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The complex-type Fibonacci p-Sequences

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Abstract. In this paper, we define a new sequence which is called the complex-type Fibonacci p-sequence and we obtain the generating matrix of this complex-type Fibonacci p-sequence. We also derive the determinantal and the permanental representations. Then, using the roots of the characteristic polynomial of the complex-type Fibonacci p-sequence, we produce the Binet formula for this defined sequence. In addition, we give the combinatorial representations, the generating function, the exponential representation and the sums of the complex-type Fibonacci p-numbers.

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1. Introduction

In [20], Stakhov and Rozin gave a generalization of the Fibonacci numbers. The generalization is called the Fibonacci p-numbers $F_p(n)$ that are given for any positive integer p by the following relation

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \text{ for } n > p$$

with the initial values $F_p(0) = 0$, $F_p(1) = \cdots F_p(p) = 1$. If we take $p = 1$, then $F_1(n) = F_n$ which are the classical Fibonacci numbers.

The complex Fibonacci sequence $\{F_n^*\}$ is defined [9] by a two-order recurrence equation:

$$F_n^* = F_n + iF_{n+1}$$

for $n \geq 0$, where $\sqrt{-1} = i$ and F_n is the n^{th} Fibonacci number (cf. [1, 10]).

Kalman [12] derived a number of closed-form formulas for the generalized sequence and he used the companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

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Also, he showed that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

The study of linear and recurrent sequences has been known for a long time and miscellaneous properties of these sequences have been studied by some authors; see, for example [6, 7, 8, 11, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24]. Further, in [4] and [5], the authors defined the new sequences using the quaternions and complex numbers and then they gave miscellaneous properties. In this paper, we define the complex-type Fibonacci p-sequence and we obtain the generating matrix of the complex-type Fibonacci p-sequence. Also, we derive the determinantal and the permanental representations by using certain matrices which are obtained from the generating matrix of these numbers. Then, we produce the Binet formula for this defined sequence. Finally, we give the combinatorial representations, the generating function, the exponential representation and the sums of the complex-type Fibonacci p-numbers.

2. The complex-type Fibonacci p-sequences

Now, we define a new sequence called the complex-type Fibonacci p-sequence. The recurrence relation of the complex-type Fibonacci p-sequence is

$$F_{p,i}^{*}(n+p+1) = i^{p+1} \cdot F_{p,i}^{*}(n+p) + i \cdot F_{p,i}^{*}(n) \tag{1}$$

for any given $p\,(p = 2, 3, \ldots)$ and $n \geq 0$, with the initial conditions $F_{p,i}^{*}(0) = \cdots = F_{p,i}^{*}(p-1) = 0$, $F_{p,i}^{*}(p) = 1$ and $\sqrt{-1} = i$.

By equation (1), we obtain the companion matrix of the complex-type Fibonacci p-sequence as:

$$C_p^F = \begin{bmatrix} i^{p+1} & 0 & \cdots & 0 & i \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}_{(p+1)\times(p+1)}.$$

and the nth power of the matrix C_p^F is

$$\left(C_p^F\right)^n = \begin{bmatrix} F_{p,i}^{*}(n+p) & iF_{p,i}^{*}(n) & iF_{p,i}^{*}(n+1) & \cdots & iF_{p,i}^{*}(n+p-1) \\ F_{p,i}^{*}(n+p-1) & iF_{p,i}^{*}(n-1) & iF_{p,i}^{*}(n) & \cdots & iF_{p,i}^{*}(n+p-2) \\ \vdots & \vdots & \vdots & & \vdots \\ F_{p,i}^{*}(n+1) & iF_{p,i}^{*}(n-p+1) & iF_{p,i}^{*}(n-p+2) & \cdots & iF_{p,i}^{*}(n) \\ F_{p,i}^{*}(n) & iF_{p,i}^{*}(n-p) & iF_{p,i}^{*}(n-p+1) & \cdots & iF_{p,i}^{*}(n-1) \end{bmatrix} \tag{2}$$

for $n \geq p$. This companion matrix C_p^F is said to be the complex-type Fibonacci p-matrix.

From the above matrix, we can easily obtain the following relationships between the complex-type Fibonacci p-numbers and the Fibonacci p-numbers for $n \geq 2p-1$

such that every even p integer:

$$\left(C_p^F\right)^n = \begin{bmatrix} \left(i^{p+1}\right)^n F_p(n+1) & \left(i^{p+1}\right)^{n+1} F_p(n-p+1) & \left(i^{p+1}\right)^{n+2} F_p(n-p+2) & \cdots & \left(i^{p+1}\right)^{n+p} F_p(n) \\ \left(i^{p+1}\right)^{n-1} F_p(n) & \left(i^{p+1}\right)^{n} F_p(n-p) & \left(i^{p+1}\right)^{n+1} F_p(n-p+1) & \cdots & \left(i^{p+1}\right)^{n+p-1} F_p(n-1) \\ \vdots & \vdots & \vdots & & \vdots \\ \left(i^{p+1}\right)^{n-p+1} F_p(n-p+2) & \left(i^{p+1}\right)^{n-p+2} F_p(n-2p+2) & \left(i^{p+1}\right)^{n-p+3} F_p(n-2p+3) & \cdots & \left(i^{p+1}\right)^{n+1} F_p(n-p+1) \\ \left(i^{p+1}\right)^{n-p} F_p(n-p+1) & \left(i^{p+1}\right)^{n-p+1} F_p(n-2p+1) & \left(i^{p+1}\right)^{n-p+2} F_p(n-2p+2) & \cdots & \left(i^{p+1}\right)^{n} F_p(n-p) \end{bmatrix}.$$

Now, we consider the permanental representations for the complex-type k-Fibonacci numbers.

Definition 2.1. A $u \times v$ real matrix $M = [m_{i,j}]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.

Suppose that x_1, x_2, …,x_u are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1) \times (v-1)$ matrix $M_{ij:k}$ is obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [2], Brualdi and Gibson obtained that $per\,(M) = per\,(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Let $p \geq 2$ be a positive integer and let $G_{p,r}^i = \left[g_{k,j}^{(p,i,r)}\right]$ be the $r \times r$ super-diagonal matrix, defined by

$$\begin{array}{c} p+1\text{th} \\ \downarrow \end{array}$$

$$G_{p,r}^i = \begin{bmatrix} i^{p+1} & 0 & 0 & \cdots & 0 & i & 0 & \cdots & 0 & 0 & 0 \\ 1 & i^{p+1} & 0 & 0 & \cdots & 0 & i & 0 & \cdots & 0 & 0 \\ 0 & 1 & i^{p+1} & 0 & 0 & \cdots & 0 & i & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & i^{p+1} & 0 & 0 & \cdots & 0 & i & 0 \\ 0 & 0 & \cdots & 0 & 1 & i^{p+1} & 0 & 0 & \cdots & 0 & i \\ 0 & 0 & 0 & \cdots & 0 & 1 & i^{p+1} & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & i^{p+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & i^{p+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & i^{p+1} \end{bmatrix}$$

for $r > p+1$.

Theorem 2.1. *For $r > p+1$ and $p \geq 2$,*

$$perG_{p,r}^i = F_{p,i}^*(r+p).$$

Proof. We prove this by induction on r. Suppose that the equation holds for $r > p+1$. Now, we consider $r+1$. If we expand the $perG^i_{p,r}$ by the Laplace expansion of a permanent with respect to the first row, then we obtain

$$perG^i_{p,r+1} = i^{p+1} \cdot perG^i_{p,r} + i \cdot perG^i_{p,r-p}.$$

Since $perG^i_{p,r} = F^*_{p,i}(r+p)$ and $perG^i_{p,r-p} = F^*_{p,i}(r)$, it is clear that $perG^i_{p,r+1} = F^*_{p,i}(r+p+1)$. Thus the result of the theorem holds. □

Let $r > p+1$ such that $p \geq 2$. Define the $r \times r$ matrix $H^i_{p,r} = \left[h^{(p,i,r)}_{k,j}\right]$ as shown:

$$h^{(p,i,r)}_{k,j} \begin{cases} i^{p+1} & \text{if } k = t \text{ and } j = t \text{ for } 1 \leq t \leq r-p-1 \\ i & \text{if } k = t \text{ and } j = t+p \text{ for } 1 \leq t \leq r-p, \\ 1 & \begin{matrix} \text{if } k = t+1 \text{ and } j = t \text{ for } 1 \leq t \leq r-p-2 \\ \text{and} \\ \text{if } k = t \text{ and } j = t \text{ for } r-p \leq t \leq r \end{matrix} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that the $r \times r$ matrix $K^i_{p,r} = \left[k^{(p,i,r)}_{k,j}\right]$ is defined by

$$K^i_{p,r} = \begin{array}{c} \begin{matrix} & & (r-p-1)\text{th} & & & \\ & & \downarrow & & & \end{matrix} \\ \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & H^i_{p,r-1} & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix} \end{array}.$$

Theorem 2.2. *Let $F^*_{p,i}(n)$ be the nth element of a complex-type Fibonacci p-number. Then*

i.For $r > p+1$,

$$perH^i_{p,r} = F^*_{p,i}(r-1).$$

ii.For $r > p+2$,

$$perK^i_{p,r} = \sum_{n=0}^{r-2} F^*_{p,i}(n).$$

Proof. We will use the induction method on r.

i. Now assume that $perH^i_{p,r} = F^*_{p,i}(r-1)$ for $r > p+1$. We examine the case $r+1$. If we expand the $perH^i_{p,r}$ by the Laplace expansion of a permanent with respect to the first row, by the definition of the matrix $H^i_{p,r}$, we obtain

$$perH^i_{p,r+1} = i^{p+1} \cdot perH^i_{p,r} + i \cdot perH^i_{p,r-p}.$$

By our assumption and the recurrence relation of the complex-type Fibonacci p-numbers,

$$perH^i_{p,r+1} = F^*_{p,i}(r) = i^{p+1} \cdot F^*_{p,i}(r-1) + i \cdot F^*_{p,i}(r-p-1).$$

So the result holds.

ii. If we expand the $perK^i_{p,r}$ by the Laplace expansion of a permanent with respect to the first row, then we write

$$perK^i_{p,r} = perK^i_{p,r-1} + H^i_{p,r-1}.$$

Thus, by the results and an inductive argument, the proof is easily seen. □

Let the notation $M \circ K$ denotes the Hadamard product of M and K. A matrix M is called convertible if there is an $n \times n$ $(1,-1)$-matrix K such that $perM = \det(M \circ K)$.

Let $r > p+2$ and let L be the $r \times r$ matrix, defined by

$$L = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 2.3. *For $r > p+2$,*

$$\det\left(G^{i}_{p,r} \circ L\right) = F^{*}_{p,i}(r+p),$$
$$\det\left(H^{i}_{p,r} \circ L\right) = F^{*}_{p,i}(r-1)$$

and

$$\det\left(K^{i}_{p,r} \circ L\right) = \sum_{n=0}^{r-2} F^{*}_{p,i}(n).$$

Proof. Since $perG^{i}_{p,r} = \det\left(G^{i}_{p,r} \circ L\right)$, $perH^{i}_{p,r} = \det\left(H^{i}_{p,r} \circ L\right)$ and $perK^{i}_{p,r} = \det\left(K^{i}_{p,r} \circ L\right)$ for $r > p+2$,, by Theorem 2.1 and Theorem 2.2, the results are obvious. □

Now we give the Binet formulas for the complex-type Fibonacci p-numbers.

Lemma 2.4. *The characteristic equation of the complex-type Fibonacci p-numbers $x^{p+1} - i^{p+1} \cdot x^{p} - i = 0$ does not have multiple roots for $p \geq 2$.*

Proof. Let $f(x) = x^{p+1} - i^{p+1} \cdot x^{p} - i$. Suppose that θ is a multiple root of $f(x) = 0$. Note that $\theta \neq 0$ and $\theta \neq 1$. When θ is a multiple root, $f(\theta) = \theta^{p+1} - i^{p+1} \cdot \theta^{p} - i = 0$ and $f'(\theta) = (p+1)\theta^{p} - \left(i^{p+1} \cdot p\right) \cdot \theta^{p-1} = 0$. Then $f'(\theta) = \theta^{p-1}\left((p+1)\theta - \left(i^{p+1} \cdot p\right)\right) = 0$. Thus, we obtain $\theta = \frac{i^{p+1} \cdot p}{p+1}$. For $p \geq 2$, $f(\theta) \neq 0$, which is a contradiction. Therefore, the equation $f(x) = 0$ does not have multiple roots. □

Let $f(z)$ be the characteristic polynomial of the matrix C^{F}_{p}. Then by Lemma 2.4, $z_1, z_2, \ldots, z_{p+1}$ are distinct eigenvalues of matrix C^{F}_{p}. Define the $(p+1) \times (p+1)$ Vandermonde matrix V^{p} as follows:

$$V^{p} = \begin{bmatrix} (z_1)^{p} & (z_2)^{p} & \ldots & (z_{p+1})^{p} \\ (z_1)^{p-1} & (z_2)^{p-1} & \ldots & (z_{p+1})^{p-1} \\ \vdots & \vdots & & \vdots \\ z_1 & z_2 & \ldots & z_{p+1} \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Let d^{p}_{k} be $(p+1) \times (1)$matrix

$$d^{p}_{k} = \begin{bmatrix} (z_1)^{n+p+1-k} \\ (z_2)^{n+p+1-k} \\ \vdots \\ (z_{p+1})^{n+p+1-k} \end{bmatrix}.$$

and $V^p_{k,j}$ be the $(p+1)\times(p+1)$ matrix obtained from V^p by replacing the j^{th} column of V^p by d^p_k.

Theorem 2.5. *Let* $\left(C^F_p\right)^n = \left[c^{F,p.n}_{k,j}\right]$, *then*

$$c^{F,p.n}_{k,j} = \frac{\det V^p_{k,j}}{\det V^p}$$

for $n \geq p$ *and* $p \geq 2$.

Proof. Since the eigenvalues of the complex-type Fibonacci p-matrix C^F_p are distinct, C^F_p is diagonalizable. Let $D_p = diag\left(z_1, z_2, \ldots, z_{p+1}\right)$, then we may write $C^F_p V^p = V^p D_p$. Since the matrix V^p is invertible, we obtain the equation $\left(V^p\right)^{-1} C^F_p V^p = D_p$. Then, C^F_p is similar to D^p; so, $\left(C^F_p\right)^n V^p = V^p \left(D_p\right)^n$. Hence we have the following linear system of equations:

$$\begin{cases} c^{F,p.n}_{k,1}\left(z_1\right)^p + c^{F,p.n}_{k,2}\left(z_1\right)^{p-1} + \cdots + c^{F,p.n}_{k,p+1} = \left(z_1\right)^{n+p+1-k} \\ c^{F,p.n}_{k,1}\left(z_2\right)^p + c^{F,p.n}_{k,2}\left(z_2\right)^{p--1} + \cdots + c^{F,p.n}_{k,p+1} = \left(z_2\right)^{n+p+1-k} \\ \vdots \\ c^{F,p.n}_{k,1}\left(z_{p+1}\right)^p + c^{F,p.n}_{k,2}\left(z_{p+1}\right)^{p-1} + \cdots + c^{F,p.n}_{k,p+1} = \left(z_{p+1}\right)^{n+p+1-k} . \end{cases}$$

Then for each $k, j = 1, 2, \ldots, p+1$, we conclude that

$$c^{F,p.n}_{k,j} = \frac{\det V^p_{k,j}}{\det V^p}.$$

□

From this result we immediately deduce:

Corollary 2.6. *Let* $F^*_{p,i}(n)$ *be the nth element of a complex-type Fibonacci p-number for* $n \geq p$ *such that* $p \geq 2$, *then*

$$F^*_{p,i}(n) = \frac{\det V^p_{p+1,1}}{\det V^p}$$

and

$$\begin{aligned} F^*_{p,i}(n) &= \frac{\det V^p_{1,2}}{i \cdot \det V^p} \\ &= \frac{\det V^p_{2,3}}{i \cdot \det V^p} \\ &= \cdots \\ &= \frac{\det V^p_{p,p+1}}{i \cdot \det V^p}. \end{aligned}$$

Let $U\left(u_1, u_2, \ldots, u_m\right)$ be a $m \times m$ companion matrix as follows:

$$U\left(u_1, u_2, \ldots, u_m\right) = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2.7. *(Chen and Louck [3]) The (k,j) entry $c_{k,j}^{(n)}(c_1,c_2,\ldots,c_v)$ in the matrix $C^n(c_1,c_2,\ldots,c_v)$ is given by the following formula:*

$$c_{k,j}^{(n)}(u_1,u_2,\ldots,u_m)=\sum_{(k_1,k_2,\ldots,k_m)}\frac{k_j+k_{j+1}+\cdots+k_m}{k_1+k_2+\cdots+k_m}\times\binom{k_1+\cdots+k_m}{k_1,\ldots,k_m}u_1^{k_1}\cdots u_m^{k_m} \tag{3}$$

where the summation is over nonnegative integers satisfying $k_1+2k_2+\cdots+mk_m=n-k+j$, $\binom{k_1+\cdots+k_m}{k_1,\ldots,k_m}=\frac{(k_1+\cdots+k_m)!}{k_1!\cdots k_m!}$ is a multinomial coefficient, and the coefficients in (3) are defined to be 1 if $n=k-j$.

Here we investigate a combinatorial representations for complex-type Fibonacci p-numbers.

Corollary 2.8. *i.*

$$F_{p,i}^{*}(n)=\sum_{(k_1,k_2,\ldots,k_{p+1})}\binom{k_1+\cdots+k_{p+1}}{k_1,\ldots,k_{p+1}}\left(i^{p+1}\right)^{k_1}(i)^{k_{p+1}}$$

where the summation is over nonnegative integers satisfying $k_1+2k_2+\cdots+(p+1)k_{p+1}=n-p$.

ii.

$$\begin{aligned}F_{p,i}^{*}(n)&=\frac{1}{i}\sum_{(k_1,k_2,\ldots,k_{p+1})}\frac{k_2+k_3+\cdots+k_{p+1}}{k_1+k_2+\cdots+k_{p+1}}\times\binom{k_1+\cdots+k_{p+1}}{k_1,\ldots,k_{p+1}}\left(i^{p+1}\right)^{k_1}(i)^{k_{p+1}}\\&=\frac{1}{i}\sum_{(k_1,k_2,\ldots,k_{p+1})}\frac{k_3+k_4+\cdots+k_{p+1}}{k_1+k_2+\cdots+k_{p+1}}\times\binom{k_1+\cdots+k_{p+1}}{k_1,\ldots,k_{p+1}}\left(i^{p+1}\right)^{k_1}(i)^{k_{p+1}}\\&=\cdots\\&=\frac{1}{i}\sum_{(k_1,k_2,\ldots,k_{p+1})}\frac{k_{p+1}}{k_1+k_2+\cdots+k_{p+1}}\times\binom{k_1+\cdots+k_{p+1}}{k_1,\ldots,k_{p+1}}\left(i^{p+1}\right)^{k_1}(i)^{k_{p+1}}\end{aligned}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(p+1)k_{p+1}=n+1$.

Proof. If we take $m=p+1$, $k=p+1$ and $j=1$, for case *i.* and $k=\alpha-1$, $j=\alpha$ such that $2\leq\alpha\leq p+1$, for case *ii.* in Theorem2.7, then the proof is immediately seen from (2). □

Now we will be concerned the exponential representation of the complex-type Fibonacci p-numbers. Using direct calculation, we obtained the generating function of $F_{p,i}^{*}(n)$ as shows:

$$g^{p,i}(x)=\frac{x^p}{1-i^{p+1}x-ix^{p+1}}.$$

Theorem 2.9. *An exponential representation of complex-type Fibonacci p-numbers is given as follows:*

$$g^{p,i}(x)=x^p\exp\left(\sum\frac{x^k}{k}\left(i^{p+1}+ix^p\right)^k\right).$$

Proof. Since

$$\begin{aligned}\ln g^{p,i}(x) &= \ln \frac{x^p}{1-i^{p+1}x-ix^{p+1}}\\ &= \ln x^p - \ln\left(1-i^{p+1}x-ix^{p+1}\right)\end{aligned}$$

and

$$\begin{aligned}-\ln\left(1-i^{p+1}x-ix^{p+1}\right) &= -\left[-x\left(i^{p+1}+ix^p\right)-\frac{1}{2}x^2\left(i^{p+1}+ix^p\right)^2-\cdots\right.\\ &\left.-\frac{1}{n}x^n\left(i^{p+1}+ix^p\right)^n-\cdots\right],\end{aligned}$$

It is clear that

$$\ln\frac{g^{p,i}(x)}{x^p} = \exp\left(\sum\frac{x^k}{k}\left(i^{p+1}+ix^p\right)^k\right).$$

Thus we have the conclusion. □

Let the sums of the complex-type Fibonacci p-numbers from 0 to n be denoted by S_n , that is,

$$S_n = \sum_{k=0}^{n} F^*_{p,i}(k)$$

and let M_p be the following $(p+2)\times(p+2)$ matrix:

$$M_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & C_p^F & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

If we use induction on n, then we obtain

$$(M_p)^n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ S_{n+p-1} & & & \\ S_{n+p-2} & & \left(C_p^F\right)^n & \\ \vdots & & & \\ S_{n-1} & & & \end{bmatrix}.$$

3. Conclusion

This paper has opened up properties of a new sequence, especially those which generalize well-known identities for the second-order sequences. In the last corollary 2.8, reference was made to combinatorial representations for these complex-type Fibonacci p-numbers. By way of conclusion for the interested reader, we note two avenues of types of further related future development by analogy with ordinary Fibonacci numbers extended for the arbitrary order situations.

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