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Derivatives with respect to the perturbations of a domain

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Abstract. The paper studies the existence of derivatives (of Gâteaux and Hadamard type) with respect to the perturbations of the domain, for the solutions of a variational problem. Such derivatives appear in shape sensitivity analysis. The method that we use is the mapping method.

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1. Introduction

Variational problems which depend on a parameter have been studied in many papers (see W. Alt and J. Kolumbán, [1], G. Kassay and J. Kolumbán, [4]). The case studied in this paper is special: the parameter is the underlying domain of the variational problem in discution. More precisely, we are interested in the behaviour of the solutions at small perturbations of the domain. This is in fact the subject of shape sensitivity analysis (see F. Murat and J. Simon [7], M.C. Delfour and J.-P. Zolésio, [2] , J. Sokolowski and J.P. Zolesio, [9]).
In the first section, after a brief presentation of the mapping method (used in shape optimization), we give a result (Theorem 2.1) concerning the stability of a variational problem at small perturbations of the domain. The next section contains the main results (Theorems 3.2 and 3.3) about the existence of Gâteaux and Hadamard derivatives of the solutions, with respect to the perturbations of the domain. Finally, we give some conditions in which the above results can be applied.

2. Stability with respect to the perturbations of the domain

Consider a family of variational problems, depending on the parameter Ω:

Find $u_\Omega \in H^1(\Omega)$ such that

$$\int_\Omega A(x, \nabla u_\Omega(x))\nabla v(x)dx + \int_\Omega a(x, u_\Omega(x))v(x)dx = 0, \ \forall\ v \in H^1(\Omega), \tag{1}$$

with $\Omega \subset \mathbb{R}^N$ a bounded, open set, where A and a are functions with properties to be mentioned later.
When the behavior of the solutions to (1) with respect to the perturbations of the domain Ω is studied, one of the difficulties is that the space in which the solutions lie ($H^1(\Omega)$), depends on the variable domain.
To overcome this, the mapping method, initiated in the 70's by A-M. Micheletti, and, in a different approach, by F. Murat and J. Simon [7], defines the admissible domains as images of a fixed set through a class of transformations. We present some notions

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and properties of the mapping method following [7] (see also [8], [2]).
Let C be a fixed, bounded, open set in $\mathbf{R}^N$, with the boundary ∂C of class $W^{i,\infty}$, $i \geq 1$ and such that int$\bar{C} = C$. The following spaces are defined:

$$W^{k,\infty}(\mathbf{R}^N)^N = \{\phi \,|\, D^\alpha \phi \in L^\infty(\mathbf{R}^N)^N, \forall\ \alpha,\ 0 \leq |\alpha| \leq k\}$$

$$\mathcal{F}^{k,\infty} = \{T : \mathbf{R}^N \to \mathbf{R}^N \mid T \text{ is bijective and } T - I, T^{-1} - I \in W^{k,\infty}(\mathbf{R}^N)^N\}$$

$$\mathcal{O}^{k,\infty} = \{\Omega \mid \Omega = T(C), T \in \mathcal{F}^{k,\infty}\}$$

$\mathcal{O}^{k,\infty}$ consists of a set of bounded open sets. The norm on $W^{k,\infty}(\mathbf{R}^N)^N$ is:

$$\|T\|_{k,\infty} = \text{ess} \sup_{x \in \mathbf{R}^N} \Big(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha T(x)\|_N^2 \Big)^{1/2}$$

It can be proved (see [7]) that there exists a metric on $\mathcal{O}^{k,\infty}$ such that the space is complete and $\Omega_n \to \Omega$ in $\mathcal{O}^{k,\infty}$ iff there exist T_n and T in $\mathcal{F}^{k,\infty}$ such that $T_n(C) = \Omega_n$, $T(C) = \Omega$ and $T_n \to T$, $T_n^{-1} \to T^{-1}$ in $W^{k,\infty}(\mathbf{R}^N)^N$.
Some important properties are given in the following:

Lemma 2.1. *(a) If $T \in \mathcal{F}^{k,\infty}$ and $\Omega = T(C)$ then: $u \in L^2(\Omega)$ $(H^1(\Omega))$ if and only if $u \circ T \in L^2(C)$ $(H^1(C))$.*
(b) Let $k \geq 1$, $u \in L^2(\mathbf{R}^N)$ (or $u \in H^1(\mathbf{R}^N)$). The mapping $T \mapsto u \circ T$ is continuous from $\mathcal{F}^{k,\infty}$ to $L^2(\mathbf{R}^N)$ (or $H^1(\mathbf{R}^N)$).
(c) Let $k \geq 1$. The mappings $T \mapsto JT^{-1}$ and $T \mapsto detJT$ from $\mathcal{F}^{k,\infty}$ to $W^{k-1,\infty}(\mathbf{R}^N)$ are continuous. (JT denotes the Jacobian matrix of T).
(e) If $\Omega_n \to \Omega$ in $\mathcal{O}^{k,\infty}$ then $1_{\Omega_n} \to 1_\Omega$ in $L^2(\mathbf{R}^N)$ (1_Ω is the characteristic function for Ω).

Using the notations above, let $T \in \mathcal{F}^{k,\infty}$ such that $\Omega = T(C)$. Making the transform $x = T(X)$ in (1), we get an equivalent problem on the fixed set C:

$$u_T \in H^1(C) \text{ such that } \int_C A(T(X), JT^{-t}(X)\nabla u_T(X))JT^{-t}(X)\nabla v(X)\text{det}JT(X)dX$$
$$+ \int_C a(T(X), u_T(X))v(X)\text{det}JT(X)dX = 0,\ \forall\ v \in H^1(C), \tag{2}$$

with $u_T = u_\Omega \circ T$.
Defining the operator $\mathcal{A} : \mathcal{F}^{k,\infty} \times H^1(C) \to (H^1(C))^*$ by

$$\langle \mathcal{A}(T,u), v\rangle = \int_C A(T(X), JT^{-t}(X)\nabla u(X))(JT^{-t}(X)\nabla v(X))\text{det}JT(X)dX$$
$$+ \int_C a(T(X), u(X))v(X)\text{det}JT(X)dX,\ \forall\ v \in H^1(C) \tag{3}$$

the variational problem can be written:

$$u_T \in H^1(C) \text{ such that } \langle \mathcal{A}(T, u_T), v\rangle = 0,\ \forall v \in H^1(C), \tag{4}$$

In this formulation, we can consider the family of variational problems (4) as depending on the parameter T.

Consider the following hypotheses on the functions A and a:
($\mathbf{H_1}$) $A = (a_1, \ldots, a_N)$ with $a_j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $a : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ having the properties:
(P1) $a_j(\cdot,\cdot)$, $j = 1, \ldots, N$ and $a(\cdot,\cdot)$ are measurable with respect to the first variable and continuous with respect to the second one,
(P2) $|a_j(x,\xi)| \leq c(k(x) + \|\xi\|_N)$ a.e. $x \in \mathbb{R}^N$, for all $\xi \in \mathbb{R}^N$ and $|a(x,\eta)| \leq c_1(k_1(x) + |\eta|)$ for all $\eta \in \mathbb{R}$, with c, c_1 positive constants and k, k_1 functions in $L^2(D)$ (for any

D a bounded and open set),

(P3) $\sum_{j=1}^{N} a_j(x,\xi)\xi_j \geq c_2\|\xi\|_N^2 - c_3$ and $a(x,\eta)\eta \geq c_4|\eta| - c_5$, a.e. $x \in \mathbb{R}^N$, for all $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}$,

(P4) $\sum_{j=1}^{N}(a_j(x,\xi) - a_j(x,\tilde{\xi}))(\xi_j - \tilde{\xi}_j) \geq \gamma_1\|\xi - \tilde{\xi}\|_N^2$, a.e. $x \in \mathbb{R}^N$, for all $\xi, \tilde{\xi} \in \mathbb{R}^N$ and $(a(x,\eta) - a(x,\tilde{\eta}))(\eta - \tilde{\eta}) \geq \gamma_2|\eta - \tilde{\eta}|^2$, a.e. $x \in \mathbb{R}^N$, for all $\eta, \tilde{\eta} \in \mathbb{R}$.

$(\mathbf{H_2})$ $|a_j(x,\xi) - a_j(\tilde{x},\tilde{\xi})| \leq \psi(x,\tilde{x})(\|\xi\|_N + \|\tilde{\xi}\|_N) + \phi(x,\tilde{x})\|\xi - \tilde{\xi}\|_N + \sigma(x,\tilde{x})$, for all $j = 1,\ldots,N$, $x,\tilde{x},\xi,\tilde{\xi} \in \mathbb{R}^N$; where $\psi(\cdot,\cdot)$, $\phi(\cdot,\cdot)$, $\sigma(\cdot,\cdot)$ are nonnegative functions belonging to $C(\mathbb{R}^N \times \mathbb{R}^N) \cap L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ and $\psi(x,\tilde{x}) = \psi(\tilde{x},x)$, $\psi(x,x) = \sigma(x,x) = 0$

$(\mathbf{H_3})$ $|a(x,\eta) - a(\tilde{x},\tilde{\eta})| \leq \chi(x,\tilde{x})(|\eta| + |\tilde{\eta}|) + \mu(x,\tilde{x})|\eta - \tilde{\eta}|$, where χ has the same properties like ψ and μ has the same properties like ϕ.

The hypotheses above inssure that, for each parameter $T \in \mathcal{F}^{k,\infty}$, the problem (4) has a unique solution.

Fix $T_0 \in \mathcal{F}^{k,\infty}$ an initial parameter.

We proved in [5], in a more general setting, using some results about parametric variational inequalities of G. Kassay and J. Kolumbán [4]:

Theorem 2.1. *Suppose that* $(\mathbf{H_1})$-$(\mathbf{H_3})$ *are satisfied. Let* $T_0 \in \mathcal{F}^{k,\infty}$ *and* $u_0 \in H^1(C)$ *fixed. If* u_0 *is a solution of the variational problem (4) for* T_0*, then this problem is stable under perturbations, that is: there exists a neighborhood* W_0 *of* T_0 *and a mapping* $u : W_0 \to H^1(C)$ *such that for each* $T \in W_0$*,* $u(T) = u_T$ *is a solution of (4) for* T*,* $u(T_0) = u_0$ *and* u *is continuous at* T_0*.*

In the following section we will also study the existence of Gâteaux and Hadamard derivatives for the function $u(T)$. To this end, we will use some ideas from W. Alt and J. Kolumbán, [1].

3. Existence of derivatives with respect to the domain

Consider a variational problem of the type (4), with an arbitrary operator $\mathcal{A} : \mathcal{F}^{k,\infty} \times H^1(C) \to (H^1(C))^*$.

For $u \in H^1(C)$, we denote $d^G\mathcal{A}_u(T_0,u_0)(u) = \lim_{\varepsilon \searrow 0} \dfrac{\mathcal{A}(T_0,u_0+\varepsilon u) - \mathcal{A}(T_0,u_0)}{\varepsilon}$ the Gâteaux derivative at the point u_0, in the direction u, of the application $u \mapsto \mathcal{A}(T_0,u)$.

It takes place, similar to W. Alt and J. Kolumbán [1]:

Lemma 3.1. *Let* $u_0 \in H^1(C)$*,* $T_0 \in \mathcal{F}^{k,\infty}$ *and suppose that there exists a neighborhood* U_0 *of* u_0 *such that the application* $u \mapsto \mathcal{A}(T_0,u)$ *is strongly monotone on* U_0 *and Gâteaux differentiable at* u_0*. Then the operator* $d^G\mathcal{A}_u(T_0,u_0)$ *is invertible.*

Proof. Let $u \in H^1(C)$ be fixed. For $\varepsilon > 0$, sufficiently small, we have, from the definition of the Gâteaux derivative in the direction u and from the strong monotonicity

$$\langle d^G\mathcal{A}_u(T_0,u_0)(u), u\rangle = \langle d^G\mathcal{A}_u(T_0,u_0)(u) - \frac{\mathcal{A}(T_0,u_0+\varepsilon u) - \mathcal{A}(T_0,u_0)}{\varepsilon}, u\rangle$$
$$+ \frac{1}{\varepsilon^2}\langle \mathcal{A}(T_0,u_0+\varepsilon u) - \mathcal{A}(T_0,u_0), \varepsilon u\rangle \geq o(\varepsilon) + \alpha\|u\|^2,$$

with $\lim_{\varepsilon \searrow 0} o(\varepsilon) = 0$ and $\alpha > 0$. We get in this way that $\langle d^G\mathcal{A}_u(T_0,u_0)(u), u\rangle \geq \alpha\|u\|^2$, for every $u \in H^1(C)$, and so $u \mapsto d^G\mathcal{A}_u(T_0,u_0)(u)$ is invertible.

□

As defined in the first section, $\mathcal{F}^{k,\infty}$ is the set of essentially bounded, with essentially bounded derivatives, perturbations of the identity. According to F. Murat and J. Simon [7], it takes place:

Theorem 3.1. *If $\|\theta\|_{W^{k,\infty}(\mathbb{R}^N)^N}$ is sufficiently small, then $I+\theta$ belongs to $\mathcal{F}^{k,\infty}$.*

In view of this, consider $\theta_0 \in W^{k,\infty}(\mathbb{R}^N)^N$ and $\theta \in W^{k,\infty}(\mathbb{R}^N)^N$, sufficiently small such that $T_0 = I + \theta_0$ belongs to $\mathcal{F}^{k,\infty}$. For a function f defined on $\mathcal{F}^{k,\infty}$ we define the Gâteaux derivative at T_0, in the direction θ:

$$d^G f(T_0)(\theta) = \lim_{\varepsilon \searrow 0} \frac{f(T_0+\varepsilon\theta) - f(T_0)}{\varepsilon}.$$

Theorem 3.2. *Consider the variational problem (4). Suppose that the following conditions take place:*
(i) u_0 is a solution of the problem corresponding to the initial parameter T_0,
(ii) $\mathcal{A}$ is consistent in T at (T_0, u_0),
(iii) the applications $\mathcal{A}(T,\cdot)$ are uniformly strongly monotone, continuous from the line segments of $H^1(C)$ to $(H^1(C))^*$ with the weak topology, for every $T \in \mathcal{F}^{k,\infty}$ and every $u \in H^1(C)$,
(iv) the application $u \mapsto \mathcal{A}(T_0, u)$ is Gâteaux differentiable at the point u_0, there exists a ball $B((T_0,u_0), r)$ such that for every (T,u) from that ball there exists the Gâteaux derivative $d^G\mathcal{A}_u(T,u)(w)$, for every $w \in H^1(C)$,
(v) for every $w \in H^1(C)$, the application $(T,u) \mapsto d^G\mathcal{A}_u(T,u)(w)$ is continuous at (T_0,u_0),
(vi) for a direction $\tilde{\theta} \in W^{k,\infty}(\mathbb{R}^N)^N$ there exists the Gâteaux derivative

$$d^G\mathcal{A}_T(T_0,u_0)(\tilde{\theta}) = \lim_{\varepsilon \searrow 0} \frac{\mathcal{A}(T_0+\varepsilon\tilde{\theta}, u_0) - \mathcal{A}(T_0,u_0)}{\varepsilon}.$$

Then, the variational problem (4) is stable with respect to the perturbations of the domain and there exists the Gâteaux derivative $d^G u(T_0)(\tilde{\theta})$ of the function u (as a function of the parameter T, see Theorem 2.1), and is given by:

$$d^G u(T_0)(\tilde{\theta}) = -d^G\mathcal{A}_u(T_0,u_0)^{-1}(d^G\mathcal{A}_T(T_0,u_0)(\tilde{\theta})).$$

Proof. We use the ideea of the proof from [1]. Lemma 3.1 assures the inversability of the application $w \mapsto d^G\mathcal{A}_u(T_0,u_0)(w)$, so we can define

$$\tilde{u} = -d^G\mathcal{A}_u(T_0,u_0)^{-1}(d^G\mathcal{A}_T(T_0,u_0)(\tilde{\theta})).$$

We will show that this is the Gâteaux derivative of u at T_0, in the direction $\tilde{\theta}$.
From the strong monotonicity we have:

$$\begin{aligned}&\alpha\|u(T_0+\varepsilon\tilde{\theta}) - u(T_0) - \varepsilon\tilde{u}\|^2\\ &\quad\le \langle \mathcal{A}(T_0+\varepsilon\tilde{\theta}, u(T_0+\varepsilon\tilde{\theta})) - \mathcal{A}(T_0+\varepsilon\tilde{\theta}, u_0+\varepsilon\tilde{u}), u(T_0+\varepsilon\tilde{\theta}) - u_0 - \varepsilon\tilde{u}\rangle.\end{aligned}$$

From $\langle \mathcal{A}(T_0+\varepsilon\tilde{\theta}, u(T_0+\varepsilon\tilde{\theta})), v\rangle = 0$ for every $v \in H^1(C)$ we get

$$\|u(T_0+\varepsilon\tilde{\theta}) - u(T_0) - \varepsilon\tilde{u}\| \le \alpha^{-1}\|\mathcal{A}(T_0+\varepsilon\tilde{\theta}, u(T_0)+\varepsilon\tilde{u})\|.$$

The proof will be complete if we show that:

$$\|\mathcal{A}(T_0+\varepsilon\tilde{\theta}, u(T_0)+\varepsilon\tilde{u})\| = o(\varepsilon),$$

for ε sufficiently small.
From (vi) we have

$$\mathcal{A}(T_0+\varepsilon\tilde{\theta}, u_0) - \mathcal{A}(T_0,u_0) - d^G\mathcal{A}_T(T_0,u_0)(\varepsilon\tilde{\theta}) = o_1(\varepsilon)\varepsilon,$$

where $\lim\limits_{\varepsilon\searrow 0}\|o_1(\varepsilon)\| = 0$.

Define the function $g : [0,\varepsilon] \to (H^1(C))^*$ by

$$g(s) = \mathcal{A}(T_0 + \varepsilon\tilde{\theta}, u_0 + s\tilde{u}) - \mathcal{A}(T_0 + \varepsilon\tilde{\theta}, u_0) - d^G\mathcal{A}_u(T_0, u_0)(s\tilde{u}).$$

From (iv), g is differentiable at the right, with

$$g'_+(s) = d^G\mathcal{A}_u(T_0 + \varepsilon\tilde{\theta}, u_0 + s\tilde{u})(\tilde{u}) - d^G\mathcal{A}_u(T_0, u_0)(\tilde{u}).$$

By (v), $\|g'_+(s)\| \le o_2(\varepsilon)$, with $\lim\limits_{\varepsilon\searrow 0} o_2(\varepsilon) = 0$.

Further on, from the Mean value Theorem, this implies:

$$\|g(\varepsilon)\| = \|g(\varepsilon) - g(0)\| \le \sup_{s\in[0,\varepsilon]} \|g'_+(s)\|\varepsilon \le o_2(\varepsilon)\varepsilon.$$

From $d^G\mathcal{A}_u(T_0, u_0)(\varepsilon\tilde{u}) = -d^G\mathcal{A}_T(T_0, u_0)(\varepsilon\tilde{\theta})$ and from $\langle\mathcal{A}(T_0, u_0), v\rangle = 0$, for every $v \in H^1(C)$, follows

$$\begin{aligned}
&\|\mathcal{A}(T_0 + \varepsilon\tilde{\theta}, u(T_0) + \varepsilon\tilde{u})\|_{(H^1(C))^*} = \sup_{\substack{v\in H^1(C)\\ \|v\|\le 1}} \langle\mathcal{A}(T_0 + \varepsilon\tilde{\theta}, u_0 + \varepsilon\tilde{u}), v\rangle \\
&= \sup_{\substack{v\in H^1(C)\\ \|v\|\le 1}} \big\{\langle\mathcal{A}(T_0 + \varepsilon\tilde{\theta}, u_0 + \varepsilon\tilde{u}) - \mathcal{A}(T_0 + \varepsilon\tilde{\theta}, u_0) - d^G\mathcal{A}_u(T_0, u_0)(\varepsilon\tilde{u}), v\rangle \\
&+ \langle\mathcal{A}(T_0 + \varepsilon\tilde{\theta}, u_0) - \mathcal{A}(T_0, u_0) - d^G\mathcal{A}_T(T_0, u_0)(\varepsilon\tilde{\theta}), v\rangle\big\} \\
&\le (o_2(\varepsilon) + \|o_1(\varepsilon)\|)\varepsilon = o(\varepsilon).
\end{aligned}$$

which concludes the proof. □

A similar result takes place for the Hadamard derivative with respect to the parameter T.

Theorem 3.3. *If, for the variational problem (4) the conditions (i)-(v) from Theorem 3.2 are satisfied and moreover:*
(vi)′ for a direction $\tilde{\theta}$ there exists the Hadamard derivative

$$d^H\mathcal{A}_T(T_0, u_0)(\tilde{\theta}) = \lim_{\substack{\varepsilon\searrow 0\\ \theta\to\tilde{\theta}}} \frac{\mathcal{A}(T_0 + \varepsilon\theta, u_0) - \mathcal{A}(T_0, u_0)}{\varepsilon},$$

then there exists the Hadamard derivative $d^H u(T_0)(\tilde{\theta})$ and is given by the formula

$$d^H u(T_0)(\tilde{\theta}) = -d^G\mathcal{A}_u(T_0, u_0)^{-1}(d^H\mathcal{A}_T(T_0, u_0)(\tilde{\theta})). \tag{5}$$

3.1. Conditions that assure the existence of the Hadamard derivative. We state in what follows some supplementary hypotheses on the functions $a : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $A = (a_1, \dots, a_N) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ which will imply the conditions from Theorem 3.3 and will permit applying this theorem to the problem (4), with the operator $\mathcal{A}$ given by (3), that is problem (2):

($\mathbf{H_4}$) $a(x,\cdot)$ is differentiable with respect to the second variable and $|\frac{\partial}{\partial\eta}a(x,\eta)| \le M_1$, for every $\eta \in \mathbb{R}$, $x \in \mathbb{R}^N$,

($\mathbf{H_5}$) for every $i = 1, \dots, N$, the function $a_i(x,\cdot)$ is partially differentiable with respect to each ξ_j , with continuous partial derivatives and $\|\nabla_\xi a_i(x,\xi)\|_N \le M_{2i}$, for every $\xi, x \in \mathbb{R}^N$,

($\mathbf{H_6}$) $|\frac{\partial}{\partial\eta}a(x,\eta) - \frac{\partial}{\partial\eta}a(\tilde{x},\tilde{\eta})| \le M_3\|\tilde{x} - x\|_N + M_4$,

$\|\nabla_\xi a_i(x,\xi) - \nabla_\xi a_i(\tilde{x},\tilde{\xi})\|_N \leq M_{5i}\|\tilde{x}-x\|_N + M_6$, for every $x, \tilde{x}, \xi, \tilde{\xi} \in \mathbb{R}^N$, $\eta, \tilde{\eta} \in \mathbb{R}$,
$(\mathbf{H_7})$ $a(\cdot,\eta)$ and $a_i(\cdot,\xi)$, $i=1,\dots,N$ are partially differentiable with respect to each x_j, with continuous partial derivatives and: $\left|\frac{\partial a}{\partial x_j}(x,\eta)\right| \leq M_7$,
$\left|\frac{\partial a_i}{\partial x_j}(x,\xi)\right| \leq M_{8ij}$, for every $x, \xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}$,
$(\mathbf{H_8})$ the functions k and k_1 from the hypothesis **(P2)** belong to the space L^∞.

Theorem 3.4. *Consider the variational problem (4):*

$$u_T \in H^1(C) \text{ such that } \langle \mathcal{A}(T,u_T), v\rangle = 0 \quad \forall v \in H^1(C)$$

where $\mathcal{A} : \mathcal{F}^{k,\infty} \times H^1(C) \to (H^1(C))^*$ *is given by (3).*
If the hypotheses $(\mathbf{H_1}) - (\mathbf{H_8})$ *are satisfied, then there exists the Hadamard derivative for the solution* u *of the variational problem (as a function of the parameter* T*), at* T_0*, in the direction* $\tilde{\theta}$*.*

The proof is technical, it uses the properties of the Gâteaux and Hadamard derivatives, the Theorem of dominated convergence and the hypotheses $(\mathbf{H_1}) - (\mathbf{H_8})$.

Remark 3.1. *The formula (5) can be also written in the form:*

$$-d^G\mathcal{A}_u(T_0,u_0)(d^H u(T_0)(\tilde{\theta})) = d^H\mathcal{A}_T(T_0,u_0)(\tilde{\theta}). \tag{6}$$

Remark 3.2. *In many applications the initial parameter* T_0 *is the identity* I*. We have then:*

$$\langle d^G\mathcal{A}_u(I,u_0)(w), v\rangle = \sum_{i=1}^N \int_C \nabla_\xi a_i(X,\nabla u_0(X))\nabla w(X)(\nabla v(X))_i dX$$
$$+ \int_C \frac{\partial}{\partial\eta} a(X,u_0(X))w(X)v(X)dX$$

$$\langle d^H\mathcal{A}_T(I,u_0)(\tilde{\theta}), v\rangle = \sum_{i=1}^N \int_C a_i(X,\nabla u_0(X))(\nabla v(X))_i tr(J\tilde{\theta}(X))dX$$
$$+ \sum_{i=1}^N \int_C a_i(X,\nabla u_0(X))(-J\tilde{\theta}^t(X)\nabla v(X))_i dX$$
$$+ \sum_{i=1}^N \int_C \big[\nabla_x a_i(X,\nabla u_0(X))\tilde{\theta}(X)$$
$$- \nabla_\xi a_i(X,\nabla u_0(X))(J\tilde{\theta}^t(X)\nabla u_0(X))\big](\nabla v(X))_i dX$$
$$+ \int_C a(X,u_0(X))v(X)tr(J\tilde{\theta}(X))dX + \int_C \nabla_x a(X,\nabla u_0(X))\tilde{\theta}(X)v(X)dX.$$

In what follows we will write explicitly the formula (6) for an example of linear variational problem, also studied in J.Sokolowski and J.P. Zolésio [9].

Example 3.1. *Consider the problem*

$$\int_\Omega B\nabla u(x)\nabla v(x)dx + \int_\Omega bu(x)v(x)dx = 0 \quad \forall v \in H^1(\Omega), \tag{7}$$

where B is a matrix with $\sum_{i=1}^{N}\sum_{k=1}^{N} b_{ik}\xi_k\xi_i \geq \alpha\|\xi\|_N^2$, *with $\alpha > 0$ constant and b nonnegative. We identify $A(x,\xi) = B\xi$, so $a_i = \sum_{k=1}^{N} b_{ik}\xi_k$ and $a(x,\eta) = b\eta$, which satisfies* $(\mathbf{H_1}) - (\mathbf{H_8})$.
Denoting the Hadamard derivative of u, at I, in the direction $\tilde{\theta}$: $\tilde{u} = d^H u(I)(\tilde{\theta})$, the formula (6) becomes:

$$\begin{aligned}&-\int_C B\nabla\tilde{u}(X)\nabla v(X)dX - \int_C b\tilde{u}(X)v(X)dX\\ &= \int_C B\nabla u_0(X)\nabla v(X)div\tilde{\theta}(X)dX - \int_C B\nabla u_0(X)(J\tilde{\theta}(X)\nabla v(X))dX\\ &- \int_C b(J\tilde{\theta}(X)\nabla u_0(X))\nabla v(X) + \int_C bu_0(X)v(X)div\tilde{\theta}(X)dX,\end{aligned}$$

for each $v \in H^1(C)$.
Taking $B = I_n$ and $b = 0$, we can find the Hadamard derivative of u in a direction $V(0) \in W^{k,\infty}(\mathbb{R}^N)^N$, $V(0)(X)$ being "the initial speed of the particle X" (see also [9], *pg.105.):*

$$\begin{aligned}-\int_C \nabla\tilde{u}(X)\nabla v(X)dX &= \int_C \nabla u_0(X)\nabla v(X)divV(0)dX\\ &- \int_C [JV(0)^t(X) + JV(0)(X)]\nabla u_0(X)\nabla v(X)dX.\end{aligned}$$

Appendix. Let H be a reflexive Banach space, H^* its dual, W a topological space. A map $S : W \times H \to H^*$ is called *consistent in w at* (w_0, x_0) if, for each $0 < r \leq 1$, there exists a neighborhood W_r of w_0, a function $\beta : W_r \to \mathbb{R}$, continuous at w_0, with $\beta(w_0) = 0$, such that for each $w \in W_r$, there exists $y_w \in H$ such that $\|y_w - x_0\| \leq \beta(w)$ and

$$\langle S(w, y_w), z - y_w\rangle + \beta(w)\|z - y_w\| \geq 0,$$

for each $z \in H$, with $r < \|z - y_w\| \leq 2$.
The maps $S(w,\cdot) : H \to H^*$ are called *uniformly strongly monotone* on $W_0 \subset W$ if there exists a positive constant α such that for all $w \in W_0$ and $x, y \in H$, $x \neq y$, we have

$$\langle S(w,x) - S(w,y), x - y\rangle \geq \alpha\|x - y\|^2.$$

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