Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 31, 2004, Pages 28–34
ISSN: 1223-6934

Fuzzy L-partitions generated by fuzzy L-sets

Lavinia Ciungu

Abstract. A fuzzy subset of a set M is qualified by means of its membership function $\varphi : M \to \{0,1\}$. If the interval $[0,1]$ is substituted by a lattice L, the fuzzy subset is called L-fuzzy subset. In [1] there are studied the partitions of fuzzy subsets and there are presented some methods to find this kind of partitions. In this paper we generalize these results for the partitions of L-fuzzy subsets.

2000 Mathematics Subject Classification. Primary: 94D05. Secondary: 03B52.
Key words and phrases. fuzzy partition, fuzzy logic.

1. Preliminaries

Let M be a set and L a lattice. A function $\varphi : M \to L$ is called fuzzy L-subset of M. We will consider that a boolean subset of a set is given by its characteristic function $\varphi : M \to \{0,1\}$. Consider $(L, \wedge, \vee, 0, 1)$ a complete distributive lattice and $(-)' : (L, \wedge, \vee, 0, 1) \to (L, \vee, \wedge, 1, 0)$ an involutive antiisomorphism such that $(x')' = x, (x \wedge y)' = x' \vee y', (x \vee y)' = x' \wedge y'$ for any x, y in L. Suppose that there exists an element x_0 in L such that $x_0 = x_0'$ and for any x in L $x \leq x_0$ or $x \geq x_0$.

Example 1.1. *Let $(L, \vee, \wedge, 0, 1)$ and $(L', \wedge, \vee, 0' = 1, 1' = 0)$ be the dual lattice. Denote by $\alpha : L \to L'$ the application taking any $x \in L$ to x from L'. Put $K = L \bigsqcup L'$ and define $<$ on K by $x < y$ iff ($x, y \in L$, $x < y$) or ($x, y \in L'$, $x <_{L'} y$) or ($x \in L$, $y \in L'$). For $x, y \in K$ define $x \vee y$ to be $x \vee_L y$ if $x, y \in L$, $x \vee_{L'} y$ if $x, y \in L'$ or $y(x)$ if $x \in L$, $y \in L'(y \in L, x \in L')$. The $\wedge$ of K is defined similarly. For $x \in K$ put $x' = \alpha(x)$ if $x \in L$ and $x' = \alpha^{-1}(x)$ if $x \in L'$. Then $(K, \vee, \wedge, 0, 1')$ becomes a lattice with an involutive antiisomorphism which has the above described properties. Now consider the equivalence relation on K that identifies $1 \in L$ and $0' \in L'$, that is $x \sim y$ iff $x = y$ or $\{x, y\} = \{1, 0'\}$. Then the factor set $K/\sim$ becomes a lattice with the inherited $<, \vee, \wedge$ and $(-)'$. In particular, we may consider the lattice $([0,1], \vee = sup, \wedge = inf)$ with $x' = 1 - x$, $\forall\, x$ and $x_0 = \frac{1}{2}$.*

Theorem 1.1. *The following are equivalent for a lattice K:*
(i) There is an element $x_0 \in K$ such that $x \leq x_0$ or $x_0 \leq x$, $\forall x \in K$ and $x_0 = x_0'$.
(ii) K is isomorphic to a lattice of the kind described in Example 1.1.

Received: October 29, 2004.

Proof. (ii)$\Rightarrow$(i) It is easy to see that in $K/\sim$ from Example 1.1 the element x_0 =the class of 1 and $0'$ has $x_0 = x_0'$ and $x \leq x_0$ or $x \geq x_0$, $\forall\, x \in K/\sim$.
(i)$\Rightarrow$(ii) Denote $(L = [0, x_0], \vee, \wedge, 0, x_0)$, $(L' = [x_0, 1], \wedge, \vee, 1, x_0)$ and $f : L \to L'$ the application given by $f(x) = x'$. It is easy to see that f is well defined and it is an isomorphism of latices, showing that L' is the dual of L and that K is isomorphic to the lattice K associated to L and L' described in the example above. □

Definition 1.1. *Let M be a set and φ a fuzzy subset of M. A family of fuzzy subsets (ν_i) that satisfies the following conditions:*
1) $\nu_i \leq \nu_j'$, $\forall i \neq j$
2) $\varphi \wedge (\sup\limits_i \nu_i)' \leq (\varphi \wedge (\sup\limits_i \nu_i)')'$
3) $\sup\limits_i \nu_i \leq \varphi$
is called a partition of φ.

In case $\varphi = 1_M$, we say that (ν_i) is a *complete partition.*

Remark 1.1. *In the case of boolean sets, the complement morphism satisfies $x \wedge x' = 0$; $x \vee x' = 1$. In the case of nonboolean sets these properties are not necessary verified, but the assumption of the existence of such element x_0 proves to be useful in some situations, as the above stated conditions for the boolean case are replaced by the weaker ones: $x \wedge x' \leq x_0$, $x \vee x' \geq x_0$, as we can see by a simple computation.*
Looking at the case of boolean subsets of sets, we have the following restatements:
1')$A \subseteq CB$ is equivalent to $A \cap B = \emptyset$, where CA is the complement of A.
2')$A \subseteq CA$ is equivalent to $A = \emptyset$.
3')If $A, B \subseteq X$ then $A \supseteq CB$ is equivalent to $A \cup B = X$
These conditions suggest we can use $x \leq x'$ (or equivalently, $x \leq x_0$) instead of $x = 0$ and $x \leq y'$ instead of $x \wedge y = 0$ (or $x \geq y'$ instead of $x \vee y = 1$) in the case of nonboolean sets. Moreover, any subset A of a set X is given by a function $\varphi : X \to \{0, 1\}$, so it is in particular a fuzzy L subset of X, where $L = (\{0, \frac{1}{2}, 1\}, <)$ and $x_0 = \frac{1}{2}$, allowing us to obtain the classical case of subsets of a set (and then of classical partitions) as a particular case of this theory.

Theorem 1.2. *The family (ν_i) is a complete partition if and only if it satisfies the following conditions:*
1) $\nu_i \leq \nu_j'$, $\forall i \neq j$
2) $\sup\limits_i \nu_i \geq (\sup\limits_i \nu_i)'$

Proof. Condition 1) is verified by hypothesis.
Condition 3) is obvious for $\varphi = 1_L$
For condition 2) we substitute φ with 1_L and we obtain:
$(\sup\limits_i \nu_i)' \leq ((\sup\limits_i \nu_i)')' = \sup\limits_i \nu_i$
which is true by hypothesis. □

Remark 1.2. *For some results we will work with countable families of fuzzy subsets, that is, sequences of fuzzy subsets, as we need this assumption for inductive constructions.*

Theorem 1.3. *Let* $(\varphi_n)_n$ *be a sequence of fuzzy subsets of* M. *Assuming that* $\sup_n \varphi_n > (\sup_n \varphi_n)'$, *the sequence* (ν_n) *defined bellow is a complete partition:*

$$\nu_n = \begin{cases} \varphi_1, & \text{if } n = 1 \\ \varphi_n \wedge (\sup_{k<n} \nu_k)', & \text{if } n > 1 \end{cases}$$

Proof. $\bullet \nu_n \leq \nu'_m, \forall n \neq m$:
It is enough to prove this for $n < m$, as for $m > n$, $\nu_n \leq \nu'_m \Leftrightarrow (\nu_n)' \geq (\nu'_m)' \Leftrightarrow \nu_m \leq \nu'_n$, where $m > n$. We have $\nu'_m = (\varphi_m \wedge (\bigvee_{k<m} \nu_k)')' = \varphi'_m \vee (\bigvee_{k<m} \nu_k)'' = \varphi'_m \vee \nu_1 \vee \nu_2 \vee \cdots \vee \nu_{m-1} \geq \nu_n$ because $n \in \{1, 2, \ldots, m-1\}$.
$\bullet (\bigvee_n \nu_n)' \leq \bigvee_n \nu_n$:
We have that $x' \leq x \Leftrightarrow x \geq x_0$, so we will prove that $\bigvee_n \nu_n \geq x_0$. Suppose that $\bigvee_n \nu_n < x_0$. Then we have that $\bigvee_{k<n} \nu_k \leq x_0 \, \forall n$, which is equivalent to $(\bigvee_{k<n} \nu_k)' \geq x_0 \, \forall n$. Then $\nu_n = \varphi_n \wedge (\bigvee_{k<n} \nu_k)' \geq \varphi_n \wedge x_0 \, \forall n$ and then

$$\begin{aligned} \bigvee_n \nu_n &\geq \bigvee_n (\varphi_n \wedge x_0) \\ &= (\bigvee_n \varphi_n) \wedge x_0 \geq x_0 \wedge x_0 = x_0 \ \text{(because} \, (\bigvee_n \varphi_n)' \leq \bigvee_n \varphi_n \Rightarrow \bigvee_n \varphi_n \geq x_0) \end{aligned}$$

which constitutes a contradiction with $\bigvee_n \nu_n < x_0$. □

2. Some properties of partitions

Theorem 2.1. *Let* (φ_1, φ_2) *be a pair of fuzzy subsets such that* $\varphi_2 \leq \varphi_1$ *and* (ν_i) *a partition of* φ_1. *Then the family* $(\varphi_2 \wedge \nu_i)$ *is a partition of* φ_2.

Proof. In order for $(\varphi_2 \wedge \nu_i)$ to be a partition of φ_2, it has to fulfill the conditions from Definition 1.1
From $\varphi_2 \wedge \nu_1 \leq \nu_1 \leq \nu'_k \leq (\varphi_2 \wedge \nu_k)'$, $\forall k \neq 1$ it follows that $(\varphi_2 \wedge \nu_i)$ satisfies condition 1).
Using the properties of the above defined lattice L and of its elements we prove condition 2):

$$\begin{aligned} \varphi_2 \wedge (\sup_i (\varphi_2 \wedge \nu_i))' &= \varphi_2 \wedge (\varphi_2 \wedge \sup_i \nu_i)' = \varphi_2 \wedge (\varphi'_2 \vee (\sup_i \nu_i)') \\ &= (\varphi_2 \wedge \varphi'_2) \vee (\varphi_2 \wedge (\sup_i \nu_i)') \leq x_0 \vee (\varphi_1 \wedge (\sup_i \nu_i)') \leq x_0 \vee x_0 = x_0 \end{aligned}$$

Remark 1.1 implies that

$$\varphi_2 \wedge (\sup_i \varphi_2 \wedge \nu_i)' \leq (\varphi_2 \wedge (\sup_i \varphi_2 \wedge \nu_i)')'$$

which leads to the fact that condition 2) is satisfied.
Condition 3) is verified because:

$$\sup_i \varphi_2 \wedge \nu_i \leq \varphi_2$$

We conclude that $(\varphi_2 \wedge \nu_i)$ is a partition of φ_2 and the theorem is proved. □

Theorem 2.2. *If the family (ν_i) is a complete partition, then the family $(\varphi \wedge \nu_i)$ is a partition of φ, for any fuzzy subset φ.*

Proof. It follows easily from Theorem 2.1 for $\varphi = 1_L$. □

Theorem 2.3. *Let (ν'_k) be a family that satisfies the conditions:*
1) $\nu'_i \leq \nu_j$, $\forall i \neq j$
2) $\sup_k \nu'_k \leq \varphi$
3) $\sup_i \nu_i \leq \sup_k \nu'_k$, where (ν_i) is a partition of φ.
Then (ν'_k) is also a partition of φ.

Proof. It is very clear from the hypothesis that (ν'_k) already satisfies conditions 1) and 3). It only remains to show condition 2).
We have

$$\varphi \wedge (\sup_k \nu'_k)' \leq \varphi \wedge (\sup_n \nu_n)'$$

Consequently (ν'_k) is a partition of φ. □

3. Examples of partitions

The following lemma is similar to a result in [2].

Lemma 3.1. *Let (φ_n) be an increasing sequence of fuzzy subsets and let (ν_n) be defined as follows:*

$$\nu_n = \begin{cases} \varphi_1, & \text{if } n = 1 \\ \varphi_n \wedge (\varphi_{n-1})', & \text{if } n > 1 \end{cases} \tag{1}$$

Then (ν_n) satisfies the condition:

$$(\sup_{k \leq n} \nu_k)' = \sup_{k<n}(\varphi_k \wedge \varphi'_k) \vee \varphi'_n,\ \forall n > 2$$

Lemma 3.2. *Let (φ_n) be an increasing sequence of fuzzy subsets such that:*

$$\varphi \wedge (\sup_n \varphi_n)' \leq (\varphi \wedge (\sup_n \varphi_n)')'$$

Then the sequence defined in (1) satisfies the following condition:

$$\varphi \wedge (\sup_n \nu_n)' \leq (\varphi \wedge (\sup_n \nu_n)')'$$

Proof. By basic computation we have

$$\begin{aligned}
\varphi \wedge (\sup_n \nu_n)' &= \varphi \wedge (\sup_n \sup_{k\leq n} \nu_k)' = \varphi \wedge \inf_n (\sup_{k\leq n} \nu_k)' \\
&= \varphi \wedge \inf_{n\geq 2} (\sup_{k\leq n} \nu_k)' = \varphi \wedge \inf_{n\geq 2} (\sup_{k<n} \varphi_k \wedge \varphi_k' \vee \varphi_n') \\
&\leq \varphi \wedge \inf_{n\geq 2} (x_0 \vee \varphi_n') = (\varphi \wedge x_0) \vee (\varphi \wedge \inf_{n\geq 2} \varphi_n') \\
&= (\varphi \wedge x_0) \vee (\varphi \wedge (\sup_{n\geq 2} \varphi_n)') = (\varphi \wedge x_0) \vee (\varphi \wedge (\sup_n \varphi_n)')
\end{aligned}$$

Using Remark 1.1 repeatedly we have

$$\varphi \wedge (\sup_n \varphi_n)' \leq (\varphi \wedge (\sup_n \varphi_n)')'$$

which is equivalent to $\quad \varphi \wedge (\sup_n \varphi_n)' \leq x_0$.

We obtain $\varphi \wedge (\sup_n \nu_n)' \leq (\varphi \wedge x_0) \vee x_0 = x_0$ and then

$$\varphi \wedge (\sup_n \nu_n)' \leq (\varphi \wedge (\sup_n \nu_n)')'$$

and this ends the proof of Lemma 3.2 □

Lemma 3.3. *Let* (φ_n) *be a sequence such that:*

$$\varphi \wedge (\sup_n \varphi_n)' \leq (\varphi \wedge (\sup_n \varphi_n)')'.$$

Then the sequence

$$\nu_n = \begin{cases} \varphi_1, & \text{if } n{=}1 \\ \varphi_n \wedge (\sup_{k<n} \varphi_k), & \text{if } n > 1 \end{cases} \tag{2}$$

satisfies the condition:

$$\varphi \wedge (\sup_n \nu_n)' \leq (\varphi \wedge (\sup_n \nu_n)')'.$$

Proof. We define

$$\psi_n = \begin{cases} 0_L, & \text{if n=1} \\ \sup_{k<n} \varphi_k, & \text{if } n > 1 \end{cases} \tag{3}$$

We can easily see that ψ_n is increasing and

$$\varphi \wedge (\sup_n \psi_n)' \leq (\varphi \wedge (\sup_n \psi_n)')'.$$

Using the fact that $\nu_n = \varphi_n \wedge \psi_n'$ and Lemma 3.2 we obtain:

$$\begin{aligned}
\varphi \wedge (\sup_n \nu_n)' &= \varphi \wedge (\sup_n (\sup_{m\leq n} \nu_m)) = \varphi \wedge (\sup_n (\sup_{m\leq n} (\varphi_m \wedge \psi_m')))' \\
&\leq \varphi \wedge (\sup_n (\sup_{m\leq n} (\varphi_m \wedge \psi_n')))' = \varphi \wedge (\sup_n (\psi_{n+1} \wedge \psi_n'))' \leq x_0.
\end{aligned}$$

Thus (φ_n) satisfies the required condition. □

Theorem 3.1. *Suppose that the sequence* (φ_n) *satisfies the properties:*
1)$\varphi \wedge (\sup_n \varphi_n)' \leq (\varphi \wedge (\sup_n \varphi'_n))'$
2)$\sup_n \varphi_n \leq \varphi$.
Then the sequence (ν_n) *defined in (2) is a partition of* φ.

Proof. It is straightforward to check condition 1) from Definition 1.1 because:

$$\nu_n = \varphi_n \wedge (\sup_{l<n} \varphi_l)' \leq (\sup_{l<n} \varphi_l)' \leq \varphi'_k \leq \nu'_k.$$

Condition 2) is satisfied from the hypothesis.
Condition 3) also follows immediately from the fact that
$\nu_n \leq \varphi_n$ and then $\sup_n \nu_n \leq \sup_n \varphi_n \leq \varphi$.
In conclusion, for any n, (ν_n) is a partition of φ. □

In case when ν_n is increasing, the sequences (φ_n) defined in (1) and in (2) respectively coincide and this makes the proof of the following result:

Theorem 3.2. *If the sequence* (φ_n) *is increasing and satisfies:*
1) $\varphi \wedge (\sup_n \varphi_n)' \leq (\varphi \wedge (\sup_n \varphi_n)')'$
2) $\sup_n \varphi_n \leq \varphi$
then the sequence (ν_n) *given by (1) is a partition of* φ.

Theorem 3.3. *Let* (φ_n) *be a sequence that satisfies the following conditions:*
1) $\varphi \wedge (\sup_n \varphi_n)' \leq (\varphi \wedge (\sup_n \varphi_n)')'$
2) $\sup_n \varphi_n \leq \varphi$.
Then the sequence (ν_n) *defined bellow is a partition of* φ*:*

$$\nu_n = \begin{cases} \varphi_1, & \text{if } n{=}1 \\ \varphi_n \wedge (\sup_{k<n} \nu_k)', & \text{if } n > 1 \end{cases} \tag{4}$$

Proof. Using Theorem 2.2 , it follows that (ν_n) satisfies condition 1) of Definition 1.1.
By Theorems 2.3 and 3.2 and the following computation, we obtain condition 2):

$$\sup_n \nu_n = \varphi_1 \vee \sup_{n\geq 2}(\varphi_n \wedge (\sup_{k<n} \nu_k)') \geq \varphi_1 \vee \sup_{n\geq 2}(\varphi_n \wedge (\sup_{k<n} \varphi_k)')$$

Finally, condition 3) follows as $\nu_n \leq \varphi_n$, which implies $\sup_n \nu_n \leq \varphi$. □

Theorem 3.4. *Let* (φ_k), $k = \overline{1,n}$, $n \geq 2$ *be an increasing sequence. Then* $(\varphi_{k+1} \wedge \varphi'_k)$, $k = \overline{1, n-1}$ *is a partition of* $\varphi_n \wedge \varphi'_1$.

Proof. Theorem 3.2 implies that $\varphi_{k+1} \wedge \varphi'_k \leq (\varphi_{j+1} \wedge \varphi'_j)'$, thus condition 1) from Definition 1.1 is proved.

Using Lemma 3.1 we obtain 2):

$$\begin{aligned}\varphi_n \wedge \varphi'_1 \wedge (\sup_k(\varphi_{k+1} \wedge \varphi'_k))' &= \varphi_n \wedge \varphi'_1 \wedge (\sup_{k<n} \varphi_{k+1} \wedge \varphi'_k)' \\ &= \varphi_n \wedge (\varphi_1 \vee \sup_{k<n}(\varphi_{k+1} \wedge \varphi'_k))' = \varphi_n \wedge (\sup_{k<n}(\varphi_n \wedge \varphi'_k) \vee \varphi'_n \\ &\leq \varphi_n \wedge (x_0 \vee \varphi'_n) = (\varphi_n \wedge x_0) \vee (\varphi_n \wedge \varphi'_n) \leq x_0\end{aligned}$$

For 3) it is enough to see that $\sup_n(\varphi_{k+1} \wedge \varphi'_k) = \sup_{k<n}(\varphi_{k+1} \wedge \varphi'_k) \leq \varphi_n \wedge \varphi'_1$. □

References

[1] Krzysztof Piasecki, Fuzzy partitions of sets, *Busefal J.*, **52**, 146-150 (1986).
[2] Krzysztof Piasecki, Probability of fuzzy events defined as denumerable additivity measure, *Fuzzy sets and systems*, **17**, 271-284 (1985).
[3] Esko Turunen, *Mathematics behind logic*, Springer-Verlag, 1999.

(Lavinia Ciungu) Department of Mathematics, University of Polytechnics, Splaiul Independentei 313, Bucharest, Romania
E-mail address: lavinia_ciungu@math.pub.ro