Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 31, 2004, Pages 1–19
ISSN: 1223-6934

Boolean BL- algebra of fractions

Dumitru Buşneag and Dana Piciu

Abstract. In [4] we have introduced the notions of $BL-$ algebra of fractions and maximal $BL-$ algebra of quotients. The scope of this paper is to prove that these algebras are Boolean algebras (see Proposition 4.3, Corollary 4.1 and Remark 5.1) and to define the notions of BL - *algebra of fractions* and *maximal $BL-$ algebra of quotients for a BL - algebra A relative to a Boolean subalgebra B of* A $(B \subseteq B(A))$.

In the last part of this paper, for a $BL-$ algebra A and Boolean subalgebra $B \subseteq A,$ is proved the existence of a *maximal BL - algebra of quotients for A relative to B* (which is a Boolean algebra, by Corollary 4.1) and we give explicit descriptions of this BL-algebra for some classes of BL-algebras and particular Boolean subalgebras B of A. For $B = B(A)$ we obtain the results of [4]. If $BL-$ algebra A is an $MV-$ algebra we obtain the results of [5], [6] (for $MV-$ algebras).

2000 Mathematics Subject Classification. 06D35, 03G25.
Key words and phrases. BL- algebra, MV- algebra, Boolean algebra, multiplier, BL-algebra of fractions, maximal BL- algebra of quotients.

1. Introduction

The concept of *maximal lattice of quotients* for a distributive lattice was defined by J.Schmid in [17], [18] taking as a guide-line the construction of *complete ring of quotients* by partial morphisms introduced by G. Findlay and J. Lambek (see [15], p.36). For the case of Hilbert algebras and MV-algebras see [2] and [5]. The central role in this constructions is played by the concept of *multiplier* (defined for a distributive lattice by W. H. Cornish in [11], [12]).

For some informal explanations of notion of *fraction* see [15], p. 37.

The paper is organized as follows.

In Section 2 we recall the basic definitions and put in evidence many rules of calculus in BL-algebras which we need in the rest of paper.

In Section 3 we present the $MV-$ center of a $BL-$ algebra (defined by Turunen and Sessa in [20]). This is a very important construction, which associates an MV-algebra with every BL-algebra. In this way, many properties can be transfered from MV-algebras to BL-algebras and backwards.

In Section 4 we define the notion of $B-$*multiplier* for a BL-algebra A relative to a Boolean subalgebra B of $B(A)$; also we put in evidence many results which we need in the rest of the paper (especially in Section 5).

In Section 5 we define the notions of *BL-algebra of fractions relative to* B and *maximal BL-algebra of quotients relative to* B for a BL-algebra A and a Boolean subalgebra B of $B(A)$.

In the last part of this paper for a $BL-$ algebra A is proved the existence of the maximal $BL-$algebra of quotients of A relative to a Boolean subalgebra $B \subseteq B(A)$

Received: October 18, 2004.

for a $BL-$algebra (Theorem 5.1) and we give explicit descriptions of this BL-algebra for some classes of BL-algebras A ($MV-$ algebras, local BL-algebras, BL-chains, and Boolean algebras) and particular Boolean subalgebras B of A. For $B = B(A)$ we obtain the results of [4]. If in particular $BL-$ algebra A is an $MV-$ algebras we obtain the results from [5] and [6].

2. Definitions and first properties

Definition 2.1. *([19]) An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a residuated lattice if $(L, \wedge, \vee, 0, 1)$ is a distributive lattice with 0 and 1, the operation $\odot$ is an isotone, associative and commutative binary operation on L, and for every $x, y, z \in L, x \odot y \leq z$ iff $x \leq y \rightarrow z$.*

Definition 2.2. *A BL-algebra ([13], [19]) is an algebra*

$$\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$$

of type (2,2,2,2,0,0) satisfying the following:
(a_1) *$(A, \wedge, \vee, 0, 1)$ is a bounded lattice,*
(a_2) *$(A, \odot, 1)$ is a commutative monoid,*
(a_3) *$\odot$ and $\rightarrow$ form an adjoint pair, i.e. $c \leq a \rightarrow b$ iff $a \odot c \leq b$ for all $a, b, c \in A$,*
(a_4) *$a \wedge b = a \odot (a \rightarrow b)$,*
(a_5) *$(a \rightarrow b) \vee (b \rightarrow a) = 1$, for all $a, b \in A$.*

The origin of BL-algebras is in Mathematical Logic; they where invented by Hájek in [13] in order to study the ,,Basic Logic" (BL, for short) arising from the continuous triangular norms, familiar in the framework of fuzzy set theory. They play the role of Lindenbaum algebras from classical Propositional calculus. Apart from their logical interest, BL-algebras have important algebraic properties (see [13], [14], [19]).

Remark 2.1. *$BL-$algebras are exactly the commutative residuated lattices satisfying a_4, a_5 (see Definition 2.1).*

In order to simplify the notation, a BL-algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ will be referred by its support set, A. So, in the rest of this paper by A we denote a $BL-$algebra.

A BL-algebra is *nontrivial* if $0 \neq 1$. For any BL-algebra A, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. A BL *-chain* is a totally ordered BL-algebra, i.e. a BL-algebra such that its lattice order is total.

For any $a \in A$, we define $a^* = a \rightarrow 0$ and denote $(a^*)^*$ by a^{**}. Clearly, $0^* = 1$.

We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \geq 1$. The *order* of $a \in A, a \neq 1$, in symbols $ord(a)$ is the smallest $n \in \omega$ such that $a^n = 0$; if no such n exists, then $ord(a) = \infty$.

A BL-algebra is called *locally finite* if all non unit elements in it have finite order.

Example 2.1. *Define on the real unit interval $I = [0, 1]$ the binary operations $\odot$ and $\rightarrow$ by*

$$x \odot y = \max\{0, x + y - 1\}$$

$$x \rightarrow y = \min\{1, 1 - x + y\}.$$

Then $(I, \leq, \odot, \rightarrow, 0, 1)$ is a BL-algebra (called Łukasiewicz structure).

Example 2.2. *Define on the real unit interval* $I = [0, 1]$

$$x \odot y = \min\{x, y\}$$

$$x \to y = 1 \text{ iff } x \leq y \text{ and } y \text{ otherwise.}$$

Then $(I, \leq, \odot, \to, 0, 1)$ *is a* BL*-algebra (called Gődel structure).*

Example 2.3. *Let* $\odot$ *be the usual multiplication of real numbers on the unit interval* $I = [0, 1]$ *and* $x \to y = 1$ *iff* $x \leq y$ *and* y/x *otherwise. Then* $(I, \leq, \odot, \to, 0, 1)$ *is a* BL*-algebra (called Product structure or Gaines structure).*

Remark 2.2. *Not every residuated lattice, however, is a* BL*-algebra (see* [19], *p.16). Consider, for example a residuated lattice defined on the unit interval, for all* $x, y, z \in I$*, such that*

$$x \odot y = 0, \text{ iff } x + y \leq \frac{1}{2} \text{ and } x \wedge y \text{ elsewhere}$$

$$x \to y = 1 \text{ if } x \leq y \text{ and } \max\{\frac{1}{2} - x, y\} \text{ elsewhere.}$$

Let $0 < y < x$, $x + y < \frac{1}{2}$. *Then* $y < \frac{1}{2} - x$ *and* $0 \neq y = x \wedge y$, *but* $x \odot (x \to y) = x \odot (\frac{1}{2} - x) = 0$. *Therefore* a_4 *does not hold.*

Example 2.4. *If* $(A, \wedge, \vee, \rceil, 0, 1)$ *is a Boolean algebra, then* $(A, \wedge, \vee, \odot, \to, 0, 1)$ *is a* BL*-algebra where the operation* $\odot$ *coincide with* $\wedge$ *and* $x \to y = \rceil x \vee y$, *for all* $x, y \in A$.

Example 2.5. *If* $(A, \wedge, \vee, \to, 0, 1)$ *is a relative Stone lattice (see* [1], *p.176), then* $(A, \wedge, \vee, \odot, \to, 0, 1)$ *is a* BL*-algebra where the operation* $\odot$ *coincide with* $\wedge$.

Example 2.6. *If* $(A, \oplus, ^*, 0)$ *is an* MV*-algebra (see* [10]*), then* $(A, \wedge, \vee, \odot, \to, 0, 1)$ *is a* BL*-algebra, where for* $x, y \in A$:

$$x \odot y = (x^* \oplus y^*)^*,$$

$$x \to y = x^* \oplus y, 1 = 0^*,$$

$$x \vee y = (x \to y) \to y = (y \to x) \to x \text{ and } x \wedge y = (x^* \vee y^*)^*.$$

Remark 2.3. *(*[19]*) A* $BL-$ *algebra* A *is an* $MV-$ *algebra iff* $x^{**} = x$ *for all* $x \in A$. *If in a* $BL-$ *algebra,* $x^{**} = x$ *for all* $x \in A$, *and for* $x, y \in A$ *we denote* $x \oplus y = (x^* \odot y^*)^*$ *then* $(A, \oplus, ^*, 0)$ *is an* $MV-$ *algebra.*

Example 2.7. *(*[13]*) From the logical point of view, the most important example of a* BL*-algebra is the Lindenbaum-Tarski algebra* L_{BL} *of the propositional Basic Logic* BL. *The formulas in this logic are built up of denumerable many propositional variables* $v_1, ...v_n$ *with two operations & and* $\to$ *and one constant* $\overline{0}$ *as follows:*

(i) every propositional variable is a formula;

(ii) $\overline{0}$ *is a formula;*

(iii) if ϕ, ψ *are formulas, then* $\phi \& \psi$ *and* $\phi \to \psi$ *are formulas.*

Let us denote by $Fmla$ *the set of all formulas of* BL. *Further connectives can be defined:*

$$\phi \wedge \psi := \phi \& (\phi \to \psi),$$

$$\phi \vee \psi := ((\phi \to \psi) \to \psi) \wedge ((\psi \to \phi) \to \phi),$$

$$\rceil \phi := \phi \to \overline{0},$$

$$\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi),$$

$$1 := \overline{0} \to \overline{0}.$$

The axioms of a BL *are:*

(A_1) $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi)),$

(A_2) $(\phi\&\psi) \to \phi$,
(A_3) $(\phi\&\psi) \to (\psi\&\phi)$,
(A_4) $(\phi\&(\phi \to \psi)) \to (\psi\&(\psi \to \phi))$,
(A_5) $(\phi \to (\psi \to \chi)) \to ((\phi\&\psi) \to \chi)$,
(A_6) $((\phi\&\psi) \to \chi) \to (\phi \to (\psi \to \chi))$,
(A_7) $((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)$,
(A_8) $\overline{0} \to \phi$.

The deduction rule is modus ponens: if ϕ and $\phi \to \psi$ then ψ. We say that ϕ is a theorem and we denote by $\vdash \phi$ if there is a proof of ϕ from $A_1 - A_8$ using modus ponens. The completeness theorem for BL says that $\vdash \phi$ if and only if ϕ is a tautology in every standard BL-algebra .

On the set $Fmla$ of all formulas we define the equivalence relation $\equiv$ by:

$$\phi \equiv \psi \text{ iff } \vdash \phi \leftrightarrow \psi.$$

Let us denote by $[\phi]$ the equivalence class of the formula ϕ, and L_{BL} the set of all equivalence classes. We define

$$0 := [\overline{0}],$$
$$1 := [\overline{1}],$$
$$[\phi] \wedge [\psi] := [\phi \wedge \psi],$$
$$[\phi] \vee [\psi] := [\phi \vee \psi],$$
$$[\phi] \odot [\psi] := [\phi\&\psi],$$
$$[\phi] \to [\psi] := [\phi \to \psi].$$

Then $(L_{BL}, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra.

Example 2.8. *A product algebra (or P-algebra) ([13]) is a BL-algebra A satisfying:*
(P_1) $c^{**} \leq (a \odot c \to b \odot c) \to (a \to b)$,
(P_2) $a \wedge a^* = 0$.

Product algebras are the algebraic counterparts of propositional Product Logic [13]. The standard product algebra is the Product structure.

Example 2.9. *A G-algebra ([13], Definition 4.2.12) is a BL-algebra A satisfying:*
(G) $a \odot a = a$, *for all $a \in A$.*

G-algebras are the algebraic counterpart of Gödel Logic. The standard G-algebra is the Gödel structure.

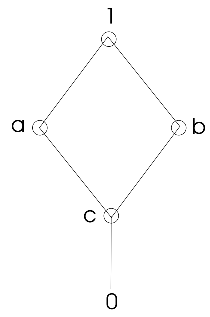
Example 2.10. *If $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra and X is a nonempty set, then the set A^X becomes a BL-algebra $(A^X, \wedge, \vee, \odot, \to, \underline{0}, \underline{1})$ with the operations defined pointwise. If $f, g \in A^X$, then*

$$(f \wedge g)(x) = f(x) \wedge g(x),$$
$$(f \vee g)(x) = f(x) \vee g(x),$$
$$(f \odot g)(x) = f(x) \odot g(x),$$
$$(f \to g)(x) = f(x) \to g(x)$$

for all $x, y \in X$ and $\underline{0}, \underline{1} : X \to A$ are the constant functions associated with $0, 1 \in A$.

Example 2.11. *([14], [16])*

We give an example of a finite BL-algebra which is not an MV-algebra. Let $A = \{0, a, b, c, 1\}$.



Define on A the following operations:

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

,

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

.

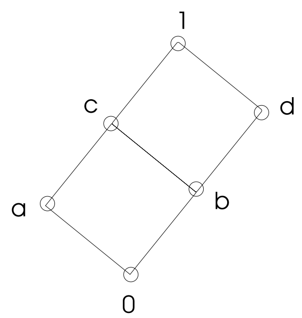
We have, $0 \leq c \leq a, b \leq 1$, but a, b are incomparable, hence A is not a $BL-$ chain. We remark that $x \odot y = x \wedge y$ for all $x, y \in A$, so $ord(x) = \infty$ for all $x \in A, x \neq 0$. It follows also that $x \odot x = x \wedge x = x$ for all $x \in A$, so A is a G-algebra. It is easy to see that $0^* = 1$ and $x^* = 0$ for all $x \in A, x \neq 0$, so $0^{**} = 0$ and $x^{**} = 1$ for all $x \in A, x \neq 0$. Thus, A is not an $MV-$ algebra.

Example 2.12. *(*[14], [16]*)*
We give an example of a finite MV-algebra which is not an MV-chain. The set

$$L_{3\times 2} = \{0, a, b, c, d, 1\} \approx L_3 \times L_2 = \{0, 1, 2\} \times \{0, 1\} =$$

$$= \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}$$

organized as lattice as in figure



and as $BL-$algebra with the operation $\rightarrow$ and

$$x \odot y = \min\{z : x \leq y \rightarrow z\} = (x \rightarrow y^*)^*, x^* = x \rightarrow 0$$

as in the following tables, is a non-linearly ordered MV -algebra

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

We have in $L_{3\times 2}$ the following operations:

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

,

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

It is easy to see that $0^* = 1, a^* = d, b^* = c, c^* = b, d^* = a, 1^* = 0$ and $x^{**} = x$, for all $x \in A$, hence $L_{3\times 2}$ is an $MV-$ algebra which is not chain.

In [3], [7], [13], [19] it is proved that if A is a BL-algebra and $a, a', a_1, ..., a_n, b, b', c, b_i \in A$, $(i \in I)$ then we have the following rules of calculus:

(c_1) $a \odot b \leq a, b$, hence $a \odot b \leq a \wedge b$ and $a \odot 0 = 0$,
(c_2) $a \leq b$ implies $a \odot c \leq b \odot c$,
(c_3) $a \leq b$ iff $a \rightarrow b = 1$,
(c_4) $1 \rightarrow a = a, a \rightarrow a = 1, a \leq b \rightarrow a, a \rightarrow 1 = 1$,
(c_5) $a \odot a^* = 0$,
(c_6) $a \odot b = 0$ iff $a \leq b^*$,
(c_7) $a \vee b = 1$ implies $a \odot b = a \wedge b$,
(c_8) $a \rightarrow (b \rightarrow c) = (a \odot b) \rightarrow c = b \rightarrow (a \rightarrow c)$,
(c_9) $(a \rightarrow b) \rightarrow (a \rightarrow c) = (a \wedge b) \rightarrow c$,
(c_{10}) $a \rightarrow (b \rightarrow c) \geq (a \rightarrow b) \rightarrow (a \rightarrow c)$,
(c_{11}) $a \leq b$ implies $c \rightarrow a \leq c \rightarrow b, b \rightarrow c \leq a \rightarrow c$ and $b^* \leq a^*$,
(c_{12}) $a \leq (a \rightarrow b) \rightarrow b$, $((a \rightarrow b) \rightarrow b) \rightarrow b = a \rightarrow b$,
(c_{13}) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$,
(c_{14}) $a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)$,
(c_{15}) $a \vee b = ((a \rightarrow b) \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a)$,
(c_{16}) $(a \wedge b)^n = a^n \wedge b^n, (a \vee b)^n = a^n \vee b^n$, hence $a \vee b = 1$ implies $a^n \vee b^n = 1$ for any $n \geq 0$,
(c_{17}) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$,
(c_{18}) $(b \wedge c) \rightarrow a = (b \rightarrow a) \vee (c \rightarrow a)$,
(c_{19}) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$,
(c_{20}) $a \rightarrow b \leq (b \rightarrow c) \rightarrow (a \rightarrow c)$,
(c_{21}) $a \rightarrow b \leq (c \rightarrow a) \rightarrow (c \rightarrow b)$,
(c_{22}) $a \rightarrow b \leq (a \odot c) \rightarrow (b \odot c)$,
(c_{23}) $a \odot (b \rightarrow c) \leq b \rightarrow (a \odot c)$,
(c_{24}) $(b \rightarrow c) \odot (a \rightarrow b) \leq a \rightarrow c$,
(c_{25}) $(a_1 \rightarrow a_2) \odot (a_2 \rightarrow a_3) \odot ... \odot (a_{n-1} \rightarrow a_n) \leq a_1 \rightarrow a_n$,
(c_{26}) $a, b \leq c$ and $c \rightarrow a = c \rightarrow b$ implies $a = b$,
(c_{27}) $a \vee (b \odot c) \geq (a \vee b) \odot (a \vee c)$, hence $a^m \vee b^n \geq (a \vee b)^{mn}$, for any $m, n \geq 0$,

(c_{28}) $(a \to b) \odot (a^{'} \to b^{'}) \leq (a \vee a^{'}) \to (b \vee b^{'})$,
(c_{29}) $(a \to b) \odot (a^{'} \to b^{'}) \leq (a \wedge a^{'}) \to (b \wedge b^{'})$,
(c_{30}) $(a \to b) \to c \leq ((b \to a) \to c) \to c$,
(c_{31}) $a \odot (\bigwedge\limits_{i\in I} b_i) \leq \bigwedge\limits_{i\in I} (a \odot b_i)$,
$a \odot (\bigvee\limits_{i\in I} b_i) = \bigvee\limits_{i\in I} (a \odot b_i)$,
$a \to (\bigwedge\limits_{i\in I} b_i) = \bigwedge\limits_{i\in I} (a \to b_i)$,
$(\bigvee\limits_{i\in I} b_i) \to a = \bigwedge\limits_{i\in I} (b_i \to a)$
$\bigvee\limits_{i\in I} (b_i \to a) \leq (\bigwedge\limits_{i\in I} b_i) \to a$,
$\bigvee\limits_{i\in I}(a \to b_i) \leq a \to (\bigvee\limits_{i\in I} b_i)$,
$a \wedge (\bigvee\limits_{i\in I} b_i) = \bigvee\limits_{i\in I} (a \wedge b_i)$; if A is a BL-chain then $a \vee (\bigwedge\limits_{i\in I} b_i) = \bigwedge\limits_{i\in I} (a \vee b_i)$,
(whenever the arbitrary meets and unions exist)
(c_{32}) $a \leq a^{**}$, $1^* = 0$, $0^* = 1, a^{***} = a^*, a^{**} \leq a^* \to a$,
(c_{33}) $(a \wedge b)^* = a^* \vee b^*$ and $(a \vee b)^* = a^* \wedge b^*$,
(c_{34}) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$, $(a \vee b)^{**} = a^{**} \vee b^{**}, (a \odot b)^{**} = a^{**} \odot b^{**}$, $(a \to b)^{**} = a^{**} \to b^{**}$,
(c_{35}) If $a^{**} \leq a^{**} \to a$, then $a^{**} = a$,
(c_{36}) $a = a^{**} \odot (a^{**} \to a)$,
(c_{37}) $a \to b^* = b \to a^* = a^{**} \to b^* = (a \odot b)^*$,
(c_{38}) $(a^{**} \to a)^* = 0, (a^{**} \to a) \vee a^{**} = 1$,
(c_{39}) $b^* \leq a$ implies $a \to (a \odot b)^{**} = b^{**}$.

In the rest of this paper by A we denote a BL-algebra; by $B(A)$ we denote the Boolean algebra of all complemented elements in $L(A)$ (hence $B(A) = B(L(A))$) and by $B \subseteq B(A)$ we denote a Boolean subalgebra of A.

Proposition 2.1. *([13], [19]) For $e \in A$, the following are equivalent:*
(i) $e \in B(A)$,
(ii) $e \odot e = e$ *and* $e = e^{**}$,
(iii) $e \odot e = e$ *and* $e^* \to e = e$,
(iv) $e \vee e^* = 1$.

Remark 2.4. *If $a \in A$ and $e \in B$, then $e \odot a = e \wedge a, a \to e = (a \odot e^*)^* = a^* \vee e$; if $e \leq a \vee a^*$, then $e \odot a \in B$.*

Proposition 2.2. *([7]) For $e \in A$, the following are equivalent:*
(i) $e \in B(A)$,
(ii) $(e \to x) \to e = e$, *for every* $x \in A$.

Lemma 2.1. *If $e, f \in B$ and $x, y \in A$, then:*
(c_{40}) $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$,
(c_{41}) $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$,
(c_{42}) $e \odot (x \to y) = e \odot [(e \odot x) \to (e \odot y)]$,
(c_{43}) $x \odot (e \to f) = x \odot [(x \odot e) \to (x \odot f)]$,
(c_{44}) $e \to (x \to y) = (e \to x) \to (e \to y)$.

Proof. (c_{40}). We have

$$(e \vee x) \odot (e \vee y) \overset{c_{13}}{=} [(e \vee x) \odot e] \vee [(e \vee x) \odot y] \overset{c_{13}}{=} [(e \vee x) \odot e] \vee [(e \odot y) \vee (x \odot y)]$$
$$= [(e \vee x) \wedge e] \vee [(e \odot y) \vee (x \odot y)] = e \vee (e \odot y) \vee (x \odot y) = e \vee (x \odot y).$$

(c_{41}). We have

$$(e \wedge x) \odot (e \wedge y) = (e \odot x) \odot (e \odot y) = (e \odot e) \odot (x \odot y) = e \odot (x \odot y) = e \wedge (x \odot y).$$

(c_{42}). By c_{22} we have $x \to y \leq (e \odot x) \to (e \odot y)$, hence $e \odot (x \to y) \leq e \odot [(e \odot x) \to (e \odot y)]$. Conversely, $e \odot [(e \odot x) \to (e \odot y)] \leq e$ and $(e \odot x) \odot [(e \odot x) \to (e \odot y)] \leq e \odot y \leq y$ so $e \odot [(e \odot x) \to (e \odot y)] \leq x \to y$. Hence $e \odot [(e \odot x) \to (e \odot y)] \leq e \odot (x \to y)$.

(c_{43}). We have $x \odot [(x \odot e) \to (x \odot f)] = x \odot [(x \odot e) \to (x \wedge f)] \overset{c_{31}}{=} x \odot [((x \odot e) \to x)$ $\wedge ((x \odot e) \to f)] = x \odot [1 \wedge ((x \odot e) \to f)] = x \odot ((x \odot e) \to f) \overset{c_8}{=} x \odot [x \to (e \to f)] =$ $x \wedge (e \to f) = x \odot (e \to f)$.

(c_{44}). Follows from c_8 and c_9 since $e \wedge x = e \odot x$.$\blacksquare$

Definition 2.3. *([13], [19]) Let A and B be $BL-$algebras. A function $f : A \to B$ is a morphism of $BL-$algebras iff it satisfies the following conditions, for every $x, y \in A$:*
(a_6) $f(0) = 0$,
(a_7) $f(x \odot y) = f(x) \odot f(y)$,
(a_8) $f(x \to y) = f(x) \to f(y)$.

Remark 2.5. *([13], [19]) It follows that:*

$$f(1) = 1,$$

$$f(x^*) = [f(x)]^*$$

$$f(x \vee y) = f(x) \vee f(y),$$

$$f(x \wedge y) = f(x) \wedge f(y),$$

for every $x, y \in A$.

If f is bijective then the morphism f is called an isomorphism of BL-algebras; in this case we write $A \approx B$.

Lemma 2.2. *If a, b, x are elements of A and $a, b \leq x$ then*
(c_{45}) $a \odot (x \to b) = b \odot (x \to a)$.

Proof. We have

$$a \odot (x \to b) = (x \wedge a) \odot (x \to b) = [x \odot (x \to a)] \odot (x \to b)$$

$$= [x \odot (x \to b)] \odot (x \to a) = (x \wedge b) \odot (x \to a) = b \odot (x \to a).\blacksquare$$

3. MV-center of a BL-algebra

In this section we prezent the *MV-center* of a BL-algebra, defined by Turunen and Sessa in [20]. This is a very important construction, which associates an MV-algebra with every BL-algebra. In this way, many properties can be transfered from MV-algebras to BL-algebras and backwards. We shall use more times this construction in our paper.

As we saw in Example 2.6, MV-algebras are BL-algebras, and more, a BL-algebra A is an MV-algebra iff $a^{**} = a$ for every $a \in A$.

The *MV-center* of a A, denoted by $MV(A)$ is defined as

$$MV(A) = \{a \in A : a^{**} = a\} = \{a^* : a \in A\}.$$

Hence, a BL-algebras A is an MV-algebra iff $A = MV(A)$.

By Proposition 2.1 follow that $B(A) \subseteq MV(A)$.

Example 3.1. *([20]) If A is a product algebra or a G-algebra, then $MV(A)$ is a Boolean BL-algebra; If A is the Product structure or the Gődel structure, then $MV(A) = \{0,1\}$;If A is the 5-element BL-algebra from Example 2.11, $MV(A) = \{0,1\}$.*

Proposition 3.1. *([20]) If A be a BL-algebra and let us define for all $a, b \in A$,*

$$a^* \oplus b^* = (a \odot b)^*.$$

Then
(i) $(MV(A), \oplus, ^*, 0)$ *is an MV-algebra,*
(ii) *the order $\leq$ of A agrees with the one of $MV(A)$, defined by*

$$a \leq_{MV} b \text{ iff } a^* \oplus b = 0, \text{ for all } a, b \in MV(A),$$

(iii) *the residuum $\to$ of A coincides with the residuum $\to_{MV}$ in $MV(A)$, defined by*

$$a \to_{MV} b = a^* \oplus b, \text{ for all } a, b \in MV(A),$$

(iv) *the product $\odot_{MV}$ on $MV(A)$ is such that*

$$a \odot_{MV} b = (a \odot b)^{**} = a \odot b, \text{ for all } a, b \in MV(A),$$

(v) *$MV(A)$ is the largest $MV-$ subalgebra of A.*

Proposition 3.2. *([8]) If A be a $BL-$ algebra, then $B(A) = B(MV(A))$.*

4. B-Multipliers on a BL-algebra

Definition 4.1. *Let $(P, \leq)$ an ordered set and $I \subseteq P$. I is an order ideal (alternative terms include down-set or decreasing set) if, whenever $x \in I, y \in P$ and $y \leq x$, we have $y \in I$. We denote by $I(P)$ the set of all order ideals of P; clearly, $I(P)$ is closed under arbitrary intersections. For a nonempty set $M \subseteq P$ we denote by $< M >_P$ the order ideal of P generated by M.*

Remark 4.1. *Is eassy to prove that for a nonempty set $M \subseteq P$,*

$$< M >_P = \{x \in P : \text{there exists } a \in M \text{ such that } x \leq a\}.$$

We denote by $Id(A)$ the set of all ideals of the lattice $L(A)$ and by $I(A)$ the set of all order ideals of A, that is:

$$I(A) = \{I \subseteq A : \text{if } x, y \in A, x \leq y \text{ and } y \in I, \text{then } x \in I\}.$$

Remark 4.2. *Clearly, $Id(A) \subseteq I(A)$ and if $I_1, I_2 \in I(A)$, then $I_1 \cap I_2 \in I(A)$. Also, if $I \in I(A)$, then $0 \in I$.*

By $B \subseteq B(A)$ we denote a Boolean subalgebra of A.

Definition 4.2. *By $B-$ partial multiplier on A we mean a map $f : I \to A$, where $I \in I(A)$, which verifies the next conditions:*
(a_9) *$f(e \odot x) = e \odot f(x)$, for every $e \in B$ and $x \in I$,*
(a_{10}) *$f(x) \leq x$, for every $x \in I$,*
(a_{11}) *If $e \in I \cap B$, then $f(e) \in B$,*
(a_{12}) *$x \wedge f(e) = e \wedge f(x)$, for every $e \in I \cap B$ and $x \in I$.*

Remark 4.3. *For every $e \in I \cap B$ and $x \in I$, $x \wedge f(e) = e \wedge f(x) \Leftrightarrow x \odot f(e) = e \odot f(x)$.*

By $dom(f) \in I(A)$ we denote the domain of f; if $dom(f) = A$, we called f *total*.

To simplify the language, we will use *multiplier* instead $B-$ *partial multiplier* using *total* to indicate that the domain of a certain multiplier is A.

Examples

1. The map $\mathbf{0} : A \to A$ defined by $\mathbf{0}(x) = 0$, for every $x \in A$ is a total multiplier on A; indeed if $x \in A$ and $e \in B$, then $\mathbf{0}(e \odot x) = 0 = e \odot 0 = e \odot \mathbf{0}(x)$ and $\mathbf{0}(x) \leq x$.

Clearly, if $e \in A \cap B = B$, then $\mathbf{0}(e) = 0 \in B$ and for $x \in A$, $x \wedge \mathbf{0}(e) = e \wedge \mathbf{0}(x) = 0$.

2. The map $\mathbf{1} : A \to A$ defined by $\mathbf{1}(x) = x$, for every $x \in A$ is also a total multiplier on A; indeed if $x \in A$ and $e \in B$, then $\mathbf{1}(e \odot x) = e \odot x = e \odot \mathbf{1}(x)$ and $\mathbf{1}(x) = x \leq x$.

The conditions $a_{11} - a_{12}$ are obviously verified.

3. For $a \in B$ and $I \in I(A)$, the map $f_a : I \to A$ defined by $f_a(x) = a \wedge x$, for every $x \in I$ is a multiplier on A (called *principal*). Indeed, for $x \in I$ and $e \in B$, we have $f_a(e \odot x) = a \wedge (e \odot x) = a \wedge (e \wedge x) = e \wedge (a \wedge x) = e \odot (a \wedge x) = e \odot f_a(x)$ and clearly $f_a(x) \leq x$.

Also, if $e \in I \cap B$, $f_a(e) = e \wedge a \in B$ and $x \wedge (a \wedge e) = e \wedge (a \wedge x)$, for every $x \in I$.

Remark 4.4. *The condition a_{12} is not a consequence of $a_9 - a_{11}$. As example, $f : I \to A, f(x) = x \wedge x^*$ for every $x \in I$, verify $a_9 - a_{11}$, but if $e \in I \cap B$ and $x \in I$, then*

$$x \wedge f(e) = x \wedge 0 \neq e \wedge (x \wedge x^*) = e \wedge f(x).$$

Remark 4.5. *In general, if consider $a \in A$, then $f_a : I \to A$ verifies only a_9, a_{10} and a_{12} but does not verify a_{11}.*

If $dom(f_a) = A$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$.

For $I \in I(A)$, we denote

$$M(I, A) = \{f : I \to A \mid f \text{ is a multiplier on } A\}$$

and

$$M(A) = \bigcup_{I \in I(A)} M(I, A).$$

If necessary, we denote $M(I, A)$ by $M_{\mathcal{BL}}(I, A)$ to indicate that we work in $BL-$ algebras; for the case of $MV-$ algebras we denote $M(I, A)$ by $M_{\mathcal{MV}}(I, A)$.

Remark 4.6. *From Propositions 3.1 and 3.2 we deduce that for every $I \in I(A)$ the algebra of multipliers $M_{\mathcal{BL}}(I, A)$ for a $BL-$ algebras is in fact a generalization of the algebra of multipliers $M_{\mathcal{MV}}(I, A)$ for $MV-$ algebras (see* [5], [6]*). Also, we deduce that if A is an $MV-$ algebra (that is $A = MV(A)$), then $M_{\mathcal{BL}}(I, A) = M_{\mathcal{MV}}(I, A)$ for every $I \in I(A)$.*

Definition 4.3. *If $I_1, I_2 \in I(A)$ and $f_i \in M(I_i, A), i = 1, 2,$ we define $f_1 \wedge f_2$, $f_1 \vee f_2$, $f_1 \boxdot f_2$, $f_1 \to f_2 : I_1 \cap I_2 \to A$ by*

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x),$$

$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$$

$$(f_1 \boxdot f_2)(x) = f_1(x) \odot [x \to f_2(x)] \overset{c_{45}}{=} f_2(x) \odot [x \to f_1(x)],$$

$$(f_1 \to f_2)(x) = x \odot [f_1(x) \to f_2(x)].$$

for every $x \in I_1 \cap I_2$.

Lemma 4.1. $f_1 \wedge f_2 \in M(I_1 \cap I_2, A)$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B$, then $(f_1 \wedge f_2)(e \odot x) = f_1(e \odot x) \wedge f_2(e \odot x) = (e \odot f_1(x)) \wedge (e \odot f_2(x)) = (e \wedge f_1(x)) \wedge (e \wedge f_2(x)) = e \wedge [f_1(x) \wedge f_2(x)] = e \odot (f_1 \wedge f_2)(x)$.

Since $f_i \in M(I_i, A), i = 1, 2$, we have $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x) \leq x \wedge x = x$, for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B$, then

$$(f_1 \wedge f_2)(e) = f_1(e) \wedge f_2(e) \in B.$$

For $e \in I_1 \cap I_2 \cap B$ and $x \in I_1 \cap I_2$ we have:

$$x \wedge (f_1 \wedge f_2)(e) = x \wedge f_1(e) \wedge f_2(e) = [x \wedge f_1(e)] \wedge [x \wedge f_2(e)] =$$

$$= [e \wedge f_1(x)] \wedge [e \wedge f_2(x)] = e \wedge (f_1 \wedge f_2)(x),$$

that is $f_1 \wedge f_2 \in M(I_1 \cap I_2, A)$. ■

Lemma 4.2. $f_1 \vee f_2 \in M(I_1 \cap I_2, A)$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B$, then $(f_1 \vee f_2)(e \odot x) = f_1(e \odot x) \vee f_2(e \odot x) = (e \odot f_1(x)) \vee (e \odot f_2(x)) \overset{c_{31}}{=} e \odot [f_1(x) \vee f_2(x)] = e \odot (f_1 \vee f_2)(x)$.

Since $f_i \in M(I_i, A), i = 1, 2$, we have $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x) \leq x \vee x = x$, for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B$, then

$$(f_1 \vee f_2)(e) = f_1(e) \vee f_2(e) \in B.$$

For $e \in I_1 \cap I_2 \cap B$ and $x \in I_1 \cap I_2$ we have:

$$x \wedge (f_1 \vee f_2)(e) = x \wedge [f_1(e) \vee f_2(e)] = [x \wedge f_1(e)] \vee [x \wedge f_2(e)] = [e \wedge f_1(x)] \vee [e \wedge f_2(x)] =$$

$$= e \wedge [f_1(x) \vee f_2(x)] = e \wedge (f_1 \vee f_2)(x),$$

that is $f_1 \vee f_2 \in M(I_1 \cap I_2, A)$. ■

Lemma 4.3. $f_1 \boxdot f_2 \in M(I_1 \cap I_2, A)$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B$, then

$$(f_1 \boxdot f_2)(e \odot x) = f_1(e \odot x) \odot [(e \odot x) \to f_2(e \odot x)] = [e \odot f_1(x)] \odot [(e \odot x) \to (e \odot f_2(x))] =$$

$$= f_1(x) \odot [e \odot ((e \odot x) \to (e \odot f_2(x)))] \overset{c_{42}}{=} f_1(x) \odot [e \odot (x \to f_2(x))] =$$

$$= e \odot [f_1(x) \odot (x \to f_2(x))] = e \odot (f_1 \boxdot f_2)(x).$$

Clearly, $(f_1 \boxdot f_2)(x) = f_1(x) \odot [x \to f_2(x)] \leq f_1(x) \leq x$, for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B$, then by Remark 2.4 we have

$$(f_1 \boxdot f_2)(e) = f_1(e) \odot [e \to f_2(e)] = f_1(e) \odot (e^* \vee f_2(e)) \in B.$$

For $e \in I_1 \cap I_2 \cap B$ and $x \in I_1 \cap I_2$ we have:

$$x \wedge (f_1 \boxdot f_2)(e) = x \wedge [f_1(e) \odot (e \to f_2(e))] =$$

$$= x \odot [f_1(e) \odot (e \to f_2(e))] = f_1(e) \odot [x \odot (e \to f_2(e))]$$

$$\overset{c_{43}}{=} f_1(e) \odot [x \odot ((x \odot e) \to (x \odot f_2(e)))] = (f_1(e) \odot x) \odot ((x \odot e) \to (x \odot f_2(e))) =$$

$$= (e \odot f_1(x)) \odot ((e \odot x) \to (e \odot f_2(x))) = f_1(x) \odot [e \odot ((e \odot x) \to (e \odot f_2(x)))]$$

$$\overset{c_{42}}{=} f_1(x) \odot [e \odot (x \to f_2(x))] = e \odot [f_1(x) \odot (x \to f_2(x))] = e \odot (f_1 \boxdot f_2)(x) = e \wedge (f_1 \boxdot f_2)(x),$$

hence

$$x \wedge (f_1 \boxdot f_2)(e) = e \wedge (f_1 \boxdot f_2)(x),$$

that is $f_1 \boxdot f_2 \in M(I_1 \cap I_2, A)$. ■

Lemma 4.4. $f_1 \to f_2 \in M(I_1 \cap I_2, A)$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B$, then

$$(f_1 \to f_2)(e \odot x) = (e \odot x) \odot [f_1(e \odot x) \to f_2(e \odot x)] = (e \odot x) \odot [(e \odot f_1(x)) \to (e \odot f_2(x))] =$$
$$= x \odot [e \odot ((e \odot f_1(x)) \to (e \odot f_2(x)))] \overset{c_{42}}{=} x \odot [e \odot (f_1(x) \to f_2(x))] =$$
$$= e \odot [x \odot (f_1(x) \to f_2(x))] = e \odot (f_1 \to f_2)(x).$$

Clearly, $(f_1 \to f_2)(x) = x \odot [f_1(x) \to f_2(x)] \leq x$, for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B$, then by Remark 2.4 we have

$$(f_1 \to f_2)(e) = e \odot [f_1(e) \to f_2(e)] = e \odot [(f_1(e))^* \vee f_2(e)] \in B.$$

For $e \in I_1 \cap I_2 \cap B$ and $x \in I_1 \cap I_2$ we have:

$$e \wedge (f_1 \to f_2)(x) = e \wedge [x \odot (f_1(x) \to f_2(x))] =$$
$$= (e \odot x) \odot [f_1(x) \to f_2(x)] = x \odot [e \odot (f_1(x) \to f_2(x))]$$
$$\overset{c_{42}}{=} x \odot [e \odot ((e \odot f_1(x)) \to (e \odot f_2(x)))] = x \odot [e \odot ((x \odot f_1(e)) \to (x \odot f_2(e)))] =$$
$$= e \odot [x \odot ((x \odot f_1(e)) \to (x \odot f_2(e)))] \overset{c_{43}}{=} e \odot [x \odot (f_1(e) \to f_2(e))] =$$
$$= x \odot [e \odot (f_1(e) \to f_2(e))] = x \odot (f_1 \to f_2)(e) = x \wedge (f_1 \to f_2)(e)$$

hence

$$x \wedge (f_1 \to f_2)(e) = e \wedge (f_1 \to f_2)(x),$$

that is $f_1 \to f_2 \in M(I_1 \cap I_2, A)$. ■

Proposition 4.1. $(M(A), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$ *is a BL-algebra.*

Proof. See [4], Proposition 13. ■

Remark 4.7. *To prove that* $(M(A), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$ *is a BL-algebra it is sufficient to ask for multipliers to verify only the axioms* a_9 *and* a_{10}.

Proposition 4.2. *If BL− algebra* $(A, \wedge, \vee, \odot, \to, 0, 1)$ *is an MV− algebra* $(A, \oplus, ^*, 0)$ *(i.e.* $x^{**} = x$, *for all* $x \in A$*), then BL− algebra* $(M(A), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$ *is an MV− algebra* $(M(A), \boxplus, ^*, \mathbf{0})$, *see* [6]. *If* $I_1, I_2 \in \mathcal{I}(A)$ *and* $f_i \in M(I_i, A), i = 1, 2$, *we have* $f_1 \boxplus f_2 : I_1 \cap I_2 \to A$,

$$(f_1 \boxplus f_2)(x) = (f_1(x) \oplus f_2(x)) \wedge x,$$

for every $x \in I_1 \cap I_2$; *for* $I \in \mathcal{I}(A)$ *and* $f \in M(I, A)$ *we have* $f^* : I \to A$

$$f^*(x) = (f \to \mathbf{0})(x) = x \odot (f(x) \to \mathbf{0}(x)) = x \odot (f(x) \to 0) = x \odot (f(x))^*,$$

for every $x \in I$.

Proof. To prove that $BL-$ algebra $M(A)$ is an $MV-$ algebra let $f \in M(I, A)$ with $I \in \mathcal{I}(A)$.

Then

$$f^{**}(x) = [(f \to \mathbf{0}) \to \mathbf{0}](x) = x \odot [(f \to \mathbf{0})(x)]^* = x \odot [x \odot (f(x))^*]^*$$
$$= x \odot [(x \odot (f(x))^*) \to 0] \overset{c_8}{=} x \odot [x \to (f(x))^{**}] = x \wedge (f(x))^{**} = x \wedge f(x) = f(x),$$

(since $f(x) \in A$ which is an $MV-$ algebra), for all $x \in I$.

So, $f^{**} = f$ and $BL-$ algebra $M(A)$ is an MV -algebra.

We have $f_1 \boxplus f_2 = (f_1^* \boxdot f_2^*)^*$ and $f^* = f \to \mathbf{0}$.

Clearly,

$$(f_1 \boxplus f_2)(x) = x \odot [f_1^*(x) \odot (x \to f_2^*(x))]^*$$
$$= x \odot [x \odot (f_1(x))^* \odot (x \to x \odot (f_2(x))^*)]^* = x \odot [(f_1(x))^* \odot x \odot (x \to x \odot (f_2(x))^*)]^*$$
$$\overset{a_4}{=} x \odot [(f_1(x))^* \odot (x \wedge x \odot (f_2(x))^*)]^* = x \odot [(f_1(x))^* \odot x \odot (f_2(x))^*]^*$$

$$= x \odot [x \odot (f_1(x))^* \odot (f_2(x))^*]^* \overset{c_{37}}{=} x \odot [x \to ((f_1(x))^* \odot (f_2(x))^*)^*]$$

$$\overset{a_4}{=} x \wedge (f_1(x) \oplus f_2(x)),$$

for all $x \in I_1 \cap I_2$. Then $(M(A), \boxplus, ^*, \mathbf{0})$ is an MV-algebra.■

Lemma 4.5. *The map $v_A : B \to M(A)$ defined by $v_A(a) = \overline{f_a}$ for every $a \in B$, is a monomorphism of BL-algebras.*

Proof. Clearly, $v_A(0) = \overline{f_0} = \mathbf{0}$. Let $a, b \in B$ and $x \in A$. We have:

$$(v_A(a) \boxdot v_A(b))(x) = v_A(a)(x) \odot (x \to v_A(b)(x)) = (a \wedge x) \odot (x \to (b \wedge x))$$

$$= (a \odot x) \odot (x \to (b \wedge x)) = a \odot [x \odot (x \to (b \wedge x))] = a \odot [x \wedge (b \wedge x)]$$

$$= a \wedge [x \wedge (b \wedge x)] = a \wedge (b \wedge x) = (a \wedge b) \wedge x = (v_A(a \wedge b))(x) = (v_A(a \odot b))(x),$$

hence

$$v_A(a \odot b) = v_A(a) \boxdot v_A(b).$$

Also,

$$(v_A(a) \to v_A(b))(x) = x \odot [v_A(a)(x) \to v_A(b)(x)] = x \odot [(a \wedge x) \to (b \wedge x)]$$

$$= x \odot [(x \odot a) \to (x \odot b)] \overset{c_{43}}{=} x \odot (a \to b) = x \wedge (a \to b)$$

(since $a \to b \in B$)

$$= v_A(a \to b)(x),$$

hence

$$v_A(a) \to v_A(b) = v_A(a \to b),$$

that is v_A is a morphism of BL-algebras.

To prove the injectivity of v_A let $a, b \in B$ such that $v_A(a) = v_A(b)$. Then $a \wedge x = b \wedge x$, for every $x \in A$, hence for $x = 1$ we obtain that $a \wedge 1 = b \wedge 1 \Rightarrow a = b$.■

Definition 4.4. *A nonempty set $I \subseteq A$ is called regular if for every $x, y \in A$ such that $x \wedge e = y \wedge e$ for every $e \in I \cap B$, then $x = y$.*

For example A is a regular subset of A (since if $x, y \in A$ and $x \wedge e = y \wedge e$ for every $e \in A \cap B = B$, then for $e = 1$ we obtain $x \wedge 1 = y \wedge 1 \Leftrightarrow x = y$).

More generally, every subset of A which contains 1 is regular.

We denote

$$R(A) = \{I \subseteq A : I \text{ is a regular subset of } A\}.$$

Remark 4.8. *The condition $I \in R(A)$ is equivalent with the condition: for every $x, y \in A$, if $f_{x|I \cap B} = f_{y|I \cap B}$, then $x = y$.*

Lemma 4.6. *If $I_1, I_2 \in I(A) \cap R(A)$, then $I_1 \cap I_2 \in I(A) \cap R(A)$.*

Proof. See [4], Lemma 15. ■

Remark 4.9. *By Lemma 4.6, we deduce that*

$$M_r(A) = \{f \in M(A) : dom(f) \in I(A) \cap R(A)\}$$

is a BL-subalgebra of $M(A)$.

Proposition 4.3. *$M_r(A)$ is a Boolean subalgebra of $M(A)$.*

Proof. Let $f : I \to A$ be a B-multiplier on A with $I \in \mathcal{I}(A) \cap \mathcal{R}(A)$. Then

$$\begin{aligned}
e \wedge [f \vee f^*](x) &= e \wedge [f(x) \vee (x \odot (f(x))^*)] = [e \wedge f(x)] \vee [e \wedge (x \odot (f(x))^*)] \\
&\overset{a_{12}}{=} [x \odot f(e)] \vee [x \odot e \odot (f(x))^*] \overset{c_{42}}{=} [x \odot f(e)] \vee [x \odot e \odot (e \odot f(x))^*] \\
&\overset{a_{12}}{=} [x \odot f(e)] \vee [x \odot e \odot (x \odot f(e))^*] = [x \odot f(e)] \vee [x \odot e \odot (x \wedge f(e))^*] \\
&\overset{c_{33}}{=} [x \odot f(e)] \vee [x \odot e \odot (x^* \vee (f(e))^*)] \overset{c_{13}}{=} [x \odot f(e)] \vee [e \odot ((x \odot x^*) \vee (x \odot (f(e))^*))] \\
&\overset{c_5}{=} [x \odot f(e)] \vee [e \odot (0 \vee (x \odot (f(e))^*))] = [x \odot f(e)] \vee [e \odot x \odot (f(e))^*] \\
&= [x \odot f(e)] \vee [x \odot (e \odot (f(e))^*)] \overset{c_{13}}{=} x \odot [f(e) \vee (e \odot (f(e))^*)] \\
&= x \odot [f(e) \vee (e \wedge (f(e))^*)] = x \odot [(f(e) \vee e) \wedge (f(e) \vee (f(e))^*)] \\
&\overset{a_{11}}{=} x \odot (e \wedge 1) = x \odot e = x \wedge e = \mathbf{1}(x) \wedge e,
\end{aligned}$$

hence $(f \vee f^*)(x) = \mathbf{1}(x)$, since $I \in \mathcal{R}(A)$, hence $f \vee f^* = \mathbf{1}$, that is $M_r(A)$ is a Boolean algebra. ■

Remark 4.10. *The axioms a_{11} and a_{12} are necessary in the proof of Proposition 4.3.*

Definition 4.5. *Given two multipliers f_1, f_2 on A, we say that f_2 extends f_1 if $dom(f_1) \subseteq dom(f_2)$ and $f_{2|dom(f_1)} = f_1$; we write $f_1 \le f_2$ if f_2 extends f_1. A multiplier f is called maximal if f can not be extended to a strictly larger domain.*

Lemma 4.7. *If $f_1, f_2 \in M(A)$, $f \in M_r(A)$ and $f \le f_1, f \le f_2$, then f_1 and f_2 agree on the $dom(f_1) \cap dom(f_2)$.*

Proof. See [4], Lemma 17. ■

Lemma 4.8. *Every multiplier $f \in M_r(A)$ can be extended to a maximal multiplier.*

Proof. See [4], Lemma 17. ■

Lemma 4.9. *Each principal multiplier f_a with $a \in B$ and $dom(f_a) \in I(A) \cap R(A)$ can be uniquely extended to the total multiplier $\overline{f_a}$ and each non-principal multiplier can be extended to a maximal non-principal one.*

Proof. See [4], Lemma 17. ■

On the Boolean algebra $M_r(A)$ we consider the relation ρ_A defined by

$(f_1, f_2) \in \rho_A$ iff f_1 and f_2 agree on the intersection of their domains.

Lemma 4.10. *ρ_A is a congruence on Boolean algebra $M_r(A)$.*

Proof. The same proof as in the case of $BL-$ algebras (see [4], Lemma 18). ■

Definition 4.6. *For $f \in M_r(A)$ with $I = dom(f) \in I(A) \cap R(A)$, we denote by $[f, I]$ the congruence class of f modulo ρ_A and $A_B = M_r(A)/\rho_A$.*

Corollary 4.1. *By Proposition 4.3 and Lemma 4.10 we deduce that A_B is a Boolean algebra.*

Remark 4.11. *If we denote by $\mathcal{F} = \mathcal{I}(A) \cap \mathcal{R}(A)$ and consider the partially ordered systems $\{\delta_{I,J}\}_{I,J \in \mathcal{F}, I \subseteq J}$ (where for $I, J \in \mathcal{F}$, $I \subseteq J, \delta_{I,J} : M(J, A) \to M(I, A)$ is defined by $\delta_{I,J}(f) = f_{|I}$), then by above construction of A_B we deduce that A_B is the inductive limit*

$$A_B = \varinjlim_{I \in \mathcal{F}} M(I, A).$$

Lemma 4.11. *Let the map $\overline{v_A} : B \to A_B$ defined by $\overline{v_A}(a) = [\overline{f_a}, A]$ for every $a \in B$. Then*

(*i*) $\overline{v_A}$ *is an injective morphism of Boolean algebras,*
(*ii*) *For every* $a \in B, [\overline{f_a}, A] \in B(A_B)$,
(*iii*) $\overline{v_A}(B) \in R(A_B)$.

Proof. (*i*). Follows from Lemma 4.5.
(*ii*). For $a \in B$ and $x \in A$ we have

$$(\overline{f_a} \boxdot \overline{f_a})(x) = \overline{f_a}(x) \odot (x \to \overline{f_a}(x)) = (a \wedge x) \odot [x \to (a \wedge x)] = (a \odot x) \odot [x \to (a \odot x)] =$$

$$= a \odot [x \odot (x \to (a \odot x))] = a \odot [x \wedge (a \odot x)] = a \odot (a \odot x) = a \wedge (a \wedge x) = a \wedge x = \overline{f_a}(x),$$

and

$$(\overline{f_a})^{**}(x) = x \odot [(\overline{f_a})^*(x) \to \mathbf{0}(x)] = x \odot [(\overline{f_a} \to \mathbf{0})(x) \to \mathbf{0}(x)] = x \odot [x \odot (\overline{f_a}(x) \to 0) \to 0] =$$

$$= x \odot [x \odot (\overline{f_a}(x))^*]^* = x \odot [x \odot (a \wedge x)^*]^* \overset{c_{33}}{=} x \odot [x \odot (a^* \vee x^*)]^* \overset{c_{13}}{=} x \odot [(x \odot a^*) \vee (x \odot x^*)]^* \overset{c_5}{=}$$

$$\overset{c_5}{=} x \odot [(x \odot a^*) \vee 0]^* = x \odot [(x \odot a^*)]^* =$$

(since $a \in B$)

$$= x \odot (x \to a) = x \wedge a = \overline{f_a}(x),$$

hence

$$\overline{f_a} \boxdot \overline{f_a} = \overline{f_a}$$

and

$$\overline{f_a}^{**} = \overline{f_a},$$

that is $[\overline{f_a}, A] \in B(A_B)$.

(*iii*). To prove $\overline{v_A}(B) \in R(A_B)$, if by contrary there exist $f_1, f_2 \in M_r(A)$ such that $[f_1, dom(f_1)] \neq [f_2, dom(f_2)]$ (that is there exists $x_0 \in dom(f_1) \cap dom(f_2)$ such that $f_1(x_0) \neq f_2(x_0)$) and $[f_1, dom(f_1)] \wedge [\overline{f_a}, A] = [f_2, dom(f_2)] \wedge [\overline{f_a}, A]$ for every $[\overline{f_a}, A] \in \overline{v_A}(B) \cap B(A_B)$ (that is by (*ii*) for every $[\overline{f_a}, A] \in \overline{v_A}(B)$ with $a \in B$), then $(f_1 \wedge \overline{f_a})(x) = (f_2 \wedge \overline{f_a})(x)$ for every $x \in dom(f_1) \cap dom(f_2)$ and every $a \in B \Leftrightarrow f_1(x) \wedge a \wedge x = f_2(x) \wedge a \wedge x$ for every $x \in dom(f_1) \cap dom(f_2)$ and every $a \in B$. For $a = 1 \in B$ and $x = x_0$ we obtain that $f_1(x_0) \wedge x_0 = f_2(x_0) \wedge x_0 \Leftrightarrow f_1(x_0) = f_2(x_0)$ which is contradictory. ■

Remark 4.12. *Since by Lemma 4.11, for every* $a, b \in B$, $[\overline{f_a}, A] = [\overline{f_b}, A]$ *iff* $a = b$, *the elements of* B *can be identified with the elements of the sets* $\{[\overline{f_a}, A] : a \in B\}$ *and* $\{\overline{f_a} : a \in B\}$. *So,* $v_A(B) \approx \overline{v_A}(B) \approx B$ *(as* $BL-$ *algebras).*

Lemma 4.12. *If* $[f, dom(f)] \in A_B$ *(with* $f \in M_r(A)$ *and* $I = dom(f) \in I(A) \cap R(A)$*), then*

$$I \cap B \subseteq \{a \in B : \overline{f_a} \wedge [f, dom(f)] \in \overline{v_A}(B)\}.$$

Proof. Let $a \in I \cap B$. Then for every $x \in I, (\overline{f_a} \wedge f)(x) = \overline{f_a}(x) \wedge f(x) = a \wedge x \wedge f(x) = a \wedge f(x) = a \odot f(x) = f(a \odot x) = x \odot f(a)$ (by a_{12}) $= x \wedge f(a)$, that is $\overline{f_a} \wedge f = \overline{f_{f(a)}} \in \overline{v_A}(B)$ (since $f(a) \in B$), that is, the required inclusion. ■

Remark 4.13. *The axiom* a_{12} *is necessary in the proof of Lemma 4.12.*

5. Boolean Maximal BL-algebra of quotients

Definition 5.1. *A BL-algebra F is called BL-algebra of fractions of A relative to B if:*

(a_{13}) *B is a BL-subalgebra of F,*

(a_{14}) *For every $a', b', c' \in F, a' \neq b'$, there exists $e \in B$ such that $e \wedge a' \neq e \wedge b'$ and $e \wedge c' \in B$.*

So, BL-algebra B is a BL-algebra of fractions of itself (since $1 \in B$).

As a notational convenience, we write $A \preceq F$ to indicate that F is a BL-algebra of fractions of A relative to B.

Remark 5.1. *If $A \preceq F$, then F is a Boolean algebra. Indeed, if by contrary, then there exists $a' \in F$ such that $a' \neq a' \odot a'$ or $(a')^{**} \neq a'$. If $a' \neq a' \odot a'$, since $A \preceq F$, then there exists $e \in B$ such that $e \wedge a' \in B$ and*

$$e \wedge a' \neq e \wedge (a' \odot a') = (e \wedge a') \odot (e \wedge a'),$$

which is contradictory!.

If $(a')^{**} \neq a'$, since $A \preceq F$, then there exists $f \in B$ such that $f \wedge a' \in B$ and

$$f \wedge a' \neq f \wedge (a')^{**} = (f \wedge a')^{**}$$

which is contradictory!

Lemma 5.1. *Let $A \preceq F$; then for every $a', b' \in F, a' \neq b'$, and any finite sequence $c'_1, ..., c'_n \in F$, there exists $e \in B$ such that $e \wedge a' \neq e \wedge b'$ and $e \wedge c'_i \in B$ for $i = 1, 2, ..., n$ $(n \geq 2)$.*

Proof. See [4], Lemma 21. ■

Lemma 5.2. *Let $A \preceq F$ and $a' \in F$. Then*

$$I_{a'} = \{e \in B : e \wedge a' \in B\} \in I(B) \cap R(A).$$

Proof. Clearly, $I_{a'} \in I(B)$.

To prove $I_{a'} \in R(A)$, let $x, y \in A$ such that $e \wedge x = e \wedge y$ for every $e \in I_{a'} \cap B$. If by contrary, $x \neq y$, since $A \preceq F$, there exists $e_0 \in B$ such that $e_0 \wedge a' \in B$ (that is $e_0 \in I_{a'}$) and $e_0 \wedge x \neq e_0 \wedge y$, which is contradictory. ■

Theorem 5.1. *For every $BL-$ algebra A, the Boolean algebra A_B in Definitin 4.6 has the following properties:*

(i) *$\overline{v_A}(B) \preceq A_B$,*

(ii) *for every BL-algebra F such that $A \preceq F$, there exists monomorphism of BL-algebras $i : F \to A_B$ which induces the canonical monomorphism $\overline{v_A}$ of B into A_B.*

Proof. The fact that $\overline{v_A}(B)$ is a BL-subalgebra of A_B follows from Lemma 4.11, (i). To prove a_{14}, let $[f, dom(f)], [g, dom(g)], [h, dom(h)] \in A_B$ with $f, g, h \in M_r(A)$ such that $[g, dom(g)] \neq [h, dom(h)]$ (that is there exists $x_0 \in dom(g) \cap dom(h)$ such that $g(x_0) \neq h(x_0)$).

Put $I = dom(f) \in I(A) \cap R(A)$ and

$$I_{[f,dom(f)]} = \{a \in B : \overline{f_a} \wedge [f, dom(f)] \in \overline{v_A}(B)\}$$

(by Lemma 4.11, $\overline{f_a} \in B(M(A))$ if $a \in B$). Then by Lemma 4.12,

$$I \cap B \subseteq I_{[f,dom(f)]}.$$

If we suppose that for every $a \in I \cap B$, $\overline{f_a} \wedge [g, dom(g)] = \overline{f_a} \wedge [h, dom(h)]$, then $[\overline{f_a} \wedge g, dom(g)] = [\overline{f_a} \wedge h, dom(h)]$, hence for every $x \in dom(g) \cap dom(h)$ we have $(\overline{f_a} \wedge g)(x) = (\overline{f_a} \wedge h)(x)$ i.e. $a \wedge g(x) = a \wedge h(x)$.

Since $I \in R(A)$ we deduce that $g(x) = h(x)$ for every $x \in dom(g) \cap dom(h)$ so $[g, dom(g)] = [h, dom(h)]$, which is contradictory.

Hence, if $[g, dom(g)] \neq [h, dom(h)]$, then there exists $a \in I \cap B$, such that $\overline{f_a} \wedge [g, dom(g)] \neq \overline{f_a} \wedge [h, dom(h)]$. But for this $a \in I \cap B$ we have

$$\overline{f_a} \wedge [f, dom(f)] \in \overline{v_A}(B)$$

(since by Lemma 4.12, $I \cap B \subseteq I_{[f,dom(f)]}$).

To prove the *maximally* of A_B, let F be a BL-algebra such that $A \preceq F$; thus $B \subseteq B(F)$

$$\begin{array}{ccc} A & \preceq & F \\ & & \swarrow i \\ & A_B & \end{array}$$

For $a' \in F, I_{a'} = \{e \in B : e \wedge a' \in B\} \in I(B) \cap R(A)$ (by Lemma 5.2).

Thus $f_{a'} : I_{a'} \to A$ defined by $f_{a'}(x) = x \wedge a'$ is a $B-$multiplier. Indeed, if $e \in B$ and $x \in I_{a'}$, then

$$f_{a'}(e \odot x) = (e \odot x) \wedge a' = (e \wedge x) \wedge a' = e \wedge (x \wedge a') = e \odot (x \wedge a') = e \odot f_{a'}(x),$$

and

$$f_{a'}(x) \leq x,$$

hence a_9 and a_{10} are verified.

To verify a_{11}, let $e \in I_{a'} \cap B = I_{a'}$. Thus, $f_{a'}(e) = e \wedge a' \in B$ (since $e \in I_{a'}$).

The condition a_{12} is obviously verified, hence $[f_{a'}, I_{a'}] \in A_B$.

We define $i : F \to A_B$, by $i(a') = [f_{a'}, I_{a'}]$, for every $a' \in F$. Clearly $i(0) = \mathbf{0}$.

For $a', b' \in F$ and $x \in I_{a'} \cap I_{b'}$, we have

$$(i(a') \boxdot i(b'))(x) = (a' \wedge x) \odot [x \to (b' \wedge x)] =$$
$$= (a' \odot x) \odot [x \to (b' \wedge x)] = a' \odot [x \odot (x \to (b' \wedge x))]$$
$$= a' \odot [x \wedge (b' \wedge x)] = a' \odot (b' \wedge x) = a' \odot (b' \odot x) = (a' \odot b') \odot x = (a' \odot b') \wedge x = i(a' \odot b')(x),$$

hence $i(a') \boxdot i(b') = i(a' \odot b')$ and

$$(i(a') \to i(b'))(x) = x \odot [i(a')(x) \to i(b')(x)]$$
$$= x \odot [(a' \wedge x) \to (b' \wedge x)] = x \odot [(x \odot a') \to (x \odot b')]$$
$$\overset{c_{43}}{=} x \odot (a' \to b') = x \wedge (a' \to b') = i(a' \to b')(x),$$

hence $i(a') \to i(b') = i(a' \to b')$, that is i is a morphism of BL-algebras.

To prove the injectivity of i, let $a', b' \in F$ such that $i(a') = i(b')$. It follows that $[f_{a'}, I_{a'}] = [f_{b'}, I_{b'}]$ so $f_{a'}(x) = f_{b'}(x)$ for every $x \in I_{a'} \cap I_{b'}$. We get $a' \wedge x = b' \wedge x$ for every $x \in I_{a'} \cap I_{b'}$. If $a' \neq b'$, by Lemma 5.1 (since $A \preceq F$), there exists $e \in B$ such that $e \wedge a', e \wedge b' \in B$ and $e \wedge a' \neq e \wedge b'$ which is contradictory (since $e \wedge a', e \wedge b' \in B$ implies $e \in I_{a'} \cap I_{b'}$). ■

The Theorem 5.1 provides the motivation for the following Definition:

Definition 5.2. *For any $BL-$ algebra A, A_B is called a maximal $BL-$ algebra of quotients of A (which by Remark 5.1 is a Boolean algebra). To range with the tradition (*[2], [5], [6], [17], [18]*) we denote A_B by $Q_B(A)$.*

Remark 5.2. *If $BL-$ algebra A is an $MV-$ algebra, then we obtain the maximal $MV-$ algebra of quotients of A (see* [6]*).*

Remark 5.3. *If A is a Boolean algebra, then $B(A) = A$. The axioms $a_9 - a_{12}$ are equivalent with a_9, hence $Q_B(A)$ is in this case just the classical Dedekind-MacNeille completion of A (see* [18]*, p.687). In contrast to the general situation, the Dedekind-MacNeille completion of a Boolean algebra is again distributive and, in fact, is a Boolean algebra (*[1]*, p.239).*

Proposition 5.1. *Let A be a BL - algebra. Then the following statements are equivalent:*

(i) *Every maximal $B-$multiplier on A has domain A,*
(ii) *For every B-multiplier $f \in M(I, A)$ there is $a \in B$ such that $f = f_a$ (that is $f(x) = a \wedge x$ for every $x \in I$),*
(iii) *$Q_B(A) \approx B$.*

Proof. $(i) \Rightarrow (ii)$. Assume (i) and for $f \in M(I, A)$ let f' its the maximal extension (by Lemma 4.7). By (i), we have $f' : A \to A$. Put $a = f'(1) \in B$ (by a_{11}), then for every $x \in I$, $f(x) = f(x) \wedge 1 \overset{a_{12}}{=} x \wedge f(1) = x \wedge a = f_a(x)$, that is $f = f_a$.

$(ii) \Rightarrow (iii)$. Follow from Lemma 4.11.

$(iii) \Rightarrow (i)$. Follow from Lemma 4.7 and Lemma 4.11.■

Definition 5.3. *If A verify one of condition of Proposition 5.1, we call A rationnaly complete.*

Example 5.1. 1. *If A is a $BL-$ algebra, $B = B(A) = \{0, 1\} = L_2$ and $A \preceq F$ then $F = \{0, 1\}$, hence $Q_B(A) = A_B \approx L_2$. Indeed, if $a, b, c \in F$ with $a \neq b$, then by a_{14} there exists $e \in B$ such $e \wedge a \neq e \wedge b$ (hence $e \neq 0$) and $e \wedge c \in B$. Then, $e = 1$, hence $c \in B$, that is, $F = B$. As examples of $BL-$ algebras with this property we have local $BL-$ algebras and $BL-$ chains.*

2. *More general, if A is a $BL-$ algebra, B is a finite Boolean subalgebra of A and $A \preceq F$, then $F = B$, hence in this case $Q_B(A) = A_B \approx B$. Indeed, since $A \preceq F$ we have $B \subseteq B(F) \subseteq F$. If consider $a \in F$, then there exists $e \in B$ such that $e \wedge x \in B$ (for example $e = 0$). B being finite, there exists a largest element $e_a \in B$ such $e_a \wedge a \in B$. Suppose $e_a \vee a \neq e_a$, then there would exists $e \in B$ such that $e \wedge (e_a \vee a) \neq e \wedge e_a$ and $e \wedge a \in B$. But $e \wedge a \in B$ implies $e \leq e_a$ and thus we obtain $e = e \wedge (e_a \vee a) \neq e \wedge e_a = e$, a contradiction. Hence $e_a \vee a = e_a$, so $a \leq e_a$, consequently $a = a \wedge e_a \in B$, that is, $F \subseteq B$. Then $F = B$, hence $Q_B(A) \approx B$.*

3. *If $B = B(A)$, then by Remark 5.3, $Q_B(A)$ is the maximal $BL-$ algebra of quotients of A defined in* [4] *(Theorem 23).*

Corollary 5.1. *If consider $MV-$ algebra $L_{3\times 2}$, from Example 2.12, then $B(A) = \{0, a, d, 1\}$ is finite. Then we obtain:*

1. *If $B = B(A) = \{0, a, d, 1\}$ then $F_{B(A)} = B(A)$ and $Q_B = Q_{B(A)}(A) = B(A) = \{0, a, d, 1\}$.*
2. *If $B = L_2 = \{0, 1\}$ then $F_{L_2} = L_2$ and $Q_{L_2}(A) = L_2$.*

Corollary 5.2. *If consider $BL-$ algebra $A = \{0, a, b, c, 1\}$, from Example 2.11, then $B(A) = \{0, 1\} = L_2$ is finite. Then we obtain $B = L_2 = \{0, 1\}$, so $F_{L_2} = L_2$ and $Q_{L_2}(A) = L_2$.*

References

[1] R. Balbes, Ph. Dwinger, *Distributive Lattices*, University of Missouri Press, 1974.

[2] D. Buşneag: Hilbert algebra of fractions and maximal Hilbert algebras of quotients, *Kobe Journal of Mathematics*, **5**, 161-172 (1988).
[3] D. Buşneag, D. Piciu, BL-algebra of fractions relative to an $\wedge$- closed system, *Analele Ştiinţifice ale Universităţii Ovidius Constanţa, Seria Matematica*, **XI**(1), 39-48 (2003).
[4] D. Buşneag, D. Piciu, BL-algebra of fractions and maximal BL-algebra of quotients, *Soft Computing (Springer - Verlag)*, DOI 10.1007/s00500-004-0372-9 (2004).
[5] D. Buşneag, D.Piciu, MV-algebra of fractions and maximal MV-algebra of quotients, *Journal of Multiple-Valued Logic and Soft Computing*, **10**, 363-383 (2004).
[6] D. Buşneag, D. Piciu, *Boolean MV-algebra of fractions,* submitted.
[7] D. Buşneag, D. Piciu, On the lattice of deductive systems of a BL-algebra, *Central European Journal of Mathematics*, **1**(2), 221-238 (2003).
[8] D. Buşneag, D. Piciu, Injective objects in the BL-algebras category (Part I), *Analele Universităţii din Craiova, Seria Matematica -Informatica*, **XXIX**, 32-39 (2002).
[9] C.C. Chang, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.*, **88**, 467-490 (1958).
[10] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, *Algebraic foundation of many -valued Reasoning*, Dordrecht, Kluwer Academic Publishers, 2000.
[11] W.H. Cornish, The multiplier extension of a distributive lattice, *Journal of Algebra*, **32**, 339-355 (1974).
[12] W.H. Cornish, A multiplier approach to implicative BCK-algebras, *Mathematics Seminar Notes, Kobe University*, **8**(1), 1980.
[13] P. Hájek, *Metamathematics of fuzzy logic*, Trends in Logic - Studia Logica Library 4, Dordrecht, Kluwer Acad. Publ., (1998).
[14] A. Iorgulescu, Classes of BCK algebras-Part III, *Preprint Series of The Institute of Mathematics of the Romanian Academy*, preprint no. **3**, 1-37 (2004).
[15] J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publishing Company, 1966.
[16] D. Piciu, *Localization of MV and BL-algebras, Ph. D. Thesis,* University of Bucharest, 2004.
[17] J. Schmid, Multipliers on distributive lattices and rings of quotients, *Houston Journal of Mathematics*, **6**(3), (1980).
[18] J. Schmid, Distributive lattices and rings of quotients, *Coll. Math. Societatis Janos Bolyai*, **33**, Szeged, Hungary, 1980.
[19] E. Turunen: *Mathematics Behind Fuzzy Logic*, Physica-Verlag, 1999.
[20] E. Turunen, S. Sessa, Local BL-algebras, *Multiple Valued Logic*, **6**(1-2), 229-249 (2001).

(Dumitru Buşneag) Department of Mathematics, University of Craiova,
Al. I. Cuza street, 13, Craiova RO-200585, Romania, Tel/Fax: 40-251412673
E-mail address: `busneag@central.ucv.ro`

(Dana Piciu) Department of Mathematics, University of Craiova,
Al. I. Cuza street, 13, Craiova RO-200585, Romania, Tel/Fax: 40-251412673
E-mail address: `danap@central.ucv.ro`