A necessary optimality condition for quasiconvex functions on closed convex sets

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Abstract. We give a necessary condition for a minimization problem of a quasiconvex function on a closed convex set. We consider both the case of a general convex set and a convex set defined as a constrained set for a quasiconvex lower semicontinuous function.

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1. Introduction

Consider the following problem

$$\begin{align*}
\text{(P)} & \quad \begin{array}{ll}
\text{minimize} & f(x), \\
\text{subject to} & x \in C \subset X, 
\end{array}
\end{align*}$$

where $X$ is a Banach space, $f: X \to \mathbb{R} \cup \{+\infty\}$ and $C$ is closed convex.

When looking in the literature for the nature of the different conditions on the objective function $f$, to solve (P), we see clearly that convexity and differentiability are among the widely used candidates. Nevertheless, even the case when $f$ is neither convex nor differentiable has been treated. In [1], Clarke considered the case of locally Lipschitz functions and in [2], Huriart-Urruty the case of directionally stable functions. The two papers may be considered as contributions to the case where $f$ enjoys some “regularity.”

A natural question is the following: what happens when $f$ is less regular, but instead possesses some kind of convexity?

In this paper, we consider the case of quasiconvex functions. The case of pseudo-convex functions is treated in the paper [3].

The paper is organized as follows. After recalling basic definitions and properties, we give in the next section a necessary condition for a minimization problem of a quasiconvex function on a closed convex set. We consider both the case of a general convex set and a convex set defined as a constrained set for a quasiconvex l.s.c. function.

As usual, $X^*$ denotes the dual space to $X$ and $\langle \cdot, \cdot \rangle$ the duality pairing. The interval $[a, b] = \{a + t(b - a); \ 0 \leq t \leq 1\}$ and $\overline{a, b} = [a, b] \setminus \{a, b\}$. The open ball centered at $x$ with radius $r$ is denoted by $B_r(x)$. We recall that $f$ is quasiconvex if for any $x, y \in X$ and any $z \in [x, y]$,

$$f(z) \leq \max\{f(x), f(y)\}.$$ 

This is equivalent to the convexity of the level sets

$$S_\lambda(f) = \{x \in X; f(x) \leq \lambda\}, \quad \forall \lambda \in \mathbb{R}.$$ 

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We will also use the notations \( \tilde{S}_\lambda(f) = \{ x \in X : f(x) < \lambda \} \), \( L_f(x_0) = \{ x \in X : f(x) = f(x_0) \} \). The mapping \( f \) is lower semicontinuous (l.s.c.) if \( \tilde{S}_\lambda(f) \) is closed for any \( \lambda \in \mathbb{R} \). When \( f \) is l.s.c., the Clarke-Rockafellar generalized derivative at \( x \) along the direction \( v \) is defined by

\[
 f'(x; v) = \sup_{\varepsilon > 0} \limsup_{y \to f(x)} \inf_{t \in \mathbb{R}} \frac{f(y + tu) - f(y)}{t},
\]

where \( y \to f \) means that \( y \to x \) and \( f(y) \to f(x) \). The Clarke-Rockafellar subdifferential of \( f \) at \( x \) is

\[
 \partial f(x) = \{ x^* \in X^* : \langle x^*, v \rangle \leq f'(x; v), \forall v \in X \}
\]

with the convention that \( \partial f(x) \) is empty if \( f \) is not finite at \( x \). And last, the normal cone of \( f \) to the convex set \( C \) at \( x_0 \) is defined by

\[
 N(C; x_0) = \{ x^* \in X^* : \langle x^*, x - x_0 \rangle \leq 0, \forall x \in C \}.
\]

2. Minimization of quasiconvex functions

The main result in this note is a necessary optimality condition for (\( \mathcal{P} \)) when \( f \) is l.s.c., quasiconvex and \( C \) is any nonempty closed convex set of \( X \).

**Theorem 2.1.** Let \( X \) be a Banach space, \( f : X \to \mathbb{R} \cup \{ +\infty \} \) a l.s.c. quasiconvex function. Consider \( x_0 \in C \) such that

(i) \( \tilde{S}_f(x_0)(f) \) is nonempty and open in \( X \).

(ii) \( \partial f(x_0) \) is nonempty and \( w^* \)-compact in \( X^* \).

(iii) There is \( \nu > 0 \) such that

\[
 \forall x \in B_\nu(x_0) \cap L(x_0), \quad 0 \notin \partial f(x).
\]  

(2.1)

Then, a necessary condition for \( x_0 \) to be a solution of (\( \mathcal{P} \)) is that

\[
 0 \in \partial f(x_0) + N(C; x_0).
\]  

(2.2)

We will need in the sequel the following technical result.

**Lemma 2.1.** Let \( X \) be a Banach space, \( f : X \to \mathbb{R} \cup \{ +\infty \} \) a l.s.c. quasiconvex function.

(i) If \( \partial f(x_0) \) is nonempty and there exists \( r > 0 \) such that \( 0 \notin \partial f(x) \) for all \( x \in B_r(x_0) \cap L_f(x_0) \), then

\[
 N(S_f(x_0)(f); x_0) = \text{Cl}(\mathbb{R}^+ \partial f(x_0)).
\]

(ii) If moreover, \( \partial f(x_0) \) is \( w^* \)-compact, then

\[
 N(S_f(x_0)(f); x_0) = \mathbb{R}^+ \partial f(x_0).
\]

**Proof.** The point (i) is [4, Proposition 2.2].

(ii) Consider a sequence \( (\lambda_n, x_n^*) \subseteq \mathbb{R}^+ \partial f(x_0) \) such that \( \lambda_n x_n^* \to y^* \). We will show that \( y^* = \lambda x^* \) for some \( \lambda \in \mathbb{R}^+ \) and \( x^* \in \partial f(x_0) \).

Since \( x_n^* \in \partial f(x_0) \) which is \( w^* \)-compact, for a subsequence still denoted \( (x_n^*) \), \( x_n^* \to x^* \in \partial f(x_0) \).

**Claim 2.1.** There is a subsequence of \( (\lambda_n) \), still denoted \( (\lambda_n) \), that is bounded.
Indeed, \(0 \notin \partial f(x_0)\). By the Hahn-Banach theorem, there is \(v \in X\) such that
\[
\langle z^*, v \rangle > 0, \quad \forall z^* \in \partial f(x).
\] (2.3)

But \(\lambda_n x_n^* \to y^*\), so there is \(M > 0\) such that
\[
M \geq \langle \lambda_n x_n^*, v \rangle.
\]
If \((\lambda_n)_n\) was not bounded, for some subsequence, still denoted \((\lambda_n)_n\), we would get
\[
\frac{M}{\lambda_n} \geq \langle x_n^*, v \rangle > 0.
\]
At the limit, we get a contradiction with (2.3).

\[\square\]

**Proof of Theorem 2.1.** Suppose that \(x_0\) minimizes \(f\) on \(C\). Then, \(C \cap \tilde{S}_f(x_0)(f) = \emptyset\).

But \(\tilde{S}_f(x_0)(f) \cap B_{\nu/4}(x_0) \neq \emptyset\) because otherwise, \(x_0\) would be a local minimum of \(f\) and hence we would get \(0 \in \partial f(x_0)\), a contradiction with (iii). Moreover,
\[
(C \cap Cl(B_{\nu/2}(x_0))) \cap (\tilde{S}_f(x_0)(f) \cap B_{\nu/2}(x_0)) = \emptyset.
\]

By (i) and using the Hahn-Banach theorem, there is \(u^* \in X^*\) such that \(u^* \neq 0\) and \(\alpha \in \mathbb{R}\) separating our two convex sets:
\[
\langle u^*, x \rangle \leq \alpha, \quad \forall x \in \tilde{S}_f(x_0)(f) \cap B_{\nu/2}(x_0),
\] (2.4)
\[
\langle u^*, x \rangle \geq \alpha, \quad \forall x \in C \cap B_{\nu/2}(x_0),
\] (2.5)

We claim that \(\langle u^*, x_0 \rangle = \alpha\). Indeed, it is clear that \(\langle u^*, x_0 \rangle \geq \alpha\). It suffices to check the other sense.

Let us first show the equality
\[
Cl(\tilde{S}_f(x_0)(f)) \cap B_{\nu/2}(x_0) = S_f(x_0)(f) \cap B_{\nu/2}(x_0).
\] (2.6)

Indeed, the sense “\(\subset\)” is obvious. For the inverse inclusion, suppose by contradiction that there is \(y \in \left(\text{Cl}(\tilde{S}_f(x_0)(f)) \cap B_{\nu/2}(x_0)\right) \setminus (S_f(x_0)(f) \cap B_{\nu/2}(x_0))\). Then, \(y \in L_f(x_0) \cap B_{\nu/2}(x_0)\) and it is a local minimum of \(f\). So \(0 \in \partial f(y)\), a contradiction with (iii).

By (2.6), there is a sequence \((x_n)_n \subset \tilde{S}_f(x_0)(f) \cap B_{\nu/2}(x_0)\) such that \(x_n \to x_0\) and hence \(\langle u^*, x_0 \rangle \leq \alpha\).

Using (2.4), \(\langle u^*, x_0 \rangle = \alpha\) and (2.6), we get
\[
u^* \in \begin{cases}
N(\tilde{S}_f(x_0)(f) \cap B_{\nu/2}(x_0); x_0) & \text{property of normal cones} \\
N(\text{Cl}(\tilde{S}_f(x_0)(f)) \cap B_{\nu/2}(x_0); x_0) & \text{by (2.6)} \\
N(S_f(x_0)(f) \cap B_{\nu/2}(x_0); x_0) & \text{property of normal cones} \\
N(S_f(x_0)(f); x_0)
\end{cases}
\]

By (ii) of Lemma 2.1, we have
\[
u^* \in \mathbb{R}^+ \partial f(x_0).
\]

And by (2.5),
\[-\nu^* \in N(C; x_0).
\]
Since \(\nu^* \neq 0\), we finally get
\[
0 \in \partial f(x_0) + N(C; x_0).
\]
\[\square\]
This theorem refines the results of Clarke [1] and Huriart-Urruty [2] when we require the quasiconvexity of the objective function $f$. In the case where the general convex set $C$, appearing in the former result, is defined as the constraint set

$$C = \{ x \in X; g(x) \leq 0 \},$$

where $g$ is quasiconvex, l.s.c. and satisfies (ii), (iii) of Theorem 1, and $g(x_0) = 0$, we obtain the following result where appears some Lagrange multiplier.

**Corollary 2.1.** A necessary condition for $x_0$ to solve (P) is

$$0 \in \partial f(x_0) + \lambda \partial g(x_0),$$

for some $\lambda > 0$.

Note that $g(x_0) = 0$ is not a problem, we could always use $h(t) = g(x) - g(x_0)$. For the proof, it suffices to use Theorem 2.1 and Lemma 2.1(ii).

The case where $f$ is pseudoconvex is investigated in an other paper [3].

**References**


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