}(B) \in R(A_B) \).

**Proof.** (i). Follows from Lemma 4.5.
(ii). For \( a \in B \) and \( x \in A \) we have
\[
(\overline{\mathcal{T}}_a \sqcup \overline{\mathcal{T}}_a)(x) = \overline{\mathcal{T}}_a(x) \circ (x \rightarrow \overline{\mathcal{T}}_a(x)) = (a \wedge x) \circ [x \rightarrow (a \wedge x)] = a \circ [x \rightarrow (a \wedge x)] = a \circ [x \wedge (a \circ x)] = a \wedge (a \circ x) = a \wedge x = \overline{\mathcal{T}}_a(x),
\]
and
\[
(\overline{\mathcal{T}}_a)^{\ast\ast}(x) = x \circ ((\overline{\mathcal{T}}_a)^{\ast}(x) \rightarrow 0(x)) = x \circ (\overline{\mathcal{T}}_a \rightarrow 0)(x) \rightarrow 0(x) = x \circ (x \circ (\overline{\mathcal{T}}_a(x) \rightarrow 0) \rightarrow 0) =
\]
\[
= x \circ (x \circ (a \wedge x)) \ast = x \circ (a \ast) \ast \ast \ast = x \circ (a \ast \ast) = x \circ (a \ast) \ast \ast \ast 
\]
(since \( a \in B \))
\[
= x \circ (x \rightarrow a) = x \wedge a = \overline{\mathcal{T}}_a(x),
\]
hence
\[
\overline{\mathcal{T}}_a \sqcup \overline{\mathcal{T}}_a = \overline{\mathcal{T}}_a
\]
and
\[
\overline{\mathcal{T}}_a^{\ast\ast} = \overline{\mathcal{T}}_a,
\]
that is \([\overline{\mathcal{T}}_a, A] \in B(A_B) \).
(iii). To prove \( \overline{\mathcal{V}}(B) \in R(A_B) \), if by contrary there exist \( f_1, f_2 \in M_r(A) \) such that \([f_1, \text{dom}(f_1)] \neq [f_2, \text{dom}(f_2)] \) (that is there exists \( x_0 \in \text{dom}(f_1) \cap \text{dom}(f_2) \) such that \( f_1(x_0) \neq f_2(x_0) \)) and \([f_1, \text{dom}(f_1)] \wedge [\overline{\mathcal{T}}_a, A] = [f_2, \text{dom}(f_2)] \wedge [\overline{\mathcal{T}}_a, A] \) for every \([\overline{\mathcal{T}}_a, A] \in \overline{\mathcal{V}}(B) \cap B(A_B) \) (that is by (ii) for every \([\overline{\mathcal{T}}_a, A] \in \overline{\mathcal{V}}(B) \) with \( a \in B \), then \([f_1, \text{dom}(f_1)] \wedge [\overline{\mathcal{T}}_a] = [f_2, \text{dom}(f_2)] \wedge [\overline{\mathcal{T}}_a] \) for every \( x \in \text{dom}(f_1) \cap \text{dom}(f_2) \)) and every \( a \in B \Rightarrow f_1(x) \wedge a \neq x = f_2(x) \wedge a \) for every \( x \in \text{dom}(f_1) \cap \text{dom}(f_2) \) and every \( a \in B \). For \( a = 1 \in B \) and \( x = x_0 \) we obtain that \( f_1(x_0) \wedge x_0 = f_2(x_0) \wedge x_0 \Rightarrow f_1(x_0) = f_2(x_0) \) which is contradictory. ■

**Remark 4.12.** Since by Lemma 4.11, for every \( a, b \in B, [\overline{\mathcal{T}}_a, A] = [\overline{\mathcal{T}}_b, A] \) iff \( a = b \), the elements of \( B \) can be identified with the elements of the sets \([\overline{\mathcal{T}}_a, A] : a \in B \) and \([\overline{\mathcal{T}}_a : a \in B \). So, \( v_A(B) \approx \overline{\mathcal{V}}(B) \approx B \) (as BL- algebras).

**Lemma 4.12.** If \([f, \text{dom}(f)] \in A_B \) (with \( f \in M_r(A) \) and \( I = \text{dom}(f) \in I(A) \cap R(A) \)), then
\[
I \cap B \subseteq \{a \in B : \overline{\mathcal{T}}_a \wedge [f, \text{dom}(f)] \in \overline{\mathcal{V}}(B) \}.
\]

**Proof.** Let \( a \in I \cap B \). Then for every \( x \in I, (\overline{\mathcal{T}}_a \wedge f)(x) = \overline{\mathcal{T}}_a(x) \wedge f(x) = a \wedge x \wedge f(x) = a \wedge f(x) = a \circ f(x) = f(a \circ x) = x \circ f(a) \) (by \( a_{12} \)) \( = x \wedge f(a) \), that is \( \overline{\mathcal{T}}_a \wedge f = f(a) \in \overline{\mathcal{V}}(B) \) (since \( f(a) \in B \)), that is, the required inclusion. ■

**Remark 4.13.** The axiom \( a_{12} \) is necessary in the proof of Lemma 4.12.
5. Boolean Maximal BL-algebra of quotients

**Definition 5.1.** A BL-algebra \( F \) is called BL-algebra of fractions of \( A \) relative to \( B \) if:

(a11) \( A \) is a BL-subalgebra of \( F \).

(a12) For every \( a', b', c' \in F, a' \neq b' \), there exists \( e \in B \) such that \( e \land a' \neq e \land b' \) and \( e \land c' \in B \).

So, BL-algebra \( B \) is a BL-algebra of fractions of itself (since \( 1 \in B \)).

As a notational convenience, we write \( A \preceq F \) to indicate that \( F \) is a BL-algebra of fractions of \( A \) relative to \( B \).

**Remark 5.1.** If \( A \preceq F \), then \( F \) is a Boolean algebra. Indeed, if by contrary, then there exists \( a' \in F \) such that \( a' \neq a' \land a' \) or \( a'^* \neq a' \). If \( a' \neq a' \land a' \), since \( A \preceq F \), then there exists \( e \in B \) such that \( e \land a' \in B \) and

\[
eq (e \land a') \lor (e \land a'),
\]

which is contradictory!

If \( a'^* \neq a' \), since \( A \preceq F \), there exists \( f \in B \) such that \( f \land a' \in B \) and

\[
f \land a' \neq f \land (a')^* = (f \land a')^*
\]

which is contradictory!

**Lemma 5.1.** Let \( A \preceq F \); then for every \( a', b' \in F, a' \neq b' \), and any finite sequence \( c_1', ..., c_n' \in F \), there exists \( e \in B \) such that \( e \land a' \neq e \land b' \) and \( e \land c_i' \in B \) for \( i = 1, 2, ..., n \) (\( n \geq 2 \)).

**Proof.** See [4], Lemma 21.

**Lemma 5.2.** Let \( A \preceq F \) and \( a' \in F \). Then

\[
I_{a'} = \{ e \in B : e \land a' \in B \} \in I(B) \cap R(A).
\]

**Proof.** Clearly, \( I_{a'} \in I(B) \).

To prove \( I_{a'} \in R(A) \), let \( x, y \in A \) such that \( e \land x = e \land y \) for every \( e \in I_{a'} \cap B \). If by contrary, \( x \neq y \), since \( A \preceq F \), there exists \( e_0 \in B \) such that \( e_0 \land a' \in B \) (that is \( e_0 \in I_{a'} \)) and \( e_0 \land x \neq e_0 \land y \), which is contradictory.

**Theorem 5.1.** For every BL-algebra \( A \), the Boolean algebra \( A_B \) in Definitin 4.6 has the following properties:

(i) \( \overline{\tau_A}(B) \succeq A_B \).

(ii) for every BL-algebra \( F \) such that \( A \preceq F \), there exists monomorphism of BL-algebras \( i : F \rightarrow A_B \) which induces the canonical monomorphism \( \overline{\tau_A} \) of \( B \) into \( A_B \).

**Proof.** The fact that \( \overline{\tau_A}(B) \) is a BL-subalgebra of \( A_B \) follows from Lemma 4.11, (i).

To prove \( \overline{\tau_A}(B) \succeq A_B \), let \( [f, \text{dom}(f)], [g, \text{dom}(g)], [h, \text{dom}(h)] \in A_B \) with \( f, g, h \in M_r(A) \) such that \( [g, \text{dom}(g)] \neq [h, \text{dom}(h)] \) (that is there exists \( x_0 \in \text{dom}(g) \cap \text{dom}(h) \) such that \( g(x_0) \neq h(x_0) \)).

Put \( I = \text{dom}(f) \in I(A) \cap R(A) \) and

\[
I_{[f, \text{dom}(f)]} = \{ a \in B : I_a \land [f, \text{dom}(f)] \in \overline{\tau_A}(B) \}
\]

(by Lemma 4.11, \( I_a \in B(M(A)) \) if \( a \in B \)). Then by Lemma 4.12,

\[
I \cap B \subseteq I_{[f, \text{dom}(f)]}.
\]
If we suppose that for every \( a \in I \cap B \), \( \overline{f} \land [g, \text{dom}(g)] = \overline{f} \land [h, \text{dom}(h)] \), then 
\[ \overline{f} \land [g, \text{dom}(g)] = [\overline{f} \land h, \text{dom}(h)] \], hence for every \( x \in \text{dom}(g) \cap \text{dom}(h) \) we have 
\[ (\overline{f} \land g)(x) = (\overline{f} \land h)(x) \] i.e. \( a \land g(x) = a \land h(x) \).

Since \( I \in R(A) \) we deduce that \( g(x) = h(x) \) for every \( x \in \text{dom}(g) \cap \text{dom}(h) \) so 
\[ [g, \text{dom}(g)] = [h, \text{dom}(h)] \], which is contradictory.

Hence, if \([g, \text{dom}(g)] \neq [h, \text{dom}(h)]\), then there exists \( a \in I \cap B \), such that \( \overline{f} \land [g, \text{dom}(g)] \neq \overline{f} \land [h, \text{dom}(h)] \). But for this \( a \in I \cap B \) we have 
\[ \overline{f} \land [f, \text{dom}(f)] \in \mathbb{P}(B) \]
(since by Lemma 4.12, \( I \cap B \subseteq \{f, \text{dom}(f)\} \)).

To prove the maximally of \( A_B \), let \( F \) be a BL-algebra such that \( A \preceq F \); thus \( B \subseteq F(B) \)
\[ A \preceq F, \quad A_B \]
\[ \overline{i} \]

For \( a' \in F, I_{a'} = \{ e \in B : e \land a' \in B \} \in I(B) \cap R(A) \) (by Lemma 5.2).

Thus \( f_{a'} : I_{a'} \rightarrow A \) defined by \( f_{a'}(x) = x \land a' \) is a \( B \)-multiplier. Indeed, if \( e \in B \) and \( x \in I_{a'} \), then 
\[ f_{a'}(e \odot x) = (e \odot x) \land a' = (e \land x) \land a' = e \land (x \land a') = e \land f_{a'}(x) \]
and 
\[ f_{a'}(x) \leq x, \]
hence \( a_9 \) and \( a_{10} \) are verified.

To verify \( a_{11} \), let \( e \in I_{a'} \cap B = I_{a'} \). Thus, \( f_{a'}(e) = e \land a' \in B \) (since \( e \in I_{a'} \)).

The condition \( a_{12} \) is obviously verified, hence \([f_{a'}, I_{a'}] \in A_B \).

We define \( i : F \rightarrow A_B \), by \( i(a') = [f_{a'}, I_{a'}] \), for every \( a' \in F \). Clearly \( i(0) = 0 \).

For \( a', b' \in F \) and \( x \in I_{a'} \cap I_{b'} \), we have 
\[ (i(a')) \square i(b')(x) = (a' \land x) \odot [x \rightarrow (b' \land x)] = \]
\[ = (a' \odot x) \odot [x \rightarrow (b' \land x)] = a' \odot [x \circ (x \rightarrow (b' \land x))] = \]
\[ = a' \circ [x \land (b' \land x)] = a' \circ (b' \land x) = (a' \land b') \circ x = (a' \circ b') \land x = i(a' \circ b')(x), \]
hence \( i(a') \square i(b') = i(a' \circ b') \) and
\[ (i(a') \rightarrow i(b'))(x) = x \odot [i(a')(x) \rightarrow i(b'(x))] = \]
\[ = x \odot [(a' \land x) \rightarrow (b' \land x)] = x \odot [(x \circ a') \rightarrow (x \circ b')] = \]
\[ = x \odot (a' \land b') = x \land (a' \land b') = i(a' \land b')(x), \]
hence \( i(a') \rightarrow i(b') = i(a' \rightarrow b') \), that is \( i \) is a morphism of BL-algebras.

To prove the injectivity of \( i \), let \( a', b' \in F \) such that \( i(a') = i(b') \). It follows that 
\([f_{a'}, I_{a'}] = [f_{b'}, I_{b'}] \) so \( f_{a'}(x) = f_{b'}(x) \) for every \( x \in I_{a'} \cap I_{b'} \). We get \( a' \land x = b' \land x \) for every \( x \in I_{a'} \cap I_{b'} \). If \( a' \neq b' \), by Lemma 5.1 (since \( A \preceq F \)), there exists \( e \in B \) such that \( e \land a', e \land b' \in B \) and \( e \land a' \neq e \land b' \) which is contradictory (since \( e \land a', e \land b' \in B \) implies \( e \in I_{a'} \cap I_{b'} \)).

The Theorem 5.1 provides the motivation for the following Definition:

**Definition 5.2.** For any BL-algebra \( A \), \( A_B \) is called a maximal BL-algebra of quotients of \( A \) (which by Remark 5.1 is a Boolean algebra). To range with the tradition ([2], [5], [6], [17], [18]) we denote \( A_B \) by \( Q_B(A) \).

**Remark 5.2.** If BL-algebra \( A \) is an MV-algebra, then we obtain the maximal MV-algebra of quotients of \( A \) (see [6]).
Remark 5.3. If $A$ is a Boolean algebra, then $B(A) = A$. The axioms $a_9 - a_{12}$ are equivalent with $a_9$, hence $Q_B(A)$ is in this case just the classical Dedekind-MacNeille completion of $A$ (see [18], p.687). In contrast to the general situation, the Dedekind-MacNeille completion of a Boolean algebra is again distributive and, in fact, is a Boolean algebra ([1], p.239).

Proposition 5.1. Let $A$ be a BL-algebra. Then the following statements are equivalent:

(i) Every maximal $B-$multiplier on $A$ has domain $A$,
(ii) For every $B$-multiplier $f \in M(I, A)$ there is $a \in B$ such that $f = f_a$ (that is $f(x) = a \land x$ for every $x \in I$),
(iii) $Q_B(A) \approx B$.

Proof. (i) $\Rightarrow$ (ii). Assume (i) and for $f \in M(I, A)$ let $f'$ its the maximal extension (by Lemma 4.7). By (ii), we have $f' : A \rightarrow A$. Put $a = f'(1) \in B$ (by $a_{11}$), then for every $x \in I$ , $f(x) = f(x) \land 1 \overset{a_{12}}{\Rightarrow} x \land f(1) = x \land a = f_a(x)$, that is $f = f_a$.

(ii) $\Rightarrow$ (iii). Follow from Lemma 4.11.

(iii) $\Rightarrow$ (i). Follow from Lemma 4.7 and Lemma 4.11.

Definition 5.3. If $A$ verify one of condition of Proposition 5.1, we call $A$ rationnaly complete.

Example 5.1. 1. If $A$ is a BL-algebra, $B = B(A) = \{0, 1\} = L_2$ and $A \leq F$ then $F = \{0, 1\}$, hence $Q_B(A) = A_B \approx L_2$. Indeed, if $a, b, c \in F$ with $a \neq b$, then by $a_{14}$ there exists $e \in B$ such $e \land a \neq e \land b$ (hence $e \neq 0$) and $e \land c \in B$. Then, $e = 1$, hence $c \in B$, that is, $F = B$. As examples of BL-algebras with this property we have local BL-algebras and BL-chains.

2. More general, if $A$ is a BL-algebra, $B$ is a finite Boolean subalgebra of $A$ and $A \leq F$, then $F = B$, hence in this case $Q_B(A) = A_B \approx B$. Indeed, since $A \leq F$ we have $B \subset B(F) \subset F$. If consider $a \in F$, then there exists $e \in B$ such that $e \land x \in B$ (for example $e = 0$). $B$ being finite, there exists a largest element $e_a \in B$ such $e_a \land a \in B$. Suppose $e_a \land a \neq e_a$, then there would exists $e \in B$ such that $e \land (e_a \land a) \neq e \land e_a$ and $e \land a \in B$. But $e \land a \in B$ implies $e \leq e_a$ and thus we obtain $e = e \land (e_a \land a) \neq e \land e_a = e$, a contradiction. Hence $e_a \land a = e_a$, so $a \leq e_a$, consequently $a = a \land e_a \in B$, that is, $F \subset B$. Then $F = B$, hence $Q_B(A) \approx B$.

3. If $B = B(A)$, then by Remark 5.3, $Q_B(A)$ is the maximal BL-algebra of quotients of $A$ defined in [4] (Theorem 23).

Corollary 5.1. If consider MV-algebra $L_{3 \times 2}$, from Example 2.12, then $B(A) = \{0, a, d, 1\}$ is finite. Then we obtain:

1. If $B = B(A) = \{0, a, d, 1\}$ then $F_{B(A)} = B(A)$ and $Q_B = Q_{B(A)}(A) = B(A) = \{0, a, d, 1\}$.

2. If $B = L_2 = \{0, 1\}$ then $F_{L_2} = L_2$ and $Q_{L_2}(A) = L_2$.

Corollary 5.2. If consider BL-algebra $A = \{0, a, b, c\}$, from Example 2.11, then $B(A) = \{0, 1\} = L_2$ is finite. Then we obtain $B = L_2 = \{0, 1\}$, so $F_{L_2} = L_2$ and $Q_{L_2}(A) = L_2$.

References


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