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A note on LM_n - algebras of fractions

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Abstract. For an LM_n-algebra L and an $\wedge-$closed system $S \subseteq L$, in [2] I defined the LM_n-algebra of fractions of L relative to S (denoted by $L[S]$). Also, in [4] I defined the LM_n - algebra of localization of L relative to a topology $\mathcal{F}$ on L (denoted by $L_{\mathcal{F}}$).

The aim of this paper is to prove that $L[S]$ is an LM_n - algebra of localization of L relative to the topology $\mathcal{F}_S = \{I \in Idn(L) : I \cap S \cap C(L) \neq \emptyset\}$.

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The concept of multiplier for distributive lattices was defined by W. H. Cornish in [7]. J. Schmid used multipliers in order to give a non–standard construction of the maximal lattice of quotients for a distributive lattice (see [12]). A direct treatment of the lattices of quotients can be found in [13]. In [9], G. Georgescu exhibited the localization lattice $L_{\mathcal{F}}$ of a distributive lattice L with respect to a topology $\mathcal{F}$ on L mimicking the familiar construction for rings (see [11]) or monoids (see [14]). In [4] the author defines, for an LM_n-algebra L, the concept of LM_n - algebra of localization relative to a topology $\mathcal{F}$ on L (as in the case of lattices).

Two concepts of LM_n - algebra of fractions relative to an $\wedge-$ closed system was defined by the author in [2], [4].

1. Definitions and preliminaries

Let n be an integer, $n \geq 2$.

Definition 1.1. *([1])An $n-$valued Lukasiewicz$-$Moisil algebra (shortly, LM_n - algebra) is an algebra $\mathcal{L} = (L, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ of type $(2, 2, 1, 0, 0, \{1\}_{1\leq i\leq n-1})$ satisfying the following conditions:*

(1.1) $(L, \wedge, \vee, N, 0, 1)$ *is a De Morgan algebra,*

(1.2) $\varphi_1, ..., \varphi_{n-1} : L \to L$ *are bounded lattice morphisms such that for every* $x, y \in L$:

(1.2.1) $\varphi_i(x) \vee N\varphi_i(x) = 1$ *for every* $i = 1, ..., n-1$,

(1.2.2) $\varphi_i(x) \wedge N\varphi_i(x) = 0$ *for every* $i = 1, ..., n-1$,

(1.2.3) $\varphi_i\varphi_j(x) = \varphi_j(x)$ *for every* $i, j = 1, ..., n-1$,

(1.2.4) $\varphi_i(Nx) = N\varphi_j(x)$ *for every* $i, j = 1, ..., n-1$ *with* $i + j = n$,

(1.2.5) $\varphi_1(x) \leq \varphi_2(x) \leq ... \leq \varphi_{n-1}(x)$,

(1.2.6) *If* $\varphi_i(x) = \varphi_i(y)$ *for every* $i = 1, ..., n-1$, *then* $x = y$.

The relation (1.2.6) is called *the determination principle.* As consequences of *the determination principle* we obtain:

(1.2.7) If $x, y \in L$, then $x \leq y$ iff $\varphi_i(x) \leq \varphi_i(y)$ for all $i = 1, ..., n-1$,

(1.2.8.) $\varphi_1(x) \leq x \leq \varphi_{n-1}(x)$ for all $x \in L$.

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We denote an LM_n-algebra $\mathcal{L} = (L, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ by its universe L.

Remark 1.1. *The endomorphisms* $\{\varphi_i\}_{1\leq i\leq n-1}$ *are called chrysippian endomorphisms.*

Examples:

1. Let $L_n = \{0, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, 1\}$.We define $x\vee y = \max\{x, y\}, x\wedge y = \min\{x, y\}, Nx = 1 - x$ $(N(\frac{j}{n-1}) = \frac{n-1-j}{n-1})$ and $\varphi_i : L_n \to L_n, \varphi_i(\frac{j}{n-1}) = 0$ if $i + j < n$ and 1 if $i + j \geq n$, for $i, j = 1, ..., n - 1$.
   Then $(L_n, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ is an LM_n-algebra.
2. If $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra, then $(B, \wedge, \vee, ', 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ is an LM_n-algebra, where $\varphi_i = 1_B$ for every $1 \leq i \leq n - 1$.
3. Let $(B, \vee, \wedge, ', 0, 1)$ a Boolean algebra and $D(B) = \{(x_1, ..., x_{n-1}) \in B^{n-1} : x_1 \leq ... \leq x_{n-1}\}$. We define pointwise the infimum and the supremum, $N(x_1, ..., x_{n-1}) = (x'_{n-1}, ..., x'_1)$ and $\varphi_i(x_1, ..., x_{n-1}) = (x_i, ..., x_i)$ for all $i = 1, ..., n - 1$.

Then $(D(B), \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ is an LM_n-algebra.

In the rest of this paper, by L we denote an LM_n*-algebra.*

We denote by $C(L)$ the set of all *complemented elements* of L and we call it *the center of* L; it is easy to see that $(C(L), \vee, \wedge, N, 0, 1)$ is a Boolean algebra.

Lemma 1.1. *([1])Let L be an* LM_n*-algebra.The following are equivalent:*

(i) $e \in C(L)$,
(ii) there are $i \in \{1, ..., n - 1\}$ *and* $x \in L$ *such that* $e = \varphi_i(x)$,
(iii) there is $i \in \{1, ..., n - 1\}$ *such that* $e = \varphi_i(e)$,
(iv) $e = \varphi_i(e)$ *for every* $i = 1, ..., n - 1$,
(v) $\varphi_i(e) = \varphi_j(e)$ *for every* $i, j = 1, ..., n - 1$.

Remark 1.2. *If* $x \in L$, *then* $\varphi_i(x) \in C(L)$ *for every* $i = 1, ..., n - 1$.

Lemma 1.2. *([1])Let L be an* LM_n*-algebra.The following are equivalent:*

(i) $e \in C(L)$,
(ii) $N\ e \in C(L)$,
(iii) $e \wedge Ne = 0$,
(iv) $e \vee Ne = 1$.

Lemma 1.3. *If L is an* LM_n*-algebra, then for every* $x \in L$, $x \wedge \varphi_1(Nx) = 0$ *which is equivalent to* $x \wedge N\varphi_{n-1}(x) = 0$.

Proof. For every $x \in L$ we have $x \leq \varphi_{n-1}(x)$, so

$$x \wedge \varphi_1(Nx) = x \wedge N\varphi_{n-1}(x) \leq \varphi_{n-1}(x) \wedge N\varphi_{n-1}(x) = 0 \text{ (by(1.2.2))},$$

hence $x \wedge \varphi_1(Nx) = 0$. □

Theorem 1.1. *([1]) For an* LM_n*-algebra L (with* $0 \neq 1$*), the following are equivalent:*

(i) $C(L) = \{0, 1\}$,
(ii) L is a chain,
(iii) L is subdirectly irreducible.

Corollary 1.1. *([1]) Every chain which is an* LM_n*-algebra is finite.*

Definition 1.2. *([1])Let L and* L' *be* LM_n*-algebras. A function* $f : L \to L'$ *is a morphism of* LM_n*-algebras iff it satisfies the following conditions, for every* $x, y \in L$:

(i) $f(x \vee y) = f(x) \vee f(y)$,

(ii) $f(x \wedge y) = f(x) \wedge f(y)$,
(iii) $f(0) = 0, f(1) = 1$,
(iv) $f(\varphi_i(x)) = \varphi_i(f(x))$ *for every* $i = 1, ..., n-1$.

Remark 1.3. *It follows (from* 1.2.4 *and* 1.2.6*) that*

$$f(Nx) = Nf(x)$$

for every $x \in L$.

We denote by $\mathbf{LM}_n$ the *category of* LM_n*-algebras.*

Definition 1.3. *(*[1]*) Let* L *an* LM_n*-algebra. We say that a nonempty subset* $I \subseteq L$ *in an* $n - ideal$ *if* I *is an ideal of the lattice* L *and if* $x \in I$, *then* $\varphi_{n-1}(x) \in I$.

Remark 1.4. *From (1.2.5) we deduce that if* $I \subseteq L$ *is an n-ideal and* $x \in I$, *then* $\varphi_i(x) \in I$ *for every* $i \in \{1, ..., n-1\}$.

We denote by $Idn(L)$ the set of all $n - ideals$ of the LM_n- algebra L.

If $X \subseteq L$ is a nonempty set, we denote by $< X >$ the n-ideal generated by X. We have that (see [1]):

$$< X >= \{y \in L : \text{there exist } p \geq 1 \text{ and } x_1, ..., x_p \in X \text{ such that } y \leq \varphi_{n-1}(\bigvee_{i=1}^{p} x_i)\}.$$

In particular, for $a \in L$, $< a >= \{x \in L : x \leq \varphi_{n-1}(a)\}$ and if $a \in C(L)$, then $< a >= \{x \in L : x \leq a\} = (a]$.

Let I be an n-ideal and $x \in L$. We denote by $(I : x) = \{y \in L : x \wedge y \in I\}$.

Lemma 1.4. *The set* $(I : x)$ *is an n-ideal.*

Proof. Let $y_1, y_2 \in (I : x)$. Then $x \wedge y_1, x \wedge y_2 \in I$, hence $x \wedge (y_1 \vee y_2) = (x \wedge y_1) \vee (x \wedge y_2) \in I$, that is, $y_1 \vee y_2 \in (I : x)$.

If $y_1 \in (I : x)$ and $y_2 \leq y_1$, then $x \wedge y_1 \in I$ and $x \wedge y_2 \leq x \wedge y_1$, hence $x \wedge y_2 \in I$, that is, $y_2 \in (I : x)$.

If $y \in (I : x)$ then $x \wedge y \in I$, hence $\varphi_{n-1}(x) \wedge \varphi_{n-1}(y) = \varphi_{n-1}(x \wedge y) \in I$. But $x \wedge \varphi_{n-1}(y) \leq \varphi_{n-1}(x) \wedge \varphi_{n-1}(y)$, so $x \wedge \varphi_{n-1}(y) \in I$, that is, $\varphi_{n-1}(y) \in (I : x)$. □

Definition 1.4. *(*[1]*) A congruence on an* LM_n*-algebra* L *is an equivalence relation on* L *compatible with the operations* $\wedge, \vee, N, \varphi_i$, *for every* $i = 1, ..., n-1$.

Proposition 1.1. *(*[1]*)For an equivalence relation* ρ *on an* LM_n*-algebra* L, *the following conditions are equivalent:*
(1) ρ *is a congruence on* L,
(2) ρ *is compatible with* $\wedge, \vee, \varphi_i$, *for every* $i = 1, ..., n-1$.

2. Topologies on an $\mathbf{LM}_n$-algebra

Definition 2.1. *(*[9]*) A non-empty set* $\mathcal{F}$ *of n-ideals of* L *will be called a topology on* L *if the following properties hold:*
(T_1) *If* $I \in \mathcal{F}$, $x \in L$ *then* $(I : x) \in \mathcal{F}$,
(T_2) *If* $I_1, I_2 \in Idn(L)$, $I_2 \in \mathcal{F}$ *and* $(I_1 : x) \in \mathcal{F}$ *for all* $x \in I_2$, *then* $I_1 \in \mathcal{F}$.

Lemma 2.1. *(*[9]*) If* $\mathcal{F}$ *is a topology on* L, *then:*

(i) *If* $I_1 \in \mathcal{F}$ *and* I_2 *is an n-ideal such that* $I_1 \subseteq I_2$, *then* $I_2 \in \mathcal{F}$,
(ii) *If* I_1, $I_2 \in \mathcal{F}$, *then* $I_1 \cap I_2 \in \mathcal{F}$,

(iii) $(\mathcal{F} \cup \{\emptyset\}, L)$ *is a topological space.*

Definition 2.2. *([2]) A nonempty subset* $S \subseteq L$ *is called* $\wedge-$ *closed system in* L *if* $1 \in S$ *and* $x, y \in S$ *implies* $x \wedge y \in S$.

For any $\wedge-$ closed system S of L we set

$$\mathcal{F}_S = \{I \in Idn(L) : I \cap S \cap C(L) \neq \emptyset\}.$$

Proposition 2.1. $\mathcal{F}_S$ *is a topology on* L *.*

Proof. Let $I \in \mathcal{F}_S$ and $x \in L$. Then $I \cap S \cap C(L) \neq \emptyset$, so, because $I \subseteq (I : x)$, we have that $(I : x) \cap S \cap C(L) \neq \emptyset$, that is, $(I : x) \in \mathcal{F}_S$.

Let $I_1, I_2 \in Idn(L)$ such that $I_2 \in \mathcal{F}_S$ and $(I_1 : x) \in \mathcal{F}_S$ for every $x \in I_2$. But $I_2 \in \mathcal{F}_S$ implies that there exists $x_0 \in I_2 \cap S \cap C(L)$, hence $(I_1 : x_0) \in \mathcal{F}_S$, that is, $(I_1 : x_0) \cap S \cap C(L) \neq \emptyset$. Then, there exists $y_0 \in (I_1 : x_0) \cap S \cap C(L)$, so $x_0 \wedge y_0 \in I_1 \cap S \cap C(L)$, that is, $I_1 \in \mathcal{F}_S$. □

Definition 2.3. *The topology* $\mathcal{F}_S$ *is called the topology associated with the* $\wedge-$ *closed system* S.

3. $\mathcal{F}$-multipliers and localization LM$_n$-algebra

We recall the construction of LM_n*-algebra of localization of* L *relative to a topology* $\mathcal{F}$.

We consider the relation $\theta_{\mathcal{F}}$ of L

$(x, y) \in \theta_{\mathcal{F}}$ iff there exists $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I$.

Lemma 3.1. *([4])* $\theta_{\mathcal{F}}$ *is a congruence on* L.

We shall denote by $x/\theta_{\mathcal{F}}$ the congruence class of an element $x \in L$, by $L/\theta_{\mathcal{F}}$ the quotient MV-algebra and by

$$p_{\mathcal{F}} : L \to L/\theta_{\mathcal{F}}$$

the canonical morphism of LM_n-algebras. We denote the chrysippian endomorphisms of $L/\theta_{\mathcal{F}}$ by $\overline{\varphi}_i$ and we have $\overline{\varphi}_i(x/\theta_{\mathcal{F}}) = \varphi_i(x)/\theta_{\mathcal{F}}$ for every $x \in L$ $(i = 1, ..., n-1)$.

Proposition 3.1. *([4]) For* $a \in L, a/\theta_{\mathcal{F}} \in C(L/\theta_{\mathcal{F}})$ *iff there exists* $I \in \mathcal{F}$ *and* $i \in \{1, ..., n-1\}$ *such that* $e \wedge \varphi_i(a) = e \wedge a$ *for every* $e \in I$. *So, if* $a \in C(L)$, *then* $a/\theta_{\mathcal{F}} \in C(L/\theta_{\mathcal{F}})$.

Definition 3.1. *Let* $\mathcal{F}$ *be a topology on* L. *By an* $\mathcal{F}$ *-multiplier on* L *we mean a map* $f : I \to L/\theta_{\mathcal{F}}$, *where* $I \in \mathcal{F}$, *which verifies the following condition:*

(3.1) $f(e \wedge x) = e/\theta_{\mathcal{F}} \wedge f(x)$, *for every* $e \in L$ *and* $x \in I$.

By $dom(f) \in Idn(L)$ we denote the domain of f; if $dom(f) = L$, we called f *total*.

If $\mathcal{F} = \{L\}$, then $\theta_{\mathcal{F}}$ is the identity congruence of L and an $\mathcal{F}-$ multiplier is a total multiplier of L in the sense defined in [3].

The maps $\mathbf{0}, \mathbf{1} : L \to L/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x \in L$ are multipliers in the sense of Definition 3.1 (see [3] for the case of multipliers).

Also, for $a \in L$ and $I \in \mathcal{F}$, $f_a : I \to L/\theta_{\mathcal{F}}$ defined by $f_a(x) = a/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}$ for every $x \in I$, is an $\mathcal{F}-$ multiplier (see [3] for the case of multipliers of L). If $dom(f_a) = L$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$.

We shall denote by $M(I, L/\theta_{\mathcal{F}})$ the set of all the $\mathcal{F}-$ multipliers having the domain $I \in \mathcal{F}$ and

$$M(L/\theta_{\mathcal{F}}) = \underset{I\in\mathcal{F}}{\cup} M(I, L/\theta_{\mathcal{F}}).$$

If $I_1, I_2 \in \mathcal{F}$, $I_1 \subseteq I_2$, we have a canonical mapping

$$\varphi_{I_1,I_2} : M(I_2, L/\theta_{\mathcal{F}}) \to M(I_1, L/\theta_{\mathcal{F}})$$

defined by

$$\varphi_{I_1,I_2}(f) = f_{|I_1} \text{ for } f \in M(I_2, L/\theta_{\mathcal{F}}).$$

Let us consider the directed system of sets

$$\langle\{M(I, L/\theta_{\mathcal{F}})\}_{I\in\mathcal{F}}, \{\varphi_{I_1,I_2}\}_{I_1,I_2\in\mathcal{F},I_1\subseteq I_2}\rangle$$

and denote by $L_{\mathcal{F}}$ the inductive limit (in the category of sets):

$$L_{\mathcal{F}} = \varinjlim_{I\in\mathcal{F}} M(I, L/\theta_{\mathcal{F}}).$$

For any $\mathcal{F}-$ multiplier $f : I \to L/\theta_{\mathcal{F}}$ we shall denote by $\widehat{(I, f)}$ the equivalence class of f in $L_{\mathcal{F}}$.

Remark 3.1. *We recall that if $f_i : I_i \to L/\theta_{\mathcal{F}}$, $i = 1, 2$, are $\mathcal{F}$-multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $L_{\mathcal{F}}$) iff there exists $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.*

Definition 3.2. *If $I_1, I_2 \in Idn(L)$ and $f_i \in M(I_i, L/\theta_{\mathcal{F}}), i = 1, 2$ we define*

$$f_1 \wedge f_2, f_1 \vee f_2 : I_1 \cap I_2 \to L/\theta_{\mathcal{F}}$$

by

$$\begin{aligned}(f_1 \wedge f_2)(x) &= f_1(x) \wedge f_2(x),\\ (f_1 \vee f_2)(x) &= f_1(x) \vee f_2(x),\end{aligned}$$

for every $x \in I_1 \cap I_2$.

Let $\widehat{(I_1, f_1)} \wedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \wedge f_2)$ and $\widehat{(I_1, f_1)} \vee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2)$.

Definition 3.3. *If $I \in Idn(L)$ and $f \in M(I, L/\theta_{\mathcal{F}})$ we define $f^* : I \to L/\theta_{\mathcal{F}}$ by*

$$f^*(x) = x/\theta_{\mathcal{F}} \wedge N(f(\varphi_{n-1}(x)))$$

for any $x \in I$.

Let $\widehat{(I, f)}^* = \widehat{(I, f^*)}$.

Lemma 3.2. *([4]) If $I_1, I_2 \in Idn(L)$ and $f_i \in M(I_i, L/\theta_{\mathcal{F}}), i = 1, 2,$ then*

$$f_1 \wedge f_2, f_1 \vee f_2 \in M(I_1 \cap I_2, L/\theta_{\mathcal{F}}).$$

Remark 3.2. *([4]) For $x \in L$ we have $\mathbf{0}^*(x) = x/\theta_{\mathcal{F}} \wedge N(0/\theta_{\mathcal{F}}) = x/\theta_{\mathcal{F}} \wedge 1/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}}$, that is, $\mathbf{0}^* = \mathbf{1}$, and similarly $\mathbf{1}^* = \mathbf{0}$.*

Lemma 3.3. *([4]) If $I \in Idn(L)$ and $f \in M(I, L/\theta_{\mathcal{F}})$, then $f^* \in M(I, L/\theta_{\mathcal{F}})$.*

Definition 3.4. *For $I \in Idn(L)$ and $i = 1, ..., n - 1$ we define*

$$\tilde{\varphi}_i : M(I, L/\theta_{\mathcal{F}}) \to M(I, L/\theta_{\mathcal{F}})$$

by

$$\tilde{\varphi}_i(f)(x) = x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f(\varphi_{n-1}(x))) = x/\theta_{\mathcal{F}} \wedge \varphi_i(f(\varphi_{n-1}(x)))/\theta_{\mathcal{F}},$$

for every $f \in M(I, L/\theta_{\mathcal{F}})$ and $x \in I$.

Lemma 3.4. *([4]) If $I \in Idn(L)$, $f \in M(I, L/\theta_{\mathcal{F}})$, then $\tilde{\varphi}_i(f) \in M(I, L/\theta_{\mathcal{F}})$ for all $i = 1, ..., n-1$.*

Let $\varphi_i^{\mathcal{F}} : L_{\mathcal{F}} \to L_{\mathcal{F}}$ defined by $\varphi_i^{\mathcal{F}}(\widehat{(I,f)}) = (\widehat{I, \tilde{\varphi}_i(f)}), i = 1, ..., n-1$.

Proposition 3.2. *([4]) $(L_{\mathcal{F}}, \wedge, \vee, ^*, \mathbf{0}, \mathbf{1}, \varphi_1^{\mathcal{F}}, ..., \varphi_{n-1}^{\mathcal{F}})$ is an LM_n-algebra.*

Definition 3.5. *The LM_n-algebra $L_{\mathcal{F}}$ will be called the localization $LM_n - algebra$ of L with respect to the topology $\mathcal{F}$.*

We recall now the construction *of LM_n-algebra of fractions relative to S* from [2]. If $S \subseteq L$ is an $\wedge-$closed system of L, we consider the following congruence on L :

$$(x, y) \in \theta_S \text{ iff there exists } e \in S \cap C(L) \text{ such that } x \wedge e = y \wedge e.$$

The quotient LM_n*-algebra* $L[S] = L/\theta_S$ is called in [2] the LM_n*-algebra of fractions of L relative to the $\wedge-$closed system S*. For $x \in L$ by x/S denotes the congruence class of x relative to θ_S.

Theorem 3.1. *If $\mathcal{F}_S$ is the topology associated with the $\wedge-$closed system $S \subseteq L$, then the LM_n-algebra $L_{\mathcal{F}_S}$ is isomorphic in $\boldsymbol{LM}_n$ with $L[S]$.*

Proof. Let $x, y \in L$. If $(x, y) \in \theta_{\mathcal{F}_S}$ then there exists $I \in \mathcal{F}_S$ (hence $I \cap S \cap C(L) \neq \emptyset$) such that $x \wedge e = y \wedge e$ for any $e \in I$. Since $I \cap S \cap C(L) \neq \emptyset$ there exists $e_0 \in I \cap S \cap C(L)$ such that $x \wedge e_0 = y \wedge e_0$, that is, $(x, y) \in \theta_S$. So, $\theta_{\mathcal{F}_S} \subseteq \theta_S$.

If $(x, y) \in \theta_S$, there exists $e_0 \in S \cap C(L)$ such that $x \wedge e_0 = y \wedge e_0$. If we set $I_0 =< e_0 >= \{x \in L : x \leq e_0\} = (e_0]$, then $I_0 \in Idn(L)$. Since $e_0 \in I_0$, we have that $e_0 \in I_0 \cap S \cap C(L)$, hence $I_0 \cap S \cap C(L) \neq \emptyset$, that is, $I_0 \in \mathcal{F}_S$. For every $e \in I_0$, $e \leq e_0$, then $e = e \wedge e_0$, so $x \wedge e = x \wedge (e \wedge e_0) = (x \wedge e_0) \wedge e = (y \wedge e_0) \wedge e = y \wedge (e \wedge e_0) = y \wedge e$, hence $(x, y) \in \theta_{\mathcal{F}_S}$, that is, $\theta_S \subseteq \theta_{\mathcal{F}_S}$. Therefore $\theta_{\mathcal{F}_S} = \theta_S$.

Then $L/\theta_{\mathcal{F}_S} = L/\theta_S = L[S]$, hence an $\mathcal{F}_S-$multiplier can be considered in this case (see Definition 3.1) as a mapping $f : I \to L[S]$ $(I \in \mathcal{F}_S)$ having the property $f(e \wedge x) = e/S \wedge f(x)$ for every $x \in I$ and $e \in L$.

We recall (see [2], Remark 2.1) that if $s \in S \cap C(L)$, then $s/S = \mathbf{1}$.

If $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)} \in L_{\mathcal{F}_S} = \varinjlim_{I \in \mathcal{F}} M(I, L[S])$ and $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ then there exists $I \in \mathcal{F}_S$ such that $I \subseteq I_1 \cap I_2$ and $f_{1|I} = f_{2|I}$. Since $I, I_1, I_2 \in \mathcal{F}_S$, there exists $e \in I \cap S \cap C(L), e_1 \in I_1 \cap S \cap C(L)$ and $e_2 \in I_2 \cap S \cap C(L)$. We shall prove that $f_1(e_1) = f_2(e_2)$. If we denote $e' = e \wedge e_1 \wedge e_2$, then $e' \in I \cap S \cap C(L)$ and $e' \leq e_1, e_2$. Since $e_1 \wedge e' = e_2 \wedge e' \in I$ then $f_1(e_1 \wedge e') = f_2(e_2 \wedge e')$, hence $f_1(e_1) \wedge e'/S = f_2(e_2) \wedge e'/S$, so $f_1(e_1) \wedge \mathbf{1} = f_2(e_2) \wedge \mathbf{1}$, that is, $f_1(e_1) = f_2(e_2)$. In a similar way, we can show that $f_1(s_1) = f_2(s_2)$ for any $s_1, s_2 \in I \cap S \cap C(L)$.

In accordance with these considerations we can define the mapping:

$$\alpha : L_{\mathcal{F}_S} = \varinjlim_{I \in \mathcal{F}} M(I, L[S]) \to L[S]$$

by putting

$$\alpha(\widehat{(I, f)}) = f(s) \in L[S],$$

where $s \in I \cap S \cap C(L)$.

We have $\alpha(\mathbf{0}) = \alpha(\widehat{(L, \mathbf{0})}) = \mathbf{0}(s) = 0/S = \mathbf{0}$ and $\alpha(\mathbf{1}) = \alpha(\widehat{(L, \mathbf{1})}) = \mathbf{1}(s) = s/S = \mathbf{1}$ for every $s \in S \cap C(L)$.

Also, for every $\widehat{(I_i, f_i)} \in L_{\mathcal{F}_S}, i = 1, 2$ we have:

$$\begin{aligned} \alpha(\widehat{(I_1, f_1)} \wedge \widehat{(I_2, f_2)}) &= \alpha((\widehat{I_1 \cap I_2, f_1} \wedge f_2)) = (f_1 \wedge f_2)(s) = f_1(s) \wedge f_2(s) \\ &= \alpha(\widehat{(I_1, f_1)}) \wedge \alpha(\widehat{(I_2, f_2)}), \end{aligned}$$

and

$$\begin{aligned} \alpha(\widehat{(I_1, f_1)} \vee \widehat{(I_2, f_2)}) &= \alpha((\widehat{I_1 \cap I_2, f_1} \vee f_2)) = (f_1 \vee f_2)(s) = f_1(s) \vee f_2(s) \\ &= \alpha(\widehat{(I_1, f_1)}) \vee \alpha(\widehat{(I_2, f_2)}) \end{aligned}$$

with $s \in I_1 \cap I_2 \cap S \cap C(L)$.

If $\widehat{(I, f)} \in L_{\mathcal{F}_S}$ and $s \in I \cap S \cap C(L)$, for every $i = 1, ..., n-1$ we have

$$\begin{aligned} \alpha(\varphi_i^{\mathcal{F}}(\widehat{(I, f)})) &= \alpha((\widehat{I, \widetilde{\varphi}_i(f)})) = \widetilde{\varphi}_i(f)(s) = (s/S) \wedge \overline{\varphi}_i(f(\varphi_{n-1}(s))) = \mathbf{1} \wedge \bar{\varphi}_i(f(s)) \\ &= \bar{\varphi}_i(f(s)) = \bar{\varphi}_i(\alpha(\widehat{(I, f)})). \end{aligned}$$

Therefore, this mapping is a morphism of LM_n-algebras.

We shall prove that α is injective and surjective. To prove the injectivity of α let $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)} \in L_{\mathcal{F}_S}$ such that $\alpha(\widehat{(I_1, f_1)}) = \alpha(\widehat{(I_2, f_2)})$. Then for any $e_1 \in I_1 \cap S \cap C(L)$, $e_2 \in I_2 \cap S \cap C(L)$ we have $f_1(e_1) = f_2(e_2)$. If $f_1(e_1) = x/S$ and $f_2(e_2) = y/S$ with $x, y \in L$, since $x/S = y/S$, there exists $e \in S \cap C(L)$ such that $x \wedge e = y \wedge e$.

If we consider $e' = e \wedge e_1 \wedge e_2 \in I_1 \cap I_2 \cap S \cap C(L)$, we have $x \wedge e' = y \wedge e'$ and $e' \leq e_1, e_2$. It follows that $f_1(e') = f_1(e' \wedge e_1) = f_1(e_1) \wedge (e'/S) = x/S \wedge \mathbf{1} = x/S = y/S = y/S \wedge \mathbf{1} = f_2(e_2) \wedge (e'/S) = f_2(e_2 \wedge e') = f_2(e')$. If we denote $I =< e' >= (e']$(since $e' \in C(L)$), then we obtain that $I \in \mathcal{F}_S$, $I \subseteq I_1 \cap I_2$ and $f_{1|I} = f_{2|I}$, hence $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$, that is, α is injective.

To prove the surjectivity of α, let $a/S \in L[S]$ and $\bar{f}_a : L \to L[S]$ defined by $\bar{f}_a(x) = a/S \wedge x/S = (a \wedge x)/S$ for every $x \in L$.

It is easy to see that $\bar{f}_a$ is an $\mathcal{F}_S-$multiplier and $\alpha(\widehat{(L, \bar{f}_a)}) = \bar{f}_a(s) = (a \wedge s)/S = a/S \wedge s/S = a/S \wedge \mathbf{1} = a/S$, where $s \in S \cap C(L)$. So α is surjective.

Therefore, α is an isomorphism of LM_n-algebras. □

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