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A necessary optimality condition for quasiconvex functions on closed convex sets

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Abstract. We give a necessary condition for a minimization problem of a quasiconvex function on a closed convex set.
We consider both the case of a general convex set and a convex set defined as a constrained set for a quasiconvex lower semicontinuous function.

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1. Introduction

Consider the following problem

$$(\mathcal{P}) \quad \begin{cases} \text{minimize } f(x), \\ x \in C \subset X, \end{cases}$$

where X is a Banach space, $f\colon X \to \mathbb{R} \cup \{+\infty\}$ and C is closed convex.

When looking in the literature for the nature of the different conditions on the objective function f, to solve $(\mathcal{P})$, we see clearly that convexity and differentiability are among the widely used candidates. Nevertheless, even the case when f is neither convex nor differentiable has been treated. In [1], Clarke considered the case of locally Lipschitz functions and in [2], Huriart-Urruty the case of directionally stable functions. The two papers may be considered as contributions to the case where f enjoys some "regularity."

A natural question is the following: what happens when f is less regular, but instead possesses some kind of convexity?

In this paper, we consider the case of quasiconvex functions. The case of pseudoconvex functions is treated in the paper [3].

The paper is organized as follows. After recalling basic definitions and properties, we give in the next section a necessary condition for a minimization problem of a quasiconvex function on a closed convex set. We consider both the case of a general convex set and a convex set defined as a constrained set for a quasiconvex l.s.c. function.

As usual, X^* denotes the dual space to X and $\langle ., . \rangle$ the duality pairing. The interval $[a, b] = \{a + t(b - a);\ 0 \le t \le 1\}$ and $]a, b[= [a, b] \setminus \{a, b\}$. The open ball centered at x with radius r is denoted by $B_r(x)$. We recall that f is quasiconvex if for any $x, y \in X$ and any $z \in [x, y]$,

$$f(z) \le \max\{f(x), f(y)\}.$$

This is equivalent to the convexity of the level sets

$$S_\lambda(f) = \{x \in X;\ f(x) \le \lambda\}, \qquad \forall \lambda \in \mathbb{R}.$$

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We will also use the notations $\tilde{S}_\lambda(f) = \{x \in X;\ f(x) < \lambda\}$, $L_f(x_0) = \{x \in X;\ f(x) = f(x_0)\}$. The mapping f is lower semicontinuous (l.s.c.) if $S_\lambda(f)$ is closed for any $\lambda \in \mathbb{R}$. When f is l.s.c., the Clarke-Rockafellar generalized derivative at x along the direction v is defined by

$$f^{\nearrow}(x;v) = \sup_{\varepsilon>0} \limsup_{y\to_f x,\ t\searrow 0} \inf_{u\in B_\varepsilon(v)} \frac{f(y+tu)-f(y)}{t},$$

where $y \to_f x$ means that $y \to x$ and $f(y) \to f(x)$. The Clarke-Rockafellar subdifferential of f at x is

$$\partial f(x) = \{x^* \in X^*;\ \langle x^*, v\rangle \le f^{\nearrow}(x;v), \forall v \in X\}$$

with the convention that $\partial f(x)$ is empty if f is not finite at x. And last, the normal cone of f to the convex set C at x_0 is defined by

$$N(C;x_0) = \{x^* \in X^*;\ \langle x^*, x - x_0\rangle \le 0, \forall x \in C\}.$$

2. Minimization of quasiconvex functions

The main result in this note is a necessary optimality condition for $(\mathcal{P})$ when f is l.s.c., quasiconvex and C is any nonempty closed convex set of X.

Theorem 2.1. *Let X be a Banach space, $f\colon X \to \mathbb{R}\cup\{+\infty\}$ a l.s.c. quasiconvex function. Consider $x_0 \in C$ such that*
(i) *$\tilde{S}_{f(x_0)}(f)$ is nonempty and open in X.*
(ii) *$\partial f(x_0)$ is nonempty and w^*-compact in X^*.*
(iii) *There is $\nu > 0$ such that*

$$\forall x \in B_\nu(x_0)\cap L(x_0), \quad 0 \notin \partial f(x). \tag{2.1}$$

Then, a necessary condition for x_0 to be a solution of $(\mathcal{P})$ is that

$$0 \in \partial f(x_0) + N(C;x_0). \tag{2.2}$$

We will need in the sequel the following technical result.

Lemma 2.1. *Let X be a Banach space, $f\colon X \to \mathbb{R}\cup\{+\infty\}$ a l.s.c. quasiconvex function.*
(i) *If $\partial f(x_0)$ is nonempty and there exists $r > 0$ such that $0 \notin \partial f(x)$ for all $x \in B_r(x_0)\cap L_f(x_0)$, then*

$$N(S_{f(x_0)}(f);x_0) = Cl(\mathbb{R}^+\partial f(x_0)).$$

(ii) *If moreover, $\partial f(x_0)$ is w^*-compact, then*

$$N(S_{f(x_0)}(f);x_0) = \mathbb{R}^+\partial f(x_0).$$

Proof. The point (i) is [4, Proposition 2.2].
(ii) Consider a sequence $(\lambda_n x_n^*)_n \subset \mathbb{R}^+\partial f(x_n)$ such that $\lambda_n x_n^* \rightharpoonup y^*$. We will show that $y^* = \lambda x^*$ for some $\lambda \in \mathbb{R}^+$ and $x^* \in \partial f(x_0)$.

Since $x_n^* \in \partial f(x_0)$ which is w^*-compact, for a subsequence still denoted $(x_n^*)_n$, $x_n^* \rightharpoonup x^* \in \partial f(x_0)$.

Claim 2.1. *There is a subsequence of $(\lambda_n)_n$, still denoted $(\lambda_n)_n$, that is bounded.*

Indeed, $0 \notin \partial f(x_0)$. By the Hahn-Banach theorem, there is $v \in X$ such that

$$\langle z^*, v\rangle > 0, \qquad \forall z^* \in \partial f(x). \tag{2.3}$$

But $\lambda_n x_n^* \rightharpoonup y^*$, so there is $M > 0$ such that

$$M \geq \langle \lambda_n x_n^*, v\rangle.$$

If $(\lambda_n)_n$ was not bounded, for some subsequence, still denoted $(\lambda_n)_n$, we would get

$$\frac{M}{\lambda_n} \geq \langle x_n^*, v\rangle > 0.$$

At the limit, we get a contradiction with (2.3). □

Proof of Theorem 2.1. Suppose that x_0 minimizes f on C. Then, $C \cap \tilde{S}_{f(x_0)}(f) = \varnothing$. But $\tilde{S}_{f(x_0)}(f) \cap B_{\nu/4}(x_0) \neq \varnothing$ because otherwise, x_0 would be a local minimum of f and hence we would get $0 \in \partial f(x_0)$, a contradiction with (iii). Moreover,

$$\big(C \cap Cl(B_{\nu/2}(x_0))\big) \cap \Big(\tilde{S}_{f(x_0)}(f) \cap B_{\nu/2}(x_0)\Big) = \varnothing.$$

By (i) and using the Hahn-Banach theorem, there is $u^* \in X^*$ such that $u^* \neq 0$ and $\alpha \in \mathbb{R}$ separating our two convex sets:

$$\langle u^*, x\rangle \leq \alpha, \qquad \forall x \in \tilde{S}_{f(x_0)}(f) \cap B_{\nu/2}(x_0), \tag{2.4}$$

$$\langle u^*, x\rangle \geq \alpha, \qquad \forall x \in C \cap B_{\nu/2}(x_0), \tag{2.5}$$

We claim that $\langle u^*, x_0\rangle = \alpha$. Indeed, it is clear that $\langle u^*, x_0\rangle \geq \alpha$. It suffices to check the other sense.

Let us first show the equality

$$Cl(\tilde{S}_{f(x_0)}(f)) \cap B_{\nu/2}(x_0) = S_{f(x_0)}(f) \cap B_{\nu/2}(x_0). \tag{2.6}$$

Indeed, the sense "$\subset$" is obvious. For the inverse inclusion, suppose by contradiction that there is $y \in \Big(Cl(\tilde{S}_{f(x_0)}(f)) \cap B_{\nu/2}(x_0)\Big) \backslash \big(S_{f(x_0)}(f) \cap B_{\nu/2}(x_0)\big)$. Then, $y \in L_f(x_0) \cap B_{\nu/2}(x_0)$ and it is a local minimum of f. So $0 \in \partial f(y)$, a contradiction with (iii).

By (2.6), there is a sequence $(x_n)_n \subset \tilde{S}_{f(x_0)}(f) \cap B_{\nu/2}(x_0)$ such that $x_n \to x_0$ and hence $\langle u^*, x_0\rangle \leq \alpha$.

Using (2.4), $\langle u^*, x_0\rangle = \alpha$ and (2.6), we get

$$\begin{array}{lll} u^* \in & N(\tilde{S}_{f(x_0)}(f) \cap B_{\nu/2}(x_0); x_0) = & \text{property of normal cones} \\ & N(Cl(\tilde{S}_{f(x_0)}(f)) \cap B_{\nu/2}(x_0); x_0) = & \text{by (2.6)} \\ & N(S_{f(x_0)}(f) \cap B_{\nu/2}(x_0); x_0) = & \text{property of normal cones} \\ & N(S_{f(x_0)}(f); x_0) & \end{array}$$

By (ii) of Lemma 2.1, we have

$$u^* \in \mathbb{R}^+ \partial f(x_0).$$

And by (2.5),

$$-u^* \in N(C; x_0).$$

Since $u^* \neq 0$, we finally get

$$0 \in \partial f(x_0) + N(C; x_0).$$

□

This theorem refines the results of Clarke [1] and Huriart-Urruty [2] when we require the quasiconvexity of the objective function f.

In the case where the general convex set C, appearing in the former result, is defined as the constraint set

$$C = \{x \in X;\ g(x) \leq 0\},$$

where g is quasiconvex, l.s.c. and satisfies (ii), (iii) of Theorem 1, and $g(x_0) = 0$, we obtain the following result where appears some Lagrange multiplier.

Corollary 2.1. *A necessary condition for x_0 to solve $(\mathcal{P})$ is*

$$0 \in \partial f(x_0) + \lambda \partial g(x_0), \qquad \text{for some } \lambda > 0.$$

Note that $g(x_0) = 0$ is not a problem, we could always use $h(t) = g(x) - g(x_0)$. For the proof, it suffices to use Theorem 2.1 and Lemma 2.1(ii).

The case where f is pseudoconvex is investigated in an other paper [3].

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