Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 49(2), 2022, Pages 270–290, DOI: 10.52846/ami.v49i2.1554
ISSN: 1223-6934

# Finite uniform approximation of zero-sum games defined on a product of staircase-function continuous spaces

Vadim Romanuke

Abstract. A method of finite approximation of zero-sum games defined on a product of staircase-function continuous spaces is presented. The method consists in uniformly sampling the player's pure strategy value set, solving "smaller" matrix games, each defined on a subinterval where the pure strategy value is constant, and stacking their solutions if they are consistent. The stack of the "smaller" matrix game solutions is an approximate solution to the initial staircase game. The (weak) consistency, equivalent to the approximate solution acceptability, is studied by how much the payoff and optimal situation change as the sampling density minimally increases. The consistency is decomposed into the payoff, optimal strategy support cardinality, optimal strategy sampling density, and support probability consistency. The most important parts are the payoff consistency and optimal strategy support cardinality (weak) consistency. However, it is practically reasonable to consider a relaxed payoff consistency, by which the game optimal value change in an appropriate approximation may grow at most by $\epsilon$ as the sampling density minimally increases. The weak consistency itself is a relaxation to the consistency, where the minimal decrement of the sampling density is ignored.

*2010 Mathematics Subject Classification.* 91A05, 91A50, 18F20, 65D99, 41A99.
*Key words and phrases.* Game theory, Payoff functional, Staircase-function strategy, Matrix game, Approximate solution consistency.

## 1. Introduction

Zero-sum (or antagonistic) games are models of processes where two sides referred to as persons or players interact in struggling for optimizing the to-be-paid-or-pay events. A possible action of the player is called its (pure) strategy. The strategy can be as a simple (point) action whose duration is usually short, as well as a process consisting of an order of simple actions (in particular, see [15, 6, 1]).

Whichever the pure strategy form is, the simplest zero-sum game is a matrix game. Any matrix game has optimal solutions (one, a finite number, or continuum) [16, 17, 9, 8]. A more complicated zero-sum game is the antagonistic game, in which the game kernel (payoff function) is a surface defined on a finite-dimensional compact Euclidean subspace. A simple example of the subspace is a unit square [20, 10, 17]. Even if the surface does not have a discontinuity, the optimal solution is not always determinable as opposed to matrix games [16, 17, 9, 11]. Moreover, zero-sum games defined on open (or half-open) subspaces (e. g., open square) may not have an optimal solution at all [16]. Therefore, rendering a zero-sum game to a matrix one is a crucial task in zero-sum-game modeling.

Received October 2, 2021. Accepted January 9, 2022.

Surely, a far more complicated case is a zero-sum game, in which the player's strategy is a function (e. g., of time). In such games, the payoff kernel is a functional [21, 14]. This functional maps every pair of functions (pure strategies of the players) into a real value. When each of the players possesses a finite set of such function-strategies, the game might be rendered down to a matrix game [12, 13]. Such rendering is impossible if the set of the player's strategies is either infinite or, all the more, continuous.

The paper proceeds as follows. The study motivation is briefly presented in Section 2. Section 3 presents the objective and tasks to be fulfilled. A zero-sum game with strategies as functions is formalized in Section 4. Staircase-function strategies are introduced in Section 5. Section 6 describes how the pure strategy value axis is sampled. The question of whether an approximate solution can be accepted or not is answered in Section 7. An example is presented in Section 8. In the last two sections the study is discussed and concluded.

## 2. Motivation to zero-sum game finite approximation

To render a zero-sum game with strategies as functions down to a matrix game, there are two fundamental conditions. First, a time interval, on which the pure strategy is defined, should be broken into a set of subintervals, on which the strategy could be approximately considered constant. Second, the set of possible values of the player's function-strategy should be finite.

The first fundamental condition is the time sampling condition. It can be done according to the rules of a system to be game-modeled, where the administrator (manager, controller, etc.) does always define (or constrain) the form of the strategies players will use [19, 3, 22]. Moreover, any process is interpreted static on a sufficiently short time span. Henceforward, the time sampling condition is considered fulfilled.

The second fundamental condition cannot be imposed for no particular reason. However, the number of factual actions of the players (in any game) is always finite. While the players may use strategies of whichever form they want, the number of their actions has a natural limit [7, 6, 18]. Thus, the set of function-strategies used in a zero-sum game is finite anyway (unless the game is everlasting). Therefore, any non-everlasting zero-sum game is played as if it is a matrix game.

The continuous game approximation is based on sampling the payoff kernel. Alternatively, this is fulfilled also by sampling the sets of players' pure strategies. A method of approximating continuous zero-sum games is known from [12]. An approximate solution is considered acceptable if it changes minimally by changing the sampling step minimally. Obviously, this method cannot be applied straightforwardly to a zero-sum game with staircase-function strategies. However, a part of the game considered on a time subinterval where the players' strategies are constant is directly approximated by the method. This is an aspiration and basis to develop a method of approximately solving zero-sum games defined on a product of staircase-function continuous spaces.

## 3. Objective and tasks to be fulfilled

Issued from the impossibility of solving zero-sum games defined on a product of staircase-function continuous spaces, the objective is to develop a method of finite approximation of such games. For achieving the objective, the following tasks are to be fulfilled:

1. To formalize a zero-sum game, in which the players' strategies are functions of time.
2. To formalize a zero-sum game, in which the players' strategies are staircase functions. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time. Herein, the time can be thought of as it is discrete.
3. To state conditions of sampling the set of possible values of the player's pure strategy so that the game be defined on a product of staircase-function finite spaces.
4. To state conditions of the appropriate finite approximation.
5. To discuss applicability and significance of the method for the game theory, whereupon an unbiased conclusion is to be made.

## 4. A zero-sum game with strategies as functions

A zero-sum game, in which the player's pure strategy is a function of time, can be defined as follows. Let each of the players use time-varying strategies defined almost everywhere on interval $[t_1;\ t_2]$ by $t_2 > t_1$. Denote a strategy of the first player by $x(t)$ and a strategy of the second player by $y(t)$. These functions are presumed to be bounded, i. e.

$$a_{\min} \leqslant x(t) \leqslant a_{\max} \ \text{ by } \ a_{\min} < a_{\max} \tag{1}$$

and

$$b_{\min} \leqslant y(t) \leqslant b_{\max} \ \text{ by } \ b_{\min} < b_{\max}. \tag{2}$$

Besides, the square of the function-strategy is presumed to be Lebesgue-integrable. Thus, the sets of the players' pure strategies are

$$X = \{x(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ a_{\min} \leqslant x(t) \leqslant a_{\max} \text{ by } a_{\min} < a_{\max}\} \subset \\ \subset \mathbb{L}_2[t_1;\ t_2] \tag{3}$$

and

$$Y = \{y(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ b_{\min} \leqslant y(t) \leqslant b_{\max} \text{ by } b_{\min} < b_{\max}\} \subset \\ \subset \mathbb{L}_2[t_1;\ t_2], \tag{4}$$

respectively. Each of sets (3) and (4) is a rectangular functional space of functions of time.

The first player's payoff in situation $\{x(t),\ y(t)\}$ is $K(x(t),\ y(t))$. The payoff is presumed to be an integral functional:

$$K(x(t),\ y(t)) = \int\limits_{[t_1;\ t_2]} f(x(t),\ y(t),\ t)\, d\mu(t) \tag{5}$$

with a function

$$f(x(t),\ y(t),\ t) \tag{6}$$

of $x(t)$ and $y(t)$ explicitly including $t$. Therefore, the continuous zero-sum game

$$\left\langle \{X,\ Y\},\ K(x(t),\ y(t)) \right\rangle \tag{7}$$

is defined on product

$$X \times Y \subset \mathbb{L}_2[t_1;\ t_2] \times \mathbb{L}_2[t_1;\ t_2] \tag{8}$$

of rectangular functional spaces (3) and (4) of players' pure strategies. In practical reality, zero-sum game (7) with strategies as functions is presumed to be played discretely through time interval $[t_1;\ t_2]$. Then a function-strategy becomes staircase.

## 5. A zero-sum game with staircase-function strategies

Denote by $N$ the number of subintervals at which the player's pure strategy is constant, where $N \in \mathbb{N} \backslash \{1\}$. Then the player's pure strategy is a staircase function having only $N$ different values. If $\left\{\tau^{(i)}\right\}_{i=1}^{N-1}$ are time points at which the staircase-function strategy changes its value, where

$$t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(N-1)} < \tau^{(N)} = t_2, \tag{9}$$

then

$$x_i = x\left(\tau^{(i)}\right) \text{ by } i = \overline{0,\ N} \tag{10}$$

are the values of the first player's strategy, and

$$y_i = y\left(\tau^{(i)}\right) \text{ by } i = \overline{0,\ N} \tag{11}$$

are the values of the second player's strategy. Points $\left\{\tau^{(i)}\right\}_{i=0}^{N}$ are not necessarily to be equidistant.

The staircase-function strategies are right-continuous:

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(i)} + \varepsilon\right) = x\left(\tau^{(i)}\right) \tag{12}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)} + \varepsilon\right) = y\left(\tau^{(i)}\right) \tag{13}$$

for $i = \overline{1,\ N-1}$, whereas

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(i)} - \varepsilon\right) \neq x\left(\tau^{(i)}\right) \tag{14}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)} - \varepsilon\right) \neq y\left(\tau^{(i)}\right) \tag{15}$$

for $i = \overline{1,\ N-1}$. As an exception,

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(N)} - \varepsilon\right) = x\left(\tau^{(N)}\right) \tag{16}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(N)} - \varepsilon\right) = y\left(\tau^{(N)}\right), \tag{17}$$

so

$$x_{N-1} = x_N \text{ and } y_{N-1} = y_N.$$

Constant values (10) and (11) by (9) mean that game (7) can be thought of as it is a succession of $N$ continuous zero-sum games

$$\left\langle \left\{ [a_{\min};\ a_{\max}],\ [b_{\min};\ b_{\max}] \right\},\ K(\alpha_i,\ \beta_i) \right\rangle \tag{18}$$

defined on product

$$[a_{\min};\ a_{\max}] \times [b_{\min};\ b_{\max}]$$

by

$$\begin{gathered} \alpha_i = x(t) \in [a_{\min};\ a_{\max}] \ \text{ and } \ \beta_i = y(t) \in [b_{\min};\ b_{\max}] \\ \forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \ \text{ for } \ i = \overline{1,\ N-1} \ \text{ and } \ \forall\, t \in \left[\tau^{(N-1)};\ \tau^{(N)}\right], \end{gathered} \tag{19}$$

where the factual payoff in situation $\{\alpha_i,\ \beta_i\}$ is

$$K(\alpha_i,\ \beta_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) \ \ \forall\, i = \overline{1,\ N-1} \tag{20}$$

and

$$K(\alpha_N,\ \beta_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t). \tag{21}$$

So, let such game (7) be called staircase. A pure-strategy situation in staircase game (7) is a succession of $N$ situations $\left\{\{\alpha_i,\ \beta_i\}\right\}_{i=1}^{N}$ in games (18).

**Theorem 1.** In a pure-strategy situation of staircase game (7), represented as a succession of $N$ games (18), functional (5) is re-written as a subinterval-wise sum

$$\begin{gathered} K(x(t),\ y(t)) = \sum_{i=1}^{N} K(\alpha_i,\ \beta_i) = \\ = \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t). \end{gathered} \tag{22}$$

*Proof.* Situation $\{\alpha_i,\ \beta_i\}$ is tied to half-subinterval $\left[\tau^{(i-1)};\ \tau^{(i)}\right)$ by $i = \overline{1,\ N-1}$ and to subinterval $\left[\tau^{(N-1)};\ \tau^{(N)}\right]$ by $i = N$. Function (6) in this situation is some function of time $t$. Denote this function by $\psi_i(t)$. For situation $\{\alpha_i,\ \beta_i\}$ function

$$\psi_i(t) = 0 \ \ \forall\, t \notin \left[\tau^{(i-1)};\ \tau^{(i)}\right), \tag{23}$$

and for situation $\{\alpha_N,\ \beta_N\}$ function

$$\psi_N(t) = 0 \ \ \forall\, t \notin \left[\tau^{(N-1)};\ \tau^{(N)}\right]. \tag{24}$$

Therefore,

$$f(x(t),\ y(t),\ t) = \sum_{i=1}^{N} \psi_i(t) \tag{25}$$

in a pure-strategy situation $\{x(t),\ y(t)\}$ of staircase game (7), by using (23) and (24). Consequently,

$$K(x(t),\ y(t)) = \int\limits_{[t_1;\ t_2]} f(x(t),\ y(t),\ t)\, d\mu(t) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} \psi_i(t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} \psi_N(t)\, d\mu(t) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) =$$

$$= \sum_{i=1}^{N} K(\alpha_i,\ \beta_i) \tag{26}$$

in a pure-strategy situation $\{x(t),\ y(t)\}$ of staircase game (7). $\square$

In fact, Theorem 1 does not provide a method of solving the staircase game. Nevertheless, it provides a fundamental decomposition of the game. This decomposition allows considering each game (18) separately.

## 6. Sampling along the pure strategy value axis

In game (18), the first player has its set $[a_{\min};\ a_{\max}]$ of pure strategies, and the second player's pure strategy set is $[b_{\min};\ b_{\max}]$. Let these sets be sampled uniformly with a step determined by an integer $S$, $S \in \mathbb{N}$. So,

$$A(S) = \left\{a^{(s)}\right\}_{s=1}^{S+1} = \left\{a_{\min} + \frac{s-1}{S} \cdot (a_{\max} - a_{\min})\right\}_{s=1}^{S+1} \subset [a_{\min};\ a_{\max}] \tag{27}$$

is a sampled pure strategy set of the first player, and

$$B(S) = \left\{b^{(s)}\right\}_{s=1}^{S+1} = \left\{b_{\min} + \frac{s-1}{S} \cdot (b_{\max} - b_{\min})\right\}_{s=1}^{S+1} \subset [b_{\min};\ b_{\max}] \tag{28}$$

is a sampled pure strategy set of the second player. The roughest sampling is by $S = 1$, when

$$A(1) = \left\{a^{(1)},\ a^{(2)}\right\} = \{a_{\min},\ a_{\max}\}$$

and

$$B(1) = \left\{b^{(1)},\ b^{(2)}\right\} = \{b_{\min},\ b_{\max}\}.$$

With the sampling by (27) and (28), the succession of $N$ continuous games (18) by (9)–(17) and (19) becomes a succession of $N$ matrix $(S+1) \times (S+1)$ games

$$\left\langle \left\{ \left\{a^{(m)}\right\}_{m=1}^{S+1},\ \left\{b^{(j)}\right\}_{j=1}^{S+1} \right\},\ \mathbf{K}_i(S) \right\rangle \tag{29}$$

with payoff matrices

$$\mathbf{K}_i(S) = [k_{imj}(S)]_{(S+1)\times(S+1)} \tag{30}$$

whose elements are

$$k_{imj}(S) = \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m)},\, b^{(j)},\, t\right) d\mu(t) \;\text{ for }\; i = \overline{1,\, N-1} \tag{31}$$

and

$$k_{Nmj}(S) = \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m)},\, b^{(j)},\, t\right) d\mu(t). \tag{32}$$

So, if integer $S$ for game (7) by (19) is somehow selected, the staircase game is represented as a succession of $N$ matrix $(S+1)\times(S+1)$ games.

By sampling (27) and (28) the staircase game becomes defined on product $A(S)\times B(S)$, which becomes a product of staircase-function finite spaces by running through all $i = \overline{1,\, N}$. Thus the game might be rendered to a matrix game in order to obtain a staircase solution. However, there is a much easier way to solve a finite staircase game.

**Theorem 2.** Game (7) on product (8) by conditions (1)–(5) made a finite staircase game by (19) and sampling (27), (28) is always solved as a stack of successive optimal solutions of $N$ matrix games (29) by (30)–(32).

*Proof.* A matrix game is always solved, either in pure or mixed strategies. Denote by

$$\mathbf{P}_i(S) = \left[p_i^{(m)}(S)\right]_{1\times(S+1)}$$

and

$$\mathbf{Q}_i(S) = \left[q_i^{(j)}(S)\right]_{1\times(S+1)}$$

the mixed strategies of the first and second players, respectively, in matrix game (29). The respective sets of mixed strategies of the first and second players are

$$\mathcal{P} = \left\{\mathbf{P}_i(S) \in \mathbb{R}^{S+1} : p_i^{(m)}(S) \geqslant 0,\; \sum_{m=1}^{S+1} p_i^{(m)}(S) = 1\right\} \tag{33}$$

and

$$\mathcal{Q} = \left\{\mathbf{Q}_i(S) \in \mathbb{R}^{S+1} : q_i^{(j)}(S) \geqslant 0,\; \sum_{j=1}^{S+1} q_i^{(j)}(S) = 1\right\}, \tag{34}$$

so

$$\mathbf{P}_i(S) \in \mathcal{P},\;\; \mathbf{Q}_i(S) \in \mathcal{Q},$$

and $\{\mathbf{P}_i(S),\, \mathbf{Q}_i(S)\}$ is a situation in this game. Let

$$\left\{\{\mathbf{P}_i^*(S),\, \mathbf{Q}_i^*(S)\}\right\}_{i=1}^{N} = \left\{\left\{\left[p_i^{(m)*}(S)\right]_{1\times(S+1)},\, \left[q_i^{(j)*}(S)\right]_{1\times(S+1)}\right\}\right\}_{i=1}^{N} \tag{35}$$

be a set of optimal solutions of $N$ games (29) by (30)–(32). Then

$$\max_{\mathbf{P}_i(S)\in\mathcal{P}} \min_{\mathbf{Q}_i(S)\in\mathcal{Q}} \mathbf{P}_i(S)\cdot\mathbf{K}_i(S)\cdot[\mathbf{Q}_i(S)]^{\mathrm{T}} =$$
$$= \max_{\mathbf{P}_i(S)\in\mathcal{P}} \min_{\mathbf{Q}_i(S)\in\mathcal{Q}} \sum_{m=1}^{S+1}\sum_{j=1}^{S+1} k_{imj}(S)\, p_i^{(m)}(S)\, q_i^{(j)}(S) =$$
$$= \max_{\mathbf{P}_i(S)\in\mathcal{P}} \min_{\mathbf{Q}_i(S)\in\mathcal{Q}} \sum_{m=1}^{S+1}\sum_{j=1}^{S+1} p_i^{(m)}(S)\, q_i^{(j)}(S) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu(t) =$$
$$= \sum_{m=1}^{S+1}\sum_{j=1}^{S+1} p_i^{(m)*}(S)\, q_i^{(j)*}(S) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu(t) =$$
$$= \mathbf{P}_i^*(S)\cdot\mathbf{K}_i(S)\cdot[\mathbf{Q}_i^*(S)]^{\mathrm{T}} = v_i^*(S) =$$
$$= \min_{\mathbf{Q}_i(S)\in\mathcal{Q}} \max_{\mathbf{P}_i(S)\in\mathcal{P}} \sum_{m=1}^{S+1}\sum_{j=1}^{S+1} p_i^{(m)}(S)\, q_i^{(j)}(S) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu(t) =$$
$$= \min_{\mathbf{Q}_i(S)\in\mathcal{Q}} \max_{\mathbf{P}_i(S)\in\mathcal{P}} \sum_{m=1}^{S+1}\sum_{j=1}^{S+1} k_{imj}(S)\, p_i^{(m)}(S)\, q_i^{(j)}(S) =$$
$$= \min_{\mathbf{Q}_i(S)\in\mathcal{Q}} \max_{\mathbf{P}_i(S)\in\mathcal{P}} \mathbf{P}_i(S)\cdot\mathbf{K}_i(S)\cdot[\mathbf{Q}_i(S)]^{\mathrm{T}} \quad \forall\, i = \overline{1,\ N-1} \tag{36}$$

and

$$\max_{\mathbf{P}_N(S)\in\mathcal{P}} \min_{\mathbf{Q}_N(S)\in\mathcal{Q}} \mathbf{P}_N(S)\cdot\mathbf{K}_N(S)\cdot[\mathbf{Q}_N(S)]^{\mathrm{T}} =$$
$$= \max_{\mathbf{P}_N(S)\in\mathcal{P}} \min_{\mathbf{Q}_N(S)\in\mathcal{Q}} \sum_{m=1}^{S+1}\sum_{j=1}^{S+1} k_{Nmj}(S)\, p_N^{(m)}(S)\, q_N^{(j)}(S) =$$
$$= \max_{\mathbf{P}_N(S)\in\mathcal{P}} \min_{\mathbf{Q}_N(S)\in\mathcal{Q}} \sum_{m=1}^{S+1}\sum_{j=1}^{S+1} p_N^{(m)}(S)\, q_N^{(j)}(S) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu(t) =$$
$$= \sum_{m=1}^{S+1}\sum_{j=1}^{S+1} p_N^{(m)*}(S)\, q_N^{(j)*}(S) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu(t) =$$
$$= \mathbf{P}_N^*(S)\cdot\mathbf{K}_N(S)\cdot[\mathbf{Q}_N^*(S)]^{\mathrm{T}} = v_N^*(S) =$$
$$= \min_{\mathbf{Q}_N(S)\in\mathcal{Q}} \max_{\mathbf{P}_N(S)\in\mathcal{P}} \sum_{m=1}^{S+1}\sum_{j=1}^{S+1} p_N^{(m)}(S)\, q_N^{(j)}(S) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu(t) =$$
$$= \min_{\mathbf{Q}_N(S)\in\mathcal{Q}} \max_{\mathbf{P}_N(S)\in\mathcal{P}} \sum_{m=1}^{S+1}\sum_{j=1}^{S+1} k_{Nmj}(S)\, p_N^{(m)}(S)\, q_N^{(j)}(S) =$$
$$= \min_{\mathbf{Q}_N(S)\in\mathcal{Q}} \max_{\mathbf{P}_N(S)\in\mathcal{P}} \mathbf{P}_N(S)\cdot\mathbf{K}_N(S)\cdot[\mathbf{Q}_N(S)]^{\mathrm{T}}. \tag{37}$$

Using Theorem 1 allows to conclude that

$$\max_{x(t)\in X} \min_{y(t)\in Y} K\left(x\left(t\right),\ y\left(t\right)\right) =$$

$$= \sum_{i=1}^{N-1} \left( \max_{\mathbf{P}_i(S)\in\mathcal{P}} \min_{\mathbf{Q}_i(S)\in\mathcal{Q}} \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} p_i^{(m)}\left(S\right) q_i^{(j)}\left(S\right) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \max_{\mathbf{P}_N(S)\in\mathcal{P}} \min_{\mathbf{Q}_N(S)\in\mathcal{Q}} \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} p_N^{(m)}\left(S\right) q_N^{(j)}\left(S\right) \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{i=1}^{N-1} \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} p_i^{(m)*}\left(S\right) q_i^{(j)*}\left(S\right) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu\left(t\right) +$$

$$+ \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} p_N^{(m)*}\left(S\right) q_N^{(j)*}\left(S\right) \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{i=1}^{N-1} \mathbf{P}_i^*\left(S\right) \cdot \mathbf{K}_i\left(S\right) \cdot \left[\mathbf{Q}_i^*\left(S\right)\right]^{\mathrm{T}} + \mathbf{P}_N^*\left(S\right) \cdot \mathbf{K}_N\left(S\right) \cdot \left[\mathbf{Q}_N^*\left(S\right)\right]^{\mathrm{T}} =$$

$$= \sum_{i=1}^{N-1} v_i^*\left(S\right) + v_N^*\left(S\right) =$$

$$= \sum_{i=1}^{N-1} \left( \min_{\mathbf{Q}_i(S)\in\mathcal{Q}} \max_{\mathbf{P}_i(S)\in\mathcal{P}} \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} p_i^{(m)}\left(S\right) q_i^{(j)}\left(S\right) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \min_{\mathbf{Q}_N(S)\in\mathcal{Q}} \max_{\mathbf{P}_N(S)\in\mathcal{P}} \sum_{m=1}^{S+1} \sum_{q=1}^{S+1} p_N^{(m)}\left(S\right) q_N^{(j)}\left(S\right) \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m)},\ b^{(j)},\ t\right) d\mu\left(t\right) =$$

$$= \min_{y(t)\in Y} \max_{x(t)\in X} K\left(x\left(t\right),\ y\left(t\right)\right) \tag{38}$$

and, therefore, the stack of successive solutions (35) is an optimal solution in game (7) by (19) sampled by (27), (28). □

Obviously, the solutions of the $(S+1)\times(S+1)$ matrix games are independent. They are solved in parallel, without caring of the succession. If all $N$ matrix games are solved in pure strategies, then stacking their solutions is fulfilled trivially. When there is at least an equilibrium in mixed strategies for a subinterval, the stacking is fulfilled as well implying that the resulting pure-mixed-strategy solution of staircase game (7) is realized successively, subinterval by subinterval, spending the same amount of time to implement both pure strategy and mixed strategy solutions [6, 1].

## 7. Approximate solution consistency

The conditions of the appropriate finite approximation are stated by using the known method of obtaining the approximate solution of continuous antagonistic games on unit multidimensional cube with uniform sampling [13]. There are five items of the conditions. The requirement of the smooth sampling of the payoff kernel is inapplicable here [14].

An easy-to-find condition of the finite approximation appropriateness is the game optimal value change:

$$|v_i^*(S) - v_i^*(S+1)| \leqslant |v_i^*(S-1) - v_i^*(S)| \quad \text{for} \;\; i = \overline{1, \; N}. \tag{39}$$

Condition (39) means that, as the sampling density minimally increases, the game optimal value change in an appropriate approximation should not grow.

The next condition is the change of the optimal strategy support cardinality. Denote the supports of the optimal strategies of the players by

$$\operatorname{supp} \mathbf{P}_i^*(S) = \{m_u\}_{u=1}^{U_i(S)} \subset \{m\}_{m=1}^{S+1} \tag{40}$$

by the respective support probabilities

$$\left\{ p_i^{(m_u)*}(S) \right\}_{u=1}^{U_i(S)} \tag{41}$$

and

$$\operatorname{supp} \mathbf{Q}_i^*(S) = \{j_w\}_{w=1}^{W_i(S)} \subset \{j\}_{j=1}^{S+1} \tag{42}$$

by the respective support probabilities

$$\left\{ q_i^{(j_w)*}(S) \right\}_{w=1}^{W_i(S)}. \tag{43}$$

Then inequalities

$$U_i(S+1) \geqslant U_i(S) \quad \text{for} \;\; i = \overline{1, \; N} \tag{44}$$

and

$$W_i(S+1) \geqslant W_i(S) \quad \text{for} \;\; i = \overline{1, \; N} \tag{45}$$

require that, by minimally increasing the sampling density, the cardinalities of the supports not decrease.

The third item is the support index distance. As the sampling density minimally increases, the maximal gap between the support indices should not increase. Let $m_u(S)$ and $j_w(S)$ be the respective support indices corresponding to integer $S$ on a subinterval by (19). Then inequalities

$$\max_{u=\overline{1, \; U_i(S+1)-1}} [m_{u+1}(S+1) - m_u(S+1)] \leqslant \\ \leqslant \max_{u=\overline{1, \; U_i(S)-1}} [m_{u+1}(S) - m_u(S)] \quad \text{for} \;\; i = \overline{1, \; N} \tag{46}$$

and

$$\max_{w=\overline{1, \; W_i(S+1)-1}} [j_{w+1}(S+1) - j_w(S+1)] \leqslant \\ \leqslant \max_{w=\overline{1, \; W_i(S)-1}} [j_{w+1}(S) - j_w(S)] \quad \text{for} \;\; i = \overline{1, \; N} \tag{47}$$

are required.

Denote by $h_1(i;\ m,\ S)$ a polyline whose vertices are probabilities $\left\{p_i^{(m)*}(S)\right\}_{m=1}^{S+1}$, and denote by $h_2(i;\ j,\ S)$ a polyline whose vertices are probabilities $\left\{q_i^{(j)*}(S)\right\}_{j=1}^{S+1}$. Then, by minimally increasing the sampling density, the "neighboring" polylines should not be farther from each other, i. e.

$$\max_{[0;\ 1]} |h_1(i;\ m,\ S) - h_1(i;\ m,\ S+1)| \leqslant$$
$$\leqslant \max_{[0;\ 1]} |h_1(i;\ m,\ S-1) - h_1(i;\ m,\ S)| \quad \text{for} \quad i = \overline{1,\ N} \tag{48}$$

and

$$\max_{[0;\ 1]} |h_2(i;\ j,\ S) - h_2(i;\ j,\ S+1)| \leqslant$$
$$\leqslant \max_{[0;\ 1]} |h_2(i;\ j,\ S-1) - h_2(i;\ j,\ S)| \quad \text{for} \quad i = \overline{1,\ N} \tag{49}$$

along with

$$\|h_1(i;\ m,\ S) - h_1(i;\ m,\ S+1)\| \leqslant$$
$$\leqslant \|h_1(i;\ m,\ S-1) - h_1(i;\ m,\ S)\| \quad \text{in} \quad \mathbb{L}_2[0;\ 1] \quad \text{for} \quad i = \overline{1,\ N} \tag{50}$$

and

$$\|h_2(i;\ j,\ S) - h_2(i;\ j,\ S+1)\| \leqslant$$
$$\leqslant \|h_2(i;\ j,\ S-1) - h_2(i;\ j,\ S)\| \quad \text{in} \quad \mathbb{L}_2[0;\ 1] \quad \text{for} \quad i = \overline{1,\ N}. \tag{51}$$

If inequalities (39), (44)–(51) hold for some $i$, then matrix game (29), assigned to the subinterval between $\tau^{(i-1)}$ and $\tau^{(i)}$, has a weakly consistent approximate solution to the corresponding continuous game (18) by (19). On this basis, the weak consistency of an approximate solution to a staircase game (7) is formulated.

**Definition 1.** The stack of successive solutions (35) is called a weakly $S$-consistent approximate solution of game (7) on product (8) by conditions (1)–(5) and (19) if inequalities (39), (44)–(51) hold.

Obviously, requirements (44)–(47) can be supplemented (strengthened) by considering a minimal decrement of the sampling density. Then inequalities

$$U_i(S) \geqslant U_i(S-1) \quad \text{for} \quad i = \overline{1,\ N} \tag{52}$$

and

$$W_i(S) \geqslant W_i(S-1) \quad \text{for} \quad i = \overline{1,\ N} \tag{53}$$

and

$$\max_{u=\overline{1,\ U_i(S)-1}} [m_{u+1}(S) - m_u(S)] \leqslant$$
$$\leqslant \max_{u=\overline{1,\ U_i(S-1)-1}} [m_{u+1}(S-1) - m_u(S-1)] \quad \text{for} \quad i = \overline{1,\ N} \tag{54}$$

and

$$\max_{w=\overline{1,\ W_i(S)-1}} [j_{w+1}(S) - j_w(S)] \leqslant$$
$$\leqslant \max_{w=\overline{1,\ W_i(S-1)-1}} [j_{w+1}(S-1) - j_w(S-1)] \quad \text{for} \quad i = \overline{1,\ N} \tag{55}$$

are required.

**Definition 2.** The stack of successive solutions (35) is called an $S$-consistent approximate solution of game (7) on product (8) by conditions (1)–(5) and (19) if inequalities (39), (44)–(55) hold.

The approximate solution consistency clearly proposes a better approximation than the weak consistency. To ascertain whether the stack of successive solutions (35) is weakly consistent or not, the three bunches of $N$ matrix games (29) should be solved, where the sampling density is defined by integers $S-1$, $S$, $S+1$. Nevertheless, the consistency meant by some sampling density integer $S$ does not guarantee that both the players will select such sampling density. Moreover, it is hard to find a continuous zero-sum game, for which a consistent approximate solution could be determined at appropriately small $S$. Hence, the following definitions are more relevant for the approximation.

**Definition 3.** An approximate solution (35) is called payoff-$S$-consistent if inequalities (39) hold.

**Definition 4.** An approximate solution (35) is called weakly support-cardinality-$S$-consistent if inequalities (44) and (45) hold.

**Definition 5.** An approximate solution (35) is called support-cardinality-$S$-consistent if inequalities (44), (45), (52), (53) hold.

**Definition 6.** An approximate solution (35) is called weakly sampling-density-$S$-consistent if inequalities (46) and (47) hold.

**Definition 7.** An approximate solution (35) is called sampling-density-$S$-consistent if inequalities (46), (47), (54), (55) hold.

**Definition 8.** An approximate solution (35) is called probability-$S$-consistent if inequalities (48)–(51) hold.

The weak consistency notion by Definition 1 may be thought of as it is decomposed by Definitions 3, 4, 6, 8. Thus, the consistency notion by Definition 2 is decomposed into Definitions 3, 5, 7, 8. Even if an approximate solution is not weakly consistent, it may be, e. g., payoff-consistent. This can be sufficient to accept it as an appropriate approximate solution.

## 8. A visual exemplification

To visually exemplify how a zero-sum staircase game is approximated by using the approximate solution consistency, consider a case in which $t \in [0;\ 4.5\pi]$, the set of pure strategies of the first player is

$$X = \{x(t),\ t \in [0;\ 4.5\pi]:\ 10 \leqslant x(t) \leqslant 14\} \subset \mathbb{L}_2[0;\ 4.5\pi] \tag{56}$$

and the set of pure strategies of the second player is

$$Y = \{y(t),\ t \in [0;\ 4.5\pi]:\ 23 \leqslant y(t) \leqslant 25\} \subset \mathbb{L}_2[0;\ 4.5\pi]. \tag{57}$$

The payoff functional is

$$K(x(t),\ y(t)) = \int\limits_{[0;\ 4.5\pi]} \left[\sin(0.4xt)\sin\left(0.3yt - \frac{\pi}{7}\right)\right] d\mu(t). \tag{58}$$

Each of the players is allowed to change its pure strategy value at time points

$$\left\{\tau^{(i)}\right\}_{i=1}^{8} = \{0.5\pi i\}_{i=1}^{8}.$$

So, the player possesses 9-subinterval staircase function-strategies defined on interval $[0;\ 4.5\pi]$. Hence, the zero-sum staircase game is represented as a succession of 9 zero-sum games (18). For these games, with the sampling by (27) and (28), the pure strategy set of the first player is

$$A(S) = \left\{a^{(s)}\right\}_{s=1}^{S+1} = \left\{10 + \frac{4s-4}{S}\right\}_{s=1}^{S+1} \subset [10;\ 14] \tag{59}$$

and

$$B(S) = \left\{b^{(s)}\right\}_{s=1}^{S+1} = \left\{23 + \frac{2s-2}{S}\right\}_{s=1}^{S+1} \subset [23;\ 25] \tag{60}$$

is the pure strategy set of the second player, making thus 9 matrix games

$$\left\langle \left\{ \left\{a^{(m)}\right\}_{m=1}^{S+1},\ \left\{b^{(j)}\right\}_{j=1}^{S+1} \right\},\ \mathbf{K}_i(S) \right\rangle =$$

$$= \left\langle \left\{ \left\{10 + \frac{4m-4}{S}\right\}_{m=1}^{S+1},\ \left\{23 + \frac{2j-2}{S}\right\}_{j=1}^{S+1} \right\},\ \mathbf{K}_i(S) \right\rangle \text{ for } i = \overline{1,\ 9}. \tag{61}$$

In games (61), there are 9 payoff matrices (30) whose elements are

$$k_{imj}(S) = \int\limits_{[0.5\cdot(i-1)\pi;\ 0.5\pi i)} \left[\sin\left(0.4a^{(m)}t\right)\sin\left(0.3b^{(j)}t - \frac{\pi}{7}\right)\right] d\mu(t)$$

$$\text{for } i = \overline{1,\ 8} \tag{62}$$

and

$$k_{9mj}(S) = \int\limits_{[4\pi;\ 4.5\pi]} \left[\sin\left(0.4a^{(m)}t\right)\sin\left(0.3b^{(j)}t - \frac{\pi}{7}\right)\right] d\mu(t). \tag{63}$$

The payoff kernel on each subinterval of

$$\left\{ \left\{[0.5\cdot(i-1)\pi;\ 0.5\pi i)\right\}_{i=1}^{8},\ [4\pi;\ 4.5\pi] \right\} \tag{64}$$

is shown in Figure 1 for $S = 100$.

A peculiarity of this staircase game is that the players' optimal strategies on subinterval $[0;\ 0.5\pi)$ are pure and unchanging at any $S$:

$$x^*(t) = 14 = a^{(S+1)} \text{ and } y^*(t) = 25 = b^{(S+1)}\ \forall t \in [0;\ 0.5\pi). \tag{65}$$

At $S = 1$, the players still have pure optimal strategies, apart from (65), on subintervals (Figure 2)

$$[0.5\pi;\ \pi),\ [\pi;\ 1.5\pi),\ [2.5\pi;\ 3\pi),\ [3\pi;\ 3.5\pi),\ [3.5\pi;\ 4\pi), \tag{66}$$

whereas they have mixed optimal strategies (the first player mixes $a^{(1)} = 10$ and $a^{(2)} = 14$, the second mixes $b^{(1)} = 23$ and $b^{(2)} = 25$) on subintervals

$$[1.5\pi;\ 2\pi),\ [2\pi;\ 2.5\pi),\ [4\pi;\ 4.5\pi]. \tag{67}$$

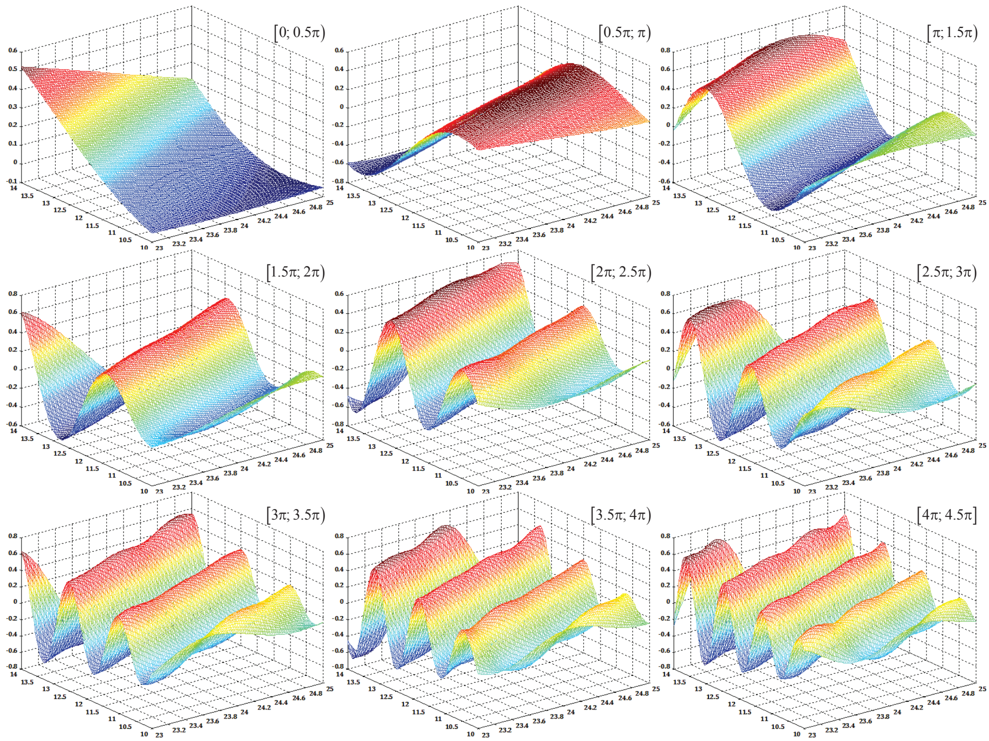


FIGURE 1. The 9 payoff kernels on subintervals (64) for $S = 100$

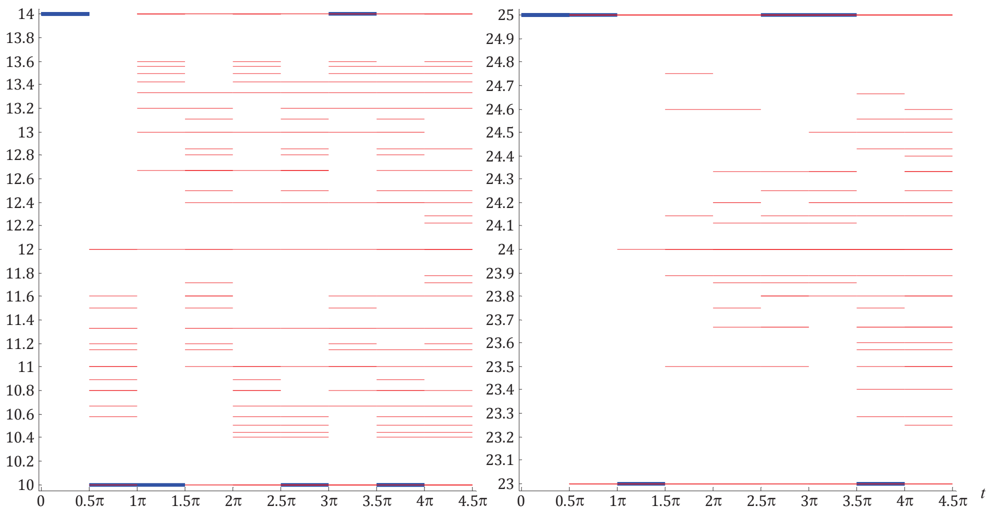


FIGURE 2. Subinterval-wise optimal strategies of the first (left) and second (right) players by $S = \overline{1,\ 10}$

At $S = 2$, along with (65), there are only mixed optimal strategies (shown as thin lines) on (66) and (67): both the players mix already three pure strategies ($a^{(1)} = 10$, $a^{(2)} = 12$, $a^{(3)} = 14$, and $b^{(1)} = 23$, $b^{(2)} = 24$, $b^{(3)} = 25$) on subintervals

$$[1.5\pi;\ 2\pi)\,,\ \ [3.5\pi;\ 4\pi)\,,\ \ [4\pi;\ 4.5\pi]\,.$$

At $S = 3$, along with (65), there springs up another optimal situation in pure strategies

$$x^*(t) = 10 = a^{(1)} \text{ and } y^*(t) = 23 = b^{(1)}\ \ \forall\, t \in [3.5\pi;\ 4\pi)\,.$$

Besides, a mixed optimal situation of four pure strategies exists on $[2\pi;\ 2.5\pi)$. At $S \geqslant 4$, apart from (65), there are no pure optimal strategies. At fewer $S$, the first player's payoff $v_i^*(S)$ (at the end of the $i$-th subinterval) and the payoff cumulative sum

$$v^{(h)*}(S) = \sum_{i=1}^{h} v_i^*(S) \text{ by } h = \overline{1,\ 9} \tag{68}$$

are also unstable (Figure 3). Note that, according to (68), $v^{(9)*}(S)$ is the optimal value in this staircase game.

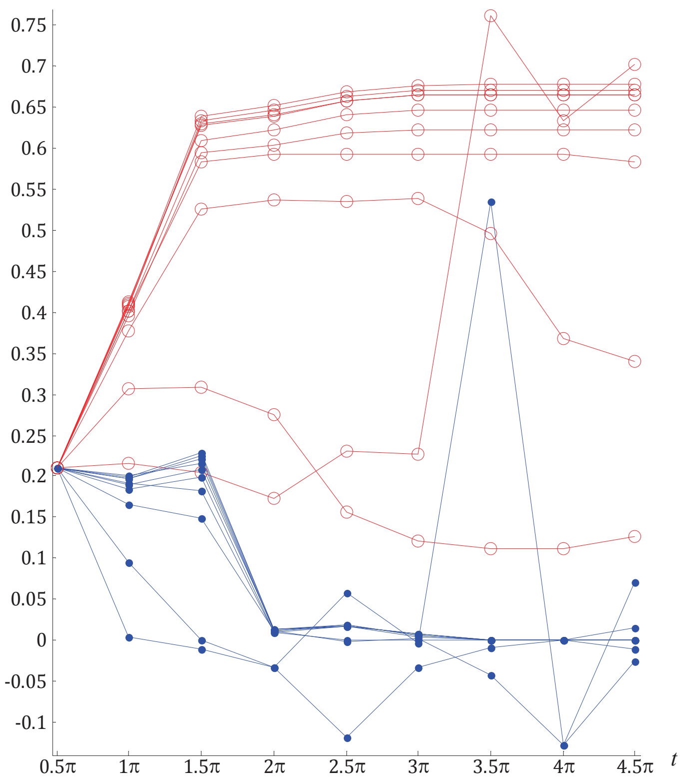


FIGURE 3. The first player's payoffs at the end of every subinterval (dots) and their cumulative sum (circles) by $S = \overline{1,\ 10}$

As the sampling density is further increased, mixed optimal strategies on subintervals (66) and (67) become more "condensed" (Figure 4). Moreover, the payoffs "condense" also (Figure 5): the subinterval payoffs run into a distinct polyline, and their cumulative sum runs into a polyline as well. A peculiarity of the second player's mixed optimal strategies consists in that pure strategies $b^{(1)} = 23$ and $b^{(S+1)} = 25$ on subintervals (66) and (67) are always selected with some probabilities

$$q_i^{(1)*}(S) > 0 \text{ and } q_i^{(S+1)*}(S) > 0,$$

where $i = \overline{2, 9}$. This peculiarity exists starting from $S = 7$.

Figure 6 shows how the "condensed" subinterval-wise optimal strategies look like by $S = \overline{91, 100}$. In fact, this is a visual approximation of how the optimal solution in the staircase game looks like. The stacks in Figure 6 do not significantly change by $S > 100$. The first player's payoffs $\{v_i^*(S)\}_{i=1}^{9}$ and the their cumulative sum (68) appear to be two quite distinct polylines (Figure 7). Strategies in Figure 4 (or even strategies in Figure 2) can be mnemonically thought of as they converge to strategies in Figure 6. Similarly, polylines in Figure 5 (or even polylines in Figure 3) can be thought of as they converge to polylines in Figure 7.

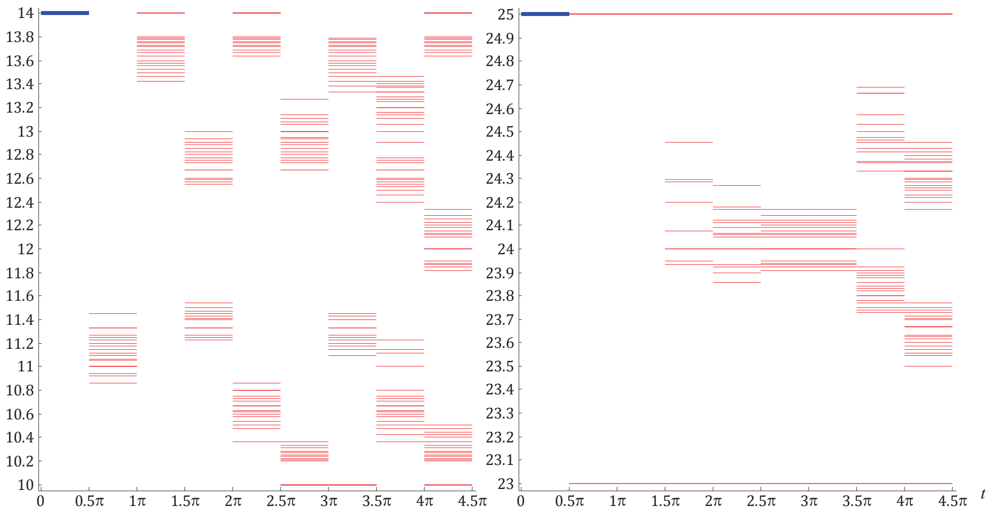

FIGURE 4. Subinterval-wise optimal strategies of the first (left) and second (right) players by $S = \overline{11, 20}$

Although Figure 7 presents a "condensed" polyline of payoff cumulative sum (68) which is clearly increasing, it is worth noting that the cumulative sum of the player's payoffs does not have to be non-decreasing (let alone increasing). For instance, if the first player's payoff at $t = 4.5\pi$ had been slightly less (than it factually is), the resulting optimal value in this staircase game would have been less than the cumulative sum at subinterval $[3.5\pi; 4\pi)$, i. e.

$$v^{(9)*}(S) < v^{(8)*}(S).$$

As it appears from Figure 5 (where the payoff polylines are already sufficiently "condensed" it is sufficient to approximate this staircase game even by $S = 11$. Amazingly enough, there are no payoff-$S$-consistent solutions even at greater $S$. This means that, as the sampling density minimally increases, the game optimal value change grows, where condition (39) is violated for some $i$. Obviously, this growth is

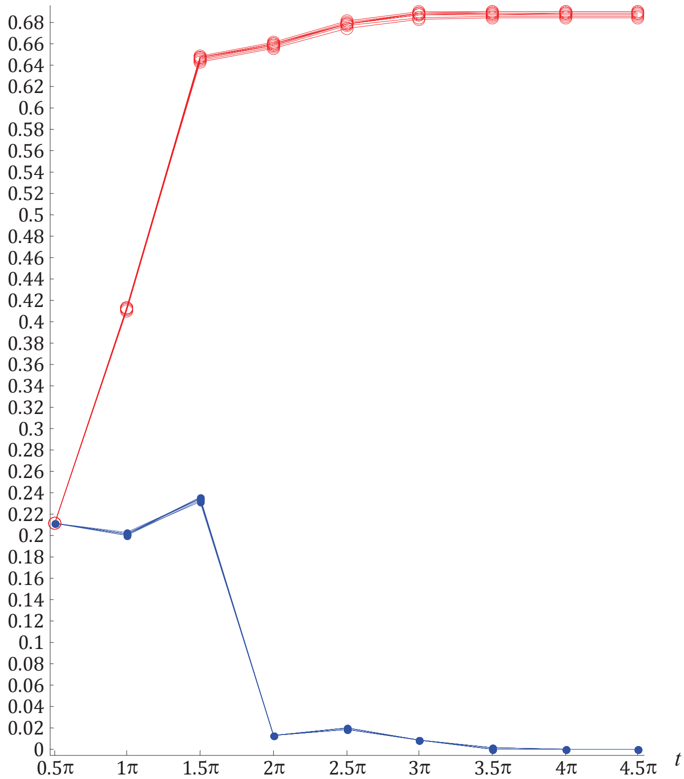


FIGURE 5. The first player's payoffs at the end of every subinterval (dots) and their cumulative sum (circles) by $S = \overline{11,\ 20}$

very small. Therefore, it is useful and practically reasonable to consider the payoff consistency adding a relaxation to (39).

**Definition 9.** An approximate solution (35) is called $\varepsilon$-payoff-$S$-consistent if inequalities

$$\begin{gathered} |v_i^*(S) - v_i^*(S+1)| - \varepsilon \leqslant |v_i^*(S-1) - v_i^*(S)| \\ \text{by some } \varepsilon > 0 \text{ for } i = \overline{1,\ N} \end{gathered} \tag{69}$$

hold.

Thus, the approximate solution is $\varepsilon$-payoff-14-consistent for

$$\varepsilon = 0.03 \cdot v_i^*(14) \quad \text{at} \quad i = \overline{1,\ 9}.$$

Moreover, it is $\varepsilon$-payoff-$S$-consistent for

$$\varepsilon = 0.03 \cdot v_i^*(S) \quad \text{at} \quad i = \overline{1,\ 9}$$

by

$$S \in \left\{\overline{14,\ 150}\right\} \setminus \{15,\ 23\}.$$

In addition, the approximate solution is support-cardinality-$S$-consistent by

$$S \in \{13,\ 14,\ 16,\ 17,\ \ldots\}.$$

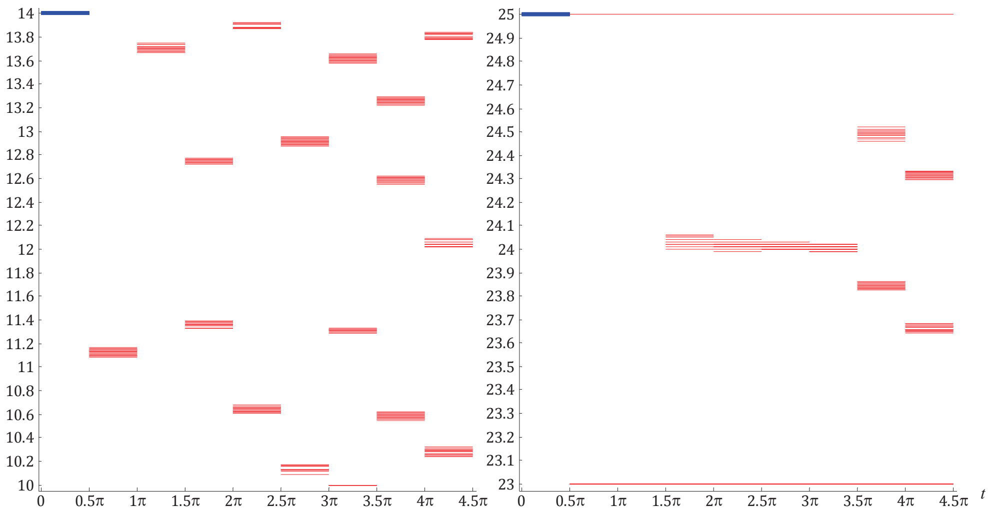


FIGURE 6. Subinterval-wise optimal strategies of the first (left) and second (right) players by $S = \overline{91,\ 100}$

But again, if an inequality in (44), (45), (52), (53) is violated, the support cardinality decreases just by 1 (by minimally increasing the sampling density). So, a similar relaxation to that in Definition 9 might be constructed for the support cardinality consistency. Then Definitions 4 and 5 are "relaxed". Definitions 6, 7, 8 might be "relaxed" in a similar way.

## 9. Discussion

Solving the sampled staircase game straightforwardly, without considering each "smaller" matrix game separately, is usually intractable. For instance, by sampling the exemplified game, where each of the players uses 9-subinterval staircase function-strategies, with $S = 14$, the resulting $15^9 \times 15^9$ matrix game cannot be solved in a reasonable time span. Therefore, solving "smaller" matrix games separately and then stacking their solutions is a far more efficient way to obtain an approximate solution of the initial staircase game. In a pessimistic case, the applicability of this way may be limited to the "smaller" matrix game size. The computation time has an exponentially-increasing dependence on the size of the square matrix. Surely, solving matrix games, in which a player has a few hundred pure strategies, may be time-consuming.

One can notice that if time $t$ is not explicitly included into the function under the integral in (5), then the payoff value depends only on the length of the subinterval. If the length does not change, every subinterval has the same matrix game. The triviality of the equal-length-subinterval solution is explained by a standstill of the players' strategies. Time variable $t$ explicitly included into (5) means that something is going on or changes within the process as time goes by (and the players develop their actions).

The (weak) consistency of an approximate solution is a criterion of its acceptability. However, a (weakly) consistent approximate solution may not exist at appropriately

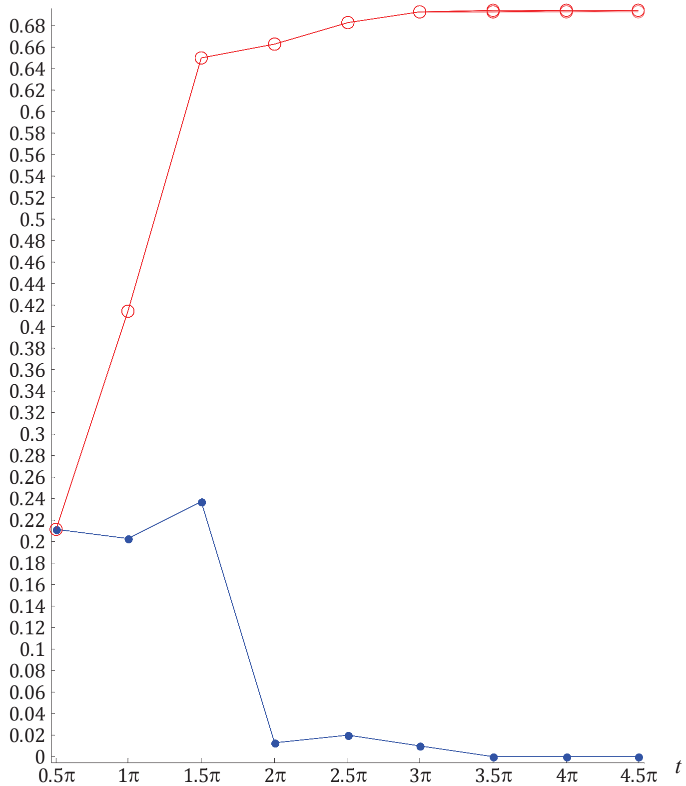


FIGURE 7. The first player's payoffs at the end of every subinterval (dots) and their cumulative sum (circles) by $S = \overline{91,\ 100}$

small (tractable) $S$. So, the consistency decomposition into parts by Definitions 3–8 and particularly isolating an $\varepsilon$-payoff consistency by Definition 9 is justified and practically applicable. There are still many open questions, though. First, it is not proved that limits

$$\lim_{S\to\infty} v_i^*(S) \quad \forall\, i = \overline{1,\ N} \tag{70}$$

exist and they are equal to the respective optimal values of the subinterval continuous games. Second, if limits (70) exist, it is not proved that this is followed by that any approximate solution (35) is $\varepsilon$-payoff-$S$-consistent for any $S \geqslant S_*$ ($S_* \in \mathbb{N}$). The inter-influence among the consistency decomposition parts by Definitions 3–8 is also uncertain yet.

Nevertheless, the presented method is a significant contribution to the antagonistic game theory. It allows approximately solving zero-sum games with staircase-function strategies in a far simpler manner. It "deeinstellungizes" the initial staircase game along with its solution interpretation [5]. Once the (weak) consistency is confirmed (the respective approximate solution should be at least $\varepsilon$-payoff consistent by Definition 9), the approximate pure-mixed-strategy solution (like that in Figure 6) can be easily implemented and practiced (e. g., see [17, 2, 11]).

## 10. Conclusion

A zero-sum game defined on a product of staircase-function continuous spaces is approximated to a matrix game by sampling the player's pure strategy value set. The set is sampled uniformly so the resulting matrix game is square. Owing to Theorem 2, the solution of the matrix game is obtained by stacking the solutions of the "smaller" matrix games, each defined on an interval where the pure strategy value is constant.

The stack of the "smaller" matrix game solutions is an approximate solution to the initial staircase game. The (weak) consistency of the approximate solution is studied by how much the payoff and optimal situation change as the sampling density minimally increases. Thus, the consistency, equivalent to the approximate solution acceptability, is decomposed into the payoff (Definition 3), optimal strategy support cardinality (Definitions 4 and 5), optimal strategy sampling density (Definitions 6 and 7), and support probability consistency (Definition 8). The most important parts are the payoff consistency and optimal strategy support cardinality (weak) consistency. They are checked in the quickest and easiest way. In addition, it is practically reasonable to consider a relaxed payoff consistency. The relaxed payoff consistency by (69) means that, as the sampling density minimally increases, the game optimal value change in an appropriate approximation may grow at most by $\varepsilon$. The weak consistency itself is a relaxation to the consistency, where the minimal decrement of the sampling density is ignored.

Therefore, the suggested method of finite approximation of staircase zero-sum games consists in the uniform sampling, solving "smaller" matrix games, and stacking their solutions if they are consistent. The finite approximation is regarded appropriate if at least the respective approximate (stacked) solution is $\varepsilon$-payoff consistent (Definition 9).

Finite uniform approximation of games on a product of staircase-function continuous spaces can be studied also for the case of non-antagonistic interests of two players. Nonetheless, an approach to solving the corresponding "smaller" bimatrix games is not straightforwardly deduced from Theorem 2. The matter is the optimality in the matrix game does not have an analogy for the bimatrix game [4, 17, 9]. This specificity will make a generalized study of two-person games dissimilar to the presented study. The consistency definitions should be generalized as well.

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(Vadim Romanuke) Faculty of Mechanical and Electrical Engineering, Polish Naval Academy, 69 Śmidowicza Street, Gdynia, 81-127, Poland
*E-mail address*: `v.romanuke@amw.gdynia.pl`