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Weigthed elliptic equation of Kirchhoff type with exponential non linear growth

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Abstract. This work is concerned with the existence of a positive ground state solution for the following non local weighted problem

$$\begin{cases} L_{(\sigma,V)}u &=& f(x,u) & \text{in } B\\ u &>& 0 & \text{in } B\\ u &=& 0 & \text{on } \partial B, \end{cases}$$

where

$$L_{(\sigma,V)}u := g(\int_B(\sigma(x)|\nabla u|^N + V(x)|u|^N)dx)\big[-\operatorname{div}(\sigma(x)|\nabla u|^{N-2}\nabla u) + V(x)|u|^{N-2}u\big],$$

B is the unit ball of $\mathbb{R}^N$, $N > 2$, $\sigma(x) = \Big(\log(\frac{e}{|x|})\Big)^{\beta(N-1)}$, $\beta \in [0,1)$ the singular logarithm weight , $V(x)$ is a positif continuous potential.The Kirchhoff function g is positive and continuous on $(0,+\infty)$. The nonlinearities are critical or subcritical growth in view of Trudinger-Moser inequalities of exponential type. We prove the existence of a positive ground state solution by using Mountain Pass theorem . In the critical case, the Euler-Lagrange function loses compactness except for a certain level. We dodge this problem by using adapted test functions to identify this level of compactness.

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1. Introduction

In this paper we study the following non local weighted problem

$$\begin{cases} L_{(\sigma,V)} &=& f(x,u) & \text{in } B\\ u &>& 0 & \text{in } B\\ u &=& 0 & \text{on } \partial B, \end{cases} \tag{1}$$

where

$$L_{(\sigma,V)} := g(\int_B(\sigma(x)|\nabla u|^N + V(x)|u|^N)dx)\big[-\operatorname{div}(\sigma(x)|\nabla u|^{N-2}\nabla u) + V(x)|u|^{N-2}u\big],$$

B is the unit ball of $\mathbb{R}^N$, $N > 2$, $f(x;t)$ is continuous in $B \times \mathbb{R}$ and behaves like $e^{\alpha t^{\frac{N}{(N-1)(1-\beta)}}}$ as $t \to +\infty$, for some $\alpha > 0$ and $V : B \to \mathbb{R}$ is a positive continuous function satisfying some conditions. The weight $\sigma(x)$ is given by

$$\sigma(x) = \Big(\log(\frac{e}{|x|})\Big)^{\beta(N-1)},\ \beta \in [0,1)\cdot \tag{2}$$

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The Kirchhoff function g is a continuous positive on $(0, +\infty)$, satisfying some mild conditions.
We give an historical survey of Kirchhoff's work. In 1883 Kirchhoff [24] studied the following parabolic problem

$$\mu \frac{\partial^2 u}{\partial t^2} - \big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\big)\frac{\partial^2 u}{\partial x^2} = 0\cdot \tag{3}$$

The last equation is an extension of the Dalembert wave equation, by including the effects of changes in string length during vibration. The parameters in equation (3) have the following meanings: L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, μ is the mass density and P_0 is the initial tension.
These kinds of problems have physical motivations. Indeed, the Kirchhoff operator $m(\big(\int_B |\nabla u|^2 dx\big))\Delta u$ (m is a kirchhoff function) also appears in the nonlinear vibration equation namely

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - m\big(\int_B |\nabla u|^2 dx\big)\mathrm{div}(\nabla u) & = & f(x,u) & \text{in} & B \times (0,T) \\ u & > & 0 & \text{in} & B \times (0,T) \\ u & = & 0 & \text{on} & \partial B \\ u(x,0) & = & u_0(x) & \text{in} & B \\ \frac{\partial u}{\partial t}(x,0) & = & u_1(x) & \text{in} & B, \end{cases} \tag{4}$$

which have focused the attention of several researchers, mainly as a result of the work of Lions [27]. We mention that non-local problems also arise in other areas, e.g. biological systems where the function u describes a process that depends on the average of itself (for example, population density), see e.g. [3, 4] and its references.
In the non weighted case ie when $\sigma(x) = 1, V(x) = 0$ and when $N = 2$, problem (1) can be seen as a stationary version of the evolution problem (4). For instance, in (1) if we set $\sigma(x) = 1, N = 2$ and $g(t) = 1$, then we find the classical Schrödinger equation $-\triangle u + V(x)u = f(x,u)$.
Also, if we take $\sigma(x) = 1$, $N = 2$, $V(x) = 0$ and $g(t) = \bar{a} + \bar{b}t$, with $\bar{a}, \bar{b} > 0$, we find Kirchhoff's classical equation which has been extensively studied. We refer to the work of Chipot [17, 18], Corrêa et al [24] and their references.
We point out that recently, in the case $g(t) = 1$, $V(x) = 0$, $N = 2$ and $\beta = 1$ the following problem

$$\begin{cases} -\mathrm{div}(\omega(x)\nabla u) & = & g(x,u) & \text{in} & B_1 \\ u & > & 0 & \text{in} & B_1 \\ u & = & 0 & \text{on} & \partial B_1, \end{cases}$$

where B_1 is the unit disk of $\mathbb{R}^2$, $\omega(x) = \log(\frac{e}{|x|})$ and the nonlinearity g behaves like $\exp\{e^{\alpha t^2}\}$ as $t \to +\infty$, for some $\alpha > 0$, was studied in [13].
In order to motivate our study, we begin by giving a brief survey on Trudinger-Moser inequalities. Since 1970, when Moser gave the famous result on the Trudinger-Moser inequality, many applications have taken place such as in the theory of conformal deformation on collectors, the study of the prescribed Gauss curvature and the mean field equations. After that, a logarithmic Trudinger-Moser inequality was used in

crucial way in [26] to study the Liouville equation of the form

$$\begin{cases} -\Delta u &= \lambda \frac{e^u}{\int_\Omega e^u} & \text{in} & \Omega \\ u &= 0 & \text{on} & \partial\Omega, \end{cases} \tag{5}$$

where Ω is an open domain of $\mathbb{R}^N$, $N \geq 2$ and λ a positive parameter.

The equation (5) has a long history and has been derived in the study of multiple condensate solution in the Chern-Simons-Higgs theory [29, 30] and also, it appeared in the study of Euler Flow [8, 9, 15, 23].

Later, the Trudinger-Moser inequality was improved to a weighted inequalities [1, 10, 11, 14]. The influence of the weight in the Sobolev norm was studied as the compact embedding in [25].

When the weight is of logarithmic type, Calanchi and Ruf [12] extend the Trudinger-Moser inequality and give some applications when $N = 2$ and for prescribed nonlinearities. After that, Calanchi et al. [13] consider a more general nonlinearities and prove the existence of radial solutions.

We should also refer to the interesting work of Figueiredo and Severo [22] which they studied the following problem

$$\begin{cases} -m\big(\int_B |\nabla u|^2 dx\big)\triangle u &= f(x,u) & \text{in} & \Omega \\ u &> 0 & \text{in} & \Omega \\ u &= 0 & \text{on} & \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, the nonlinearity f behaves like $\exp(\alpha t^2)$ as $t \to +\infty$, for some $\alpha > 0$. $m : (0, +\infty) \to (0, +\infty)$ is a continuous function satisfying some conditions. The authors proved that this problem has a positive ground state solution. The existence result was proved by combining minimax techniques and Trudinger-Moser inequality.

It should be noted that recently , the following nonhomogeneous Kirchhoff-Schrödinger equation

$$\begin{cases} -M(\int_{\mathbb{R}^2} |\nabla u|^2 + V(|x|)u^2 dx)(-\triangle u + V(|x|)u) &= Q(x)\tilde{g}(u) + \varepsilon h(x), \\ u(x) \to 0 \text{ as } |x| \to +\infty, \end{cases}$$

has been studied in [2], where ε is a positive parameter, $M : \mathbb{R}^+ \to \mathbb{R}^+$, $V, Q : (0, +\infty) \to \mathbb{R}$, are continuous functions satisfying some mild conditions. The nonlinearity $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is continuous and behaves like $\exp(\alpha t^2)$ as $t \to +\infty$, for some $\alpha > 0$. The authors proved the existence of at least two weak solutions for this equation by combining the Mountain Pass Theorem and Ekeland's Variational Principle.

Inspired by the works cited above, we investigate our problem in adapted weighted Sobolev space setting. We use Trudinger-Moser inequality to study and prove the existence of solutions to (1).

In literature, more attention has been accorded to the subspace of radial functions

$$W^{1,N}_{0,rad}(B,\sigma) = cl\{u \in C^\infty_{0,rad}(\Omega); \int_\Omega \sigma(x)|\nabla u|^N \, dx < \infty\},$$

endowed with the norm

$$\|u\|_{N,\sigma} = \Big(\int_\Omega \sigma(x)|\nabla u|^N \, dx\Big)^{\frac{1}{N}}.$$

So, we are motivated by the following double exponential inequality proved in [11], which is an improvement of the Trudinger-Moser inequality in a weighted Sobolev space.

Theorem 1.1. [11] *Let $\beta \in [0,1)$ and let σ given by (2), then*

$$\int_B e^{|u|^s} dx < +\infty, \ \forall\, u \in W^{1,N}_{0,rad}(B,\sigma), \text{ if and only if } s \le \gamma_{N,\beta} = \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta} \tag{6}$$

and

$$\sup_{\substack{u \in W^{1,N}_{0,rad}(B,\sigma) \\ \|u\|_{N,\sigma} \le 1}} \int_B e^{\alpha |u|^{\gamma_{N,\beta}}} dx < +\infty \ \Leftrightarrow \ \alpha \le \alpha_{N,\beta} = N[\omega_{N-1}^{\frac{1}{N-1}}(1-\beta)]^{\frac{1}{1-\beta}} \tag{7}$$

where ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$ and N' is the Hölder conjugate of N.

The major difficulty in this problem lies in the concurrence between the growths of g and f. To avoid this difficulty, many authors usually assume that g is increasing or bounded.(see[3, 4, 22]).
Let us now state our results.
We impose the following conditions for the Kirchhoff function g. So, we define the function

$$G(t) = \int_0^t g(s)ds,$$

where the function g is continuous on $\mathbb{R}^+$ and verifies :

(G_1): There exists $g_0 > 0$ sucht that $g(t) \ge g_0$ for all $t \ge 0$ and

$$G(t+s) \ge G(t) + G(s) \ \forall\ s,t \ge 0;$$

(G_2): There exists constants $a_1, a_2 > 0$ and $t_1 > 0$ such that for some $\delta \in \mathbb{R}$,

$$g(t) \le a_1 + a_2 t^\delta, \ \forall t \ge t_1;$$

(G_3): $\dfrac{g(t)}{t}$ is nonincreasing for $t > 0$.

As a consequence of (G_3), a simple calculation shows that

$$\frac{1}{N}G(t) - \frac{1}{2N}g(t)t \text{ is nondecreasing for } t \ge 0.$$

Consequently, one has

$$\frac{1}{N}G(t) - \frac{1}{2N}g(t)t \ge 0, \ \forall t \ge 0. \tag{8}$$

A typical example of a function g fulfilling the conditions (G_1), (G_2) and (G_3) is given by

$$g(t) = g_0 + at\ , g_0, a > 0.$$

Another example is given by $g(t) = 1 + \ln(1+t)$.
Let $\gamma := \gamma_{N,\beta} = \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta}$. In view of inequalities (5) and (6), we say that f has subcritical growth at $+\infty$ if

$$\lim_{s \to +\infty} \frac{|f(x,s)|}{e^{\alpha s^\gamma}} = 0, \text{ for all } \alpha \text{ such that } \alpha_{N,\beta} \ge \alpha > 0 \tag{9}$$

and f has critical growth at $+\infty$ if there exists some $0<\alpha_0\leq\alpha_{N,\beta}$,

$$\lim_{s\to+\infty}\frac{|f(x,s)|}{e^{\alpha s^{\gamma}}}=0,\ \forall\,\alpha \text{ such that } \alpha_0\leq\alpha\leq\alpha_{N,\beta} \text{ and } \lim_{s\to+\infty}\frac{|f(x,s)|}{e^{\alpha s^{\gamma}}}=+\infty,\ \forall\,\alpha<\alpha_0. \tag{10}$$

For this paper, we hypothesize that the nonlinearity $f(x,t)$ verifies the following assumptions.

(A_1) The non-linearity $f:\overline{B}\times\mathbb{R}\to\mathbb{R}$ is positive, continuous, radial in x, and $f(x,t)=0$ for $t\leq 0$.

(A_2) There exist $t_0>0$ and $M_0>0$ such that for all $t>t_0$ and for all $x\in B$ we have

$$0<F(x,t)\leq M_0 f(x,t),$$

where

$$F(x,t)=\int_0^t f(x,s)ds.$$

(A_3) For each $x\in B$, $\dfrac{f(x,t)}{t^{2N-1}}$ is increasing for $t>0$.

(A_4)

$$\lim_{t\to\infty}\frac{f(x,t)t}{e^{\alpha_0 t^{\gamma}}}\geq\gamma_0 \quad\text{uniformly in } x,\ \text{with } \gamma_0>\frac{g\big((\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}\big)(1-\beta)^{N-1}N^{(N-1)(1-\beta)}}{\alpha_0^{(N-1)(1-\beta)}}$$

The condition (A_2) implies that for any $\varepsilon>0$, there exists a real $t_\varepsilon>0$ such that

$$F(x,t)\leq\varepsilon t f(x,t),\ \forall|t|>t_\varepsilon,\ \text{uniformly in } x\in B. \tag{11}$$

Also, we have that the condition (A_3) leads to

$$\lim_{t\to 0}\frac{f(x,t)}{t^{\theta}}=0 \text{ for all } 0\leq\theta<2N-1. \tag{12}$$

The asymptotic condition (A_4) would be crucial to identify the minimax level of the energy associated to the problem (1).

An example of such non-linearity, is given by $f(t)=F'(t)$, with $F(t)=\frac{t^{2N+2}}{2N+2}+t^{\tau}e^{\alpha_0 t^{\gamma}}$, where $\tau>2N$. A simple calculation shows that f verifies the conditions (A_1), (A_2), (A_3) and (A_4).

The potential V is continuous on $\overline{B}$ and verifies

(V_1): $V(x)\geq V_0>0$ in B for some $V_0>0$. As a consequence we have that the function $\dfrac{1}{V}$ belongs to $L^{\frac{1}{N-1}}(B)$.

To study the solvability of the problem (1), consider the space

$$\mathbf{W}=\{u\in W^1_{0,rad}\mid\int_B V(x)|u|^N dx<+\infty\},$$

endowed with the norm

$$\|u\|=\Big(\int_B\sigma(x)|\nabla u|^N dx+\int_B V(x)|u|^N dx\Big)^{\frac{1}{N}}.$$

We say that u is a solution to the problem (1), if u is a weak solution in the following sense.

Definition 1.1. A function u is called a solution to (1) if $u \in \mathbf{W}$ and

$$g(\|u\|^N)\Big[\int_B \big(\sigma(x)|\nabla u|^{N-2}\,\nabla u\,\nabla\varphi + V|u|^{N-2}u\varphi\big)\,dx\Big] = \int_B f(x,u)\,\varphi\,dx, \text{ for all } \varphi \in \mathbf{W}.$$

It is clear that finding weak solutions of the problem (1) is equivalent to finding nonzero critical points of the following functional on $\mathbf{W}$:

$$\mathcal{E}(u) = \frac{1}{N}G(\|u\|^N) - \int_B F(x,u)dx, \tag{13}$$

where $F(x,u) = \displaystyle\int_0^u f(x,t)dt$.
In order to find critical points of the functional $\mathcal{E}$ associated with (1), one generally applies the mountain pass given by Ambrosetti and Robinowitz, see [5].

Definition 1.2. A solution u is a ground state solution of the problem (1), if u is a solution and

$$\mathcal{E}(u) = r, \text{ with } r = \inf_{u\in\mathcal{S}} \mathcal{E}(u) \text{ where } \mathcal{S} = \{u \in \mathbf{W} : \mathcal{E}'(u) = 0, u \neq 0\}, \tag{14}$$

and

$$\mathcal{E}'(u)\varphi = g(\|u_n\|^N)\int_B \big(\omega(x)\;|\nabla u|^{N-2}\nabla u\;\nabla\varphi\big)\;dx - \int_B f(x,u)\;\varphi\;dx\;,\;\;\varphi \in \mathbf{W}\cdot$$

We start by the first result. In the subcritical exponential growth, we have the following result.

Theorem 1.2. *Let $f(x,t)$ a funtion satisfying (11), (A_1), (A_2) and (A_3). In addition, suppose that (G_1), (G_2), (G_3) and (V_1) hold, then problem* (1) *has a positive ground state solution.*

In the context of the critical exponential growth, the study of the problem (1) becomes more difficult than in the subcritical case. Our Euler-Lagrange function is losing compactness at a certain level. To overcame this lack of compactness, we choose test functions, which are extremal for the Trudinger-Moser inequality (7). Our result is as follows :

Theorem 1.3. *Assume that $f(x,t)$ satisfies (12) and the conditions $(A_1), (A_2), (A_3)$ and (A_4). If in addition(G_1), (G_2), (G_3) and (V_1) are satisfied, then the problem* (1) *has a positive ground state solution.*

To the best of our knowledge, the present papers results have not been covered yet in the literature.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge about functional space. In section 3, we give some useful lemmas for the compactness analysis. In section 4, we prove that the energy $\mathcal{E}$ satisfies the two geometric properties. Section 5 is devoted to estimate the minimax level of the energy. Finally, we conclude with the proofs of the main results in section 6.
Through this paper, the constant C may change from line to another and we sometimes index the constants in order to show how they change.

2. Weighted Lebesgue and Sobolev spaces setting

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain in $\mathbb{R}^N$ and let $\sigma \in L^1(\Omega)$ be a nonnegative function. Following Drabek et al. and Kufner in [19, 25], the weighted Lebesgue space $L^p(\Omega, \sigma)$ is defined as follows:

$$L^p(\Omega, \sigma) = \{u : \Omega \to \mathbb{R} \text{ measurable; } \int_\Omega \sigma(x)|u|^p \, dx < \infty\},$$

for any real number $1 \leq p < \infty$.
This is a normed vector space equipped with the norm

$$\|u\|_{p,\sigma} = \Big(\int_\Omega \sigma(x)|u|^p \, dx\Big)^{\frac{1}{p}}$$

and for $\sigma(x) = 1$, we find the standard Lebesgue space $L^p(\Omega)$ and its norm

$$\|u\|_p = \Big(\int_\Omega |u|^p \, dx\Big)^{\frac{1}{p}}.$$

In [19], the corresponding weighted Sobolev space was defined as

$$W^{1,p}(\Omega, \sigma) = \{u \in L^p(\Omega); \ \nabla u \in L^p(\Omega, \sigma)\}$$

and equipped with the norm defined on $W^{1,p}(\Omega)$ by

$$\|u\|_{W^{1,p}(\Omega,\sigma)} = \big(\|u\|_p^p + \|\nabla u\|_{p,\sigma}^p\big)^{\frac{1}{p}}. \tag{15}$$

$L^p(\Omega, \sigma)$ and $W^{1,p}(\Omega, \sigma)$ are separable, reflexive Banach spaces provided that $\sigma(x)^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$.

Furthemore, if $\sigma(x) \in L^1_{loc}(\Omega)$, then $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega, \sigma)$ and we can introduce the space $W_0^{1,p}(\Omega, \sigma)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, \sigma)$.
The space $W_0^{1,p}(\Omega, \sigma)$ is equipped with the following norm,

$$\|u\|_{W_0^{1,p}(\sigma,\Omega)} = \Big(\int_\Omega \sigma(x)|\nabla u|^p \, dx\Big)^{\frac{1}{p}}, \tag{16}$$

which is equivalent to the one given by (15).

Also, we will use the space $W_0^{1,N}(\Omega, \sigma)$, which is the closure of $C_0^\infty(\Omega)$ in $W^{1,N}(\Omega, \sigma)$, equipped with the norm

$$\|u\|_{W_0^{1,N}(\Omega,\sigma)} = \Big(\int_\Omega \sigma(x)|\nabla u|^N \, dx\Big)^{\frac{1}{N}}.$$

Let s the real such that

$$s \in (1, +\infty) \text{ and } \sigma^{-s} \in L^1(\Omega). \tag{17}$$

The last condition gives important embedding of the space $W^{1,N}(\Omega, \sigma)$ into usual Lebesgue spaces without weight. More precisely, following [19] we have

$$W^{1,N}(\Omega, \sigma) \hookrightarrow L^N(\Omega) \text{ with compact injection} \tag{18}$$

and

$$W^{1,N}(\Omega, \sigma) \hookrightarrow L^{N+\eta}(\Omega) \text{ with compact injection for } 0 \leq \eta < N(s-1),$$

provided

$$\sigma^{-s} \in L^1(\Omega) \text{ with } s \in (1, +\infty).$$

Let the subspace

$$W^{1,N}_{0,rad}(B,\sigma) = cl\{u \in C^{\infty}_{0,rad}(\Omega); \int_{\Omega} \sigma(x)|\nabla u|^N \ dx < \infty\},$$

with $\sigma(x) = \Big(\log(\frac{1}{|x|})\Big)^{\beta(N-1)}$. Then the space

$$\mathbf{W} = \{u \in W^{1,N}_{0,rad}(B,\sigma) \mid \int_B V(x)|u|^N dx < +\infty\},$$

is a Banach and reflexive space provided (V_1) is satisfied. $\mathbf{W}$ is endowed with the norm

$$\|u\| = \Big(\int_B \sigma(x)|\nabla u|^N dx + \int_B V(x)|u|^N dx\Big)^{\frac{1}{N}},$$

which is equivalent to the following norm

$$\|u\|_{W^{1,N}_{0,rad}(B,\sigma)} = \Big(\int_{\Omega} \sigma(x)|\nabla u|^N \ dx\Big)^{\frac{1}{N}}.$$

3. Preliminary for the compactness analysis

In this section, we will present a number of technical lemmas for our future use. We begin by the radial lemma.

Lemma 3.1. *Assume that V is continuous and verifies (V_1).*
(i) [11] *Let u be a radially symmetric C^1_0 function on the unit ball B. Then we have*

$$|u(x)| \leq \frac{|\log(|x|)|^{\frac{1-\beta}{N'}}}{\omega^{\frac{1}{N}}_{N-1}(1-\beta)^{\frac{1}{N'}}}\|u\|_{W^{1,N}_{0,rad}},$$

where ω_{N-1} is the area of the unit sphere $S_{N-1} \in \mathbb{R}^N$.
(ii) *There exists a positive contant C such that for all $u \in \mathbf{W}$*

$$\int_B V|u|^N dx \leq C\|u\|^N$$

and then the norms $\|.\|$ and $\|.\|_{W^1_{0,rad}(\Omega,w)} = \Big(\int_{\Omega} w(x)|\nabla .|^N \ dx\Big)^{\frac{1}{N}}$ are equivalents.
(iii) *The following embedding is continuous*

$$E \hookrightarrow L^q(B) \textit{ for all } q \geq 1.$$

(iv) *$\mathbf{W}$ is compactly embedded in $L^q(B)$ for all $q \geq 1$.*

Proof. See [11] for the proof.
(ii) From (i) we have for all $u \in \mathbf{W}$,

$$\begin{aligned}\int_B V|u|^N dx &\leq \frac{m}{\omega^{N-1}_{N-1}(1-\beta)^{N-1}}\|u\|^N_{W^{1,N}_{0,rad}}\int_B |\log|x||^{(1-\beta)(N-1)}dx\\ &\leq C\frac{m}{\omega^{N-1}_{N-1}(1-\beta)^{N-1}}\|u\|^N \leq C\|u\|^N\end{aligned}$$

where $m = \max_{x\in\bar{B}} V(x)$, then (ii) follows.
(iii) From (i) and (ii), we have that the following embedding are continuous

$$\mathbf{W} \hookrightarrow W^1_{0,rad}(B) \hookrightarrow L^q(B)\ \forall q \geq N.$$

We have by the Hölder inequality

$$\int_B |u|dx \leq \Big(\int_B \frac{1}{V^{\frac{1}{N-1}}}dx\Big)^{\frac{N-1}{N}}\Big(\int_B V|u|^N dx\Big)^{\frac{1}{N}} \leq \Big(\int_B \frac{1}{V^{\frac{1}{N-1}}}dx\Big)^{\frac{N-1}{N}}\|u\|$$

For any $1<\beta<N$, there holds

$$\int_B |u|^\beta dx \leq \int_B (|u|+|u|^N)dx \leq \Big(\int_B \frac{1}{V^{\frac{1}{N-1}}}dx\Big)^{\frac{N-1}{N}}\|u\| + \frac{1}{V_0}\|u\|^N.$$

Thus we get the continuous embedding $\mathbf{W} \hookrightarrow L^q(B)$ for all $q \geq 1$.
(iv) The above embedding is also compact. Indeed, let $u_k \subset \mathbf{W}$ be a sequence such that $\|u_k\| \leq C$ for all k. Then $\|u_k\|_{W^1_{0,rad}} \leq C$ for all k. On the other hand, we have the following compact embedding[19] $W^1_{0,rad} \hookrightarrow L^q$ for all q such that $1 \leq q < Ns$, with $s>1$, then up to a subsequence, there exists some $u \in W^1_{0,rad}$, such that u_k convergent to u strongly in $L^q(B)$ for all q such that $1 \leq q < Ns$. Without loss of generality, we may assume that

$$\begin{cases} u_k & \rightharpoonup u & \text{weakly in } \mathbf{W} \\ u_k & \to u & \text{strongly in } L^1(B) \\ u_k(x) & \to u(x) & \text{almost everywhere in } B. \end{cases} \tag{19}$$

For $q>1$, it follows from (19) and the continuous embedding $\mathbf{W} \hookrightarrow L^p(B)$ $(p \geq 1)$ that

$$\begin{aligned} \int_B |u_k-u|^q dx &= \int_B |u_k-u|^{\frac{1}{2}}|u_k-u|^{q-\frac{1}{2}}dx \\ &\leq \Big(\int_B |u_k-u|dx\Big)^{\frac{1}{2}}\Big(\int_B |u_k-u|^{2q-1}dx\Big)^{\frac{1}{2}} \\ &\leq C\Big(\int_B |u_k-u|dx\Big)^{\frac{1}{2}} \to 0. \end{aligned}$$

This concludes the lemma. □

Next, an important lemma.

Lemma 3.2. [21] *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $f : \overline{\Omega}\times\mathbb{R}$ a continuous function. Let $\{u_n\}_n$ be a sequence in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x,u_n)$ and $f(x,u)$ are also in $L^1(\Omega)$. If*

$$\int_\Omega |f(x,u_n)u_n|dx \leq C,$$

where C is a positive constant, then

$$f(x,u_n) \to f(x,u) \text{ in } L^1(\Omega).$$

In an attempt to prove a compactness condition for the energy $\mathcal{E}$, we need a lions type result [28] about an improved Trudinger-Moser inequality when we deal with weakly convergent sequences and double exponential case.

Lemma 3.3. *Let $\{u_k\}_k$ be a sequence in $\mathbf{W}$. Suppose that $\|u_k\|=1$, $u_k \rightharpoonup u$ weakly in $\mathbf{W}$, $u_k(x)\to u(x)$ a.e $x\in B$, $\nabla u_k(x)\to\nabla u(x)$ a.e $x\in B$ and $u\not\equiv 0$. Then*

$$\sup_k \int_B e^{p\ \alpha_{N,\beta}|u_k|^\gamma}dx<+\infty$$

for all $1<p<U$ where U is given by:

$$U=\begin{cases}\dfrac{1}{(1-\|u\|^N)^{\frac{\gamma}{N}}} & \text{if } \|u\|<1\\ +\infty & \text{if } \|u\|=1\end{cases}$$

Proof. For $a,\ b\in\mathbb{R}$, $q>1$. If q' its conjugate i.e. $\frac{1}{q}+\frac{1}{q'}=1$ we have, by young inequality, that

$$e^{a+b}\le\frac{1}{q}e^{qa}+\frac{1}{q'}e^{q'b}.$$

Also, we have

$$(1+a)^q\le(1+\varepsilon)a^q+(1-\frac{1}{(1+\varepsilon)^{\frac{1}{q-1}}})^{1-q},\ \forall a\ge 0,\ \forall\varepsilon>0\ \forall q>1. \tag{20}$$

So, we get

$$\begin{aligned}|u_k|^\gamma &= |u_k-u+u|^\gamma\\ &\le (|u_k-u|+|u|)^\gamma\\ &\le (1+\varepsilon)|u_k-u|^\gamma+\big(1-\tfrac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}}\big)^{1-\gamma}|u|^\gamma.\end{aligned}$$

which implies that

$$\begin{aligned}\int_B e^{p\ \alpha_{N,\beta}|u_k|^\gamma}dx\le{}&\frac{1}{q}\int_B e^{pq\ \alpha_{N,\beta}(1+\varepsilon)|u_k-u|^\gamma}dx\\ &+\frac{1}{q'}\int_B e^{pq'\ \alpha_{N,\beta}(1-\frac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}})^{1-\gamma}|u|^\gamma}dx,\end{aligned}$$

for any $p>1$.
From (6), the last integral is finite. To complete the proof we have to prove that for every p such that $1<p<U$,

$$\sup_k\int_B e^{pq\ \alpha_{N,\beta}(1+\varepsilon)|u_k-u|^\gamma}dx<+\infty, \tag{21}$$

for some $\varepsilon>0$ and $q>1$.
In the following, we suppose that $\|u\|<1$, and in the case of $\|u\|=1$, the proof is similar.
When

$$\|u\|<1$$

and

$$p<\frac{1}{(1-\|u\|^N)^{\frac{\gamma}{N}}}.$$

So, there exists $\nu>0$ such that

$$p(1-\|u\|^N)^{\frac{\gamma}{N}}(1+\nu)<1.$$

On the other hand, by Brezis-Lieb's lemma [7] we have

$$\|u_k-u\|^N=\|u_k\|^N-\|u\|^N+o(1)\text{ where }o(1)\to 0\text{ as }k\to+\infty.$$

Then,

$$\|u_k - u\|^N = 1 - \|u\|^N + o(1),$$

hence,

$$\lim_{k\to+\infty} \|u_k - u\|^\gamma = (1 - \|u\|^N)^{\frac{\gamma}{N}}.$$

Therefore, for every $\varepsilon > 0$, there exists $k_\varepsilon \geq 1$ such that

$$\|u_k - u\|^\gamma \leq (1+\varepsilon)(1 - \|u\|^N)^{\frac{\gamma}{N}}, \ \forall\ k \geq k_\varepsilon.$$

If we take $q = 1 + \varepsilon$ with $\varepsilon = \sqrt[3]{1+\nu} - 1$, then $\forall k \geq k_\varepsilon$, we have

$$pq(1+\varepsilon)\|u_k - u\|^\gamma \leq 1.$$

Consequently,

$$\begin{aligned}
\int_B e^{pq\ \alpha_{N,\beta}(1+\varepsilon)|u_k-u|^\gamma} dx &\leq \int_B e^{(1+\varepsilon)pq\ \alpha_{N,\beta}(\frac{|u_k-u|}{\|u_k-u\|})^\gamma \|u_k-u\|^\gamma} dx \\
&\leq \int_B e^{\ \alpha_{N,\beta}(\frac{|u_k-u|}{\|u_k-u\|})^\gamma} dx \\
&\leq \sup_{\|u\|\leq 1} \int_B e^{\ \alpha_{N,\beta}|u|^\gamma} dx < +\infty.
\end{aligned}$$

Therefore, (21) hold and the lemma is proved. □

4. The mountain pass geometry of the energy

Since the nonlinearity f is critical or subcritical at $+\infty$, there exist $a,\ C > 0$ positive constants and there exists $t_2 > 1$ such for that

$$|f(x,t)| \leq Ce^{a\ t^\gamma},\ \forall\ |t| > t_2. \tag{22}$$

So the functional $\mathcal{E}$ given by (13) is well defined and of class C^1.
In order to prove the existence of a ground state solution to the problem (1), we will prove the existence of nonzero critical point of the functional $\mathcal{E}$ by using the theorem introduced by Ambrosetti and Rabionowitz in [5] (Mountain Pass Theorem) without the Palais-Smale condition.

Theorem 4.1. [5] *Let E be a Banach space and $J : E \to \mathbb{R}$ a $\mathcal{C}^1$ functional satisfying $J(0) = 0$. Suppose that there exist $\rho,\ \bar{\beta}_0 > 0$ and $e \in E$ with $\|e\| > \rho$ such that*

$$\inf_{\|u\|=\rho} J(u) \geq \beta_0 \ \text{and}\ J(e) \leq 0.$$

Then there is a sequence $(u_n) \subset E$ such

$$J(u_n) \to \bar{c} \ \text{and}\ J'(u_n) \to 0,$$

where

$$\bar{c} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \geq \bar{\beta}_0$$

and

$$\Gamma := \{\gamma \in C([0,1], E) \ \text{such that}\ \gamma(0) = 0 \ \ \text{and}\ \ \gamma(1) = e\}.$$

The number $\bar{c}$ is called mountain pass level or minimax level of the functional J.

Before starting the proof of the geometric properties for the functional $\mathcal{E}$, it follows from the continuous embedding $\mathbf{W} \hookrightarrow L^q(B)$ for all $q \geq 1$, that there exists a constant $C > 0$ such that $\|u\|_{N'q} \leq c\|u\|$, for all $u \in \mathbf{W}$.

In the next Lemmas, we prove that the functional $\mathcal{E}$ has the mountain pass geometry of the theorem 4.1.

Lemma 4.2. *Suppose that f has critical growth at $+\infty$. In addition if $(A_1), (A_3)$ (G_1) and (V_1) hold, then, there exist ρ, $\beta_0 > 0$ such that $\mathcal{J}(u) \geq \beta_0$ for all $u \in \mathbf{W}$ with $\|u\| = \rho$.*

Proof. It follows from (9) that there exists $\delta_0 > 0$

$$F(x,t) \leq \epsilon |t|^N, \text{ for } |t| < \delta_0. \tag{23}$$

From (A_3) and (22), for all $q > N$, there exist a positive constant $C > 0$ such that

$$F(x,t) \leq C|t|^q e^{a\ t^\gamma}, \ \forall\ |t| > \delta_1. \tag{24}$$

Using (23), (24) and the continuity of F, we get for all $q > N$,

$$F(x,t) \leq \epsilon |t|^N + C|t|^q e^{a\ t^\gamma}, \text{ for all } t \in \mathbb{R}. \tag{25}$$

Since

$$\mathcal{E}(u) = \frac{1}{N} G(\|u\|^N) - \int_B F(x,u)dx,$$

we get from (G_1) and (25)

$$\mathcal{E}(u) \geq \frac{g_0}{N}\|u\|^N - \varepsilon \int_B |u|^N dx - C\int_B |u|^q e^{a\ |u|^\gamma}\ dx.$$

From the Hölder inequality, we obtain

$$\mathcal{E}(u) \geq \frac{g_0}{N}\|u\|^N - \varepsilon \int_B |u|^N dx - C\Big(\int_B e^{Na\ |u|^\gamma} dx\Big)^{\frac{1}{N}} \|u\|^q_{N'q}.$$

From the Theorem 1.1, if we choose $u \in \mathbf{W}$ such that

$$a\|u\|^\gamma \leq \alpha_{N,\beta}, \tag{26}$$

we get

$$\int_B e^{a\ |u|^\gamma} dx = \int_B e^{a\ \|u\|^\gamma (\frac{|u|}{\|u\|})^\gamma}) dx < +\infty.$$

On the other hand $\|u\|_{N'q} \leq C_1\|u\|$, so

$$\mathcal{E}(u) \geq \frac{g_0}{N}\|u\|^N - \epsilon C_1 \|u\|^N - C\|u\|^q = \|u\|^N\big(\frac{g_0}{N} - \epsilon C_1 - C\|u\|^{q-N}\big),$$

for all $u \in \mathbf{W}$ satisfying (26). Since $N < q$, we can choose $\rho = \|u\| \leq \big(\frac{\alpha_{N,\beta}}{aN}\big)^{\frac{1}{\gamma}}$ and for fixed ϵ such that $\dfrac{g_0}{N} - \epsilon C_1 > 0$, there exists

$$\beta_0 = \rho^N\big(\frac{g_0}{N} - \epsilon C_1 - C\rho^{q-N}\big) > 0 \ \text{ with } \mathcal{E}(u) \geq \beta_0 > 0.$$

□

By the following Lemma, we prove the second geometric property for the functional $\mathcal{E}$.

Lemma 4.3. *Suppose that* (A_1), (A_2), (V_1) *and* (G_2) *hold. Then there exists* $e \in \mathbf{W}$ *with* $\mathcal{E}(e) < 0$ *and* $\|e\| = \rho$.

Proof. From the condition (G_2), for all $t \geq t_1$, we have that

$$G(t) \leq \begin{cases} a_0 + a_1 t + \frac{a_2}{\delta+1} t^{\delta+1} & \text{if} \quad \delta \neq -1 \\ b_0 + a_1 t + a_2 \ln t & \text{if} \quad \delta = -1, \end{cases} \tag{27}$$

where $a_0 = \int_0^{t_1} g(t)dt - a_1 t_1 - a_2 \frac{t_1^{\delta+1}}{\delta+1}$ and $b_0 = \int_0^{t_1} g(t)dt - a_1 t_1 - a_2 \ln t_1$.
It follows from the condition (A_3), that

$$\lim_{t\to+\infty} \frac{F(x,t)}{t^N} = +\infty \text{ uniformly in } \ x \in B.$$

In particular, for $p > \max(N, N(\delta+1))$ there exists C_1 and C_2 such that

$$F(x,t) \geq C_1 |t|^p - C_2 \geq C_1 |t|^p - C_2, \ \forall t \in \mathbb{R}, \ x \in B. \tag{28}$$

Next, one arbitrarily picks $\bar{u} \in \mathbf{W}$ such that $\|\bar{u}\| = 1$. Thus from (27) and (28) for all $t \geq t_1$

$$\mathcal{E}(t\bar{u}) \leq \begin{cases} \dfrac{a_0}{2} + \dfrac{a_1}{N} t^N + \dfrac{a_2}{N(\delta+1)} t^{N(\delta+1)} - C_1 \|\bar{u}\|_p^p \, t^p - \dfrac{\omega_{N-1}}{N} C_2 & \text{if} \quad \delta \neq -1, \\ \dfrac{b_0}{N} + \dfrac{a_1}{N} t^N + \dfrac{a_2}{N} \ln^N t - C_1 \|\bar{u}\|_p^p \, t^p - \dfrac{\omega_{N-1}}{N} C_2 & \text{if} \quad \delta = -1, \end{cases}$$

Therefore,

$$\lim_{t\to+\infty} \mathcal{E}(t\bar{u}) = -\infty.$$

We take $e = \bar{t}\bar{u}$, for some $\bar{t} > 0$ large enough. So, the Lemma 4.3 follows. $\square$

5. The minimax estimate of the energy

According to Lemmas 4.2 and 4.3, let

$$d := \inf_{\gamma\in\Lambda} \max_{t\in[0,1]} \mathcal{E}(\gamma(t)) > 0 \tag{29}$$

and

$$\Lambda := \{\gamma \in C([0,1], \mathbf{W}) \text{ such that } \gamma(0) = 0 \ \text{ and } \ \mathcal{E}(\gamma(1)) < 0\}.$$

We are going to estimate the minimax value of the functional $\mathcal{E}$. The idea is to construct a sequence of functions $(v_n) \in \mathbf{W}$ and estimate $\max\{\mathcal{E}(tv_n) : t \geq 0\}$. For this goal, let consider the following Moser function

$$v_n(x) = \frac{N^{1-\beta}}{\alpha_{N,\beta}^{\frac{1}{\gamma}}} \begin{cases} \dfrac{(\log \frac{1}{|x|})^{1-\beta}}{n^{\frac{1-\beta}{N}}} & \text{if} \ \ e^{-\frac{n}{N}} \leq |x| \leq 1 \\ \dfrac{1}{N^{(1-\beta)}} \, n^{\frac{1}{\gamma}} & \text{if} \ \ 0 \leq |x| \leq e^{-\frac{n}{N}}. \end{cases} \tag{30}$$

Let $v_n(x) = \dfrac{w_n(x)}{\|w_n\|}$. Then $v_n \in \mathbf{W}$ and $\|v_n\| = 1$.

5.1. Helpful lemmas. We need two technical lemmas who will help us to reach our aims and objectives.

Lemma 5.1. *Assume that V is continuous and (V_1) is satisfied. Then there holds*
(i)

$$\|v_n\|^\gamma \leq 1 + \frac{mN^{(1-\beta)N}}{(N-1)(1-\beta)\alpha_{N,\beta}^{\frac{N}{\gamma}}}\big(H+Z\big) + o_n(1),$$

$H = \omega_{N-1}n^{(1-\beta)(N-1)}e^{-n(\frac{N}{N-1})}(1-e^{-\frac{n}{N}})$, $Z = \dfrac{1}{N^{(1-\beta)N}}\dfrac{\omega_{N-1}\ n^{(N-1)(1-\beta)}e^{-n}}{N}$
and where $m = \max\limits_{x\in\bar{B}} V(x)$.
In addition,

$$\frac{1}{\|v_n\|^\gamma} \geq 1 - E(N;n;m),$$

$$E := E(N;n;m) = \frac{mN^{(1-\beta)N}}{(N-1)(1-\beta)\alpha_{N,\beta}^{\frac{N}{\gamma}}}\big(H+Z\big) + o_n(1).$$

(ii) $\forall\ x$ *such that* $|x| \leq e^{-\frac{n}{N}}$,

$$\alpha_{N,\beta}w_n^\gamma(x) \geq n(1-E).$$

Proof. (i) We have,

$$\|v_n\|^N = 1 + \int_B V(x)|v_n|^N dx \leq 1 + m\int_B |v_n|^N dx,$$

then,

$$\|v_n\|^N \leq 1 + \frac{mN^{(1-\beta)N}}{\alpha_{N,\beta}^{\frac{N}{\gamma}}}\left\{\int_{e^{\frac{-n}{N}}\leq|x|\leq 1}\frac{(\log\frac{1}{|x|})^{(1-\beta)N}}{n^{1-\beta}}dx + \int_{0\leq|x|\leq e^{\frac{-n}{N}}}\frac{1}{N^{(1-\beta)N}}n^{(N-1)(1-\beta)}dx\right\}.$$

We have

$$\begin{aligned}
\int_{e^{\frac{-n}{N}}\leq|x|\leq 1}\frac{(\log\frac{1}{|x|})^{(1-\beta)N}}{n^{1-\beta}}dx =& \frac{\omega_{N-1}}{n^{(1-\beta)}}\int_{e^{\frac{-n}{N}}}^1 r^{N-1}(\log\frac{1}{r})^{(1-\beta)N}dr \\
\leq& \frac{\omega_{N-1}}{n^{(1-\beta)}}e^{-n(\frac{N}{N-1})}n^{(1-\beta)N}\int_{e^{-\frac{n}{N}}}^1 dr \\
=& \omega_{N-1}n^{(1-\beta)(N-1)}e^{-n(\frac{N}{N-1})}(1-e^{-\frac{n}{N}}) = o_n(1).
\end{aligned}$$

Also,

$$\begin{aligned}
\int_{0\leq|x|\leq e^{-\frac{n}{N}}}\frac{1}{N^{(1-\beta)N}}n^{(N-1)(1-\beta)}dx =& \frac{1}{N^{(1-\beta)N}}\omega_{N-1}\ n^{(N-1)(1-\beta)}\int_0^{e^{-\frac{n}{N}}} r^{N-1}dr \\
=& \frac{1}{N^{(1-\beta)N}}\frac{\omega_{N-1}\ n^{(N-1)(1-\beta)}e^{-n}}{N} = o_n(1).
\end{aligned}$$

Hence,

$$\|v_n\|^N \leq 1 + \frac{mN^{(1-\beta)N}}{\alpha_{N,\beta}^{\frac{N}{\gamma}}}\big(H+Z\big).$$

Therefore,

$$\|v_n\|^{\gamma} \leq \Big(1 + \frac{mN^{(1-\beta)N}}{\alpha_{N,\beta}^{\frac{N}{\gamma}}}\big(H+Z\big)\Big)^{\frac{1}{(N-1)(1-\beta)}}.$$

So,

$$\|v_n\|^{\gamma} \leq 1 + \frac{mN^{(1-\beta)N}}{(N-1)(1-\beta)\alpha_{N,\beta}^{\frac{N}{\gamma}}}\big(H+Z\big) + o_n(1).$$

Consequently,

$$\frac{1}{\|v_n\|^{\gamma}} \geq 1 - E.$$

(ii) We have for all x such that $|x| \leq e^{\frac{-n}{N}}$,

$$\begin{aligned} \alpha_{N,\beta} w_n^{\gamma} &= \alpha_{N,\beta}\frac{|v_n|^{\gamma}}{\|v_n\|^{\gamma}} \\ &\geq n(1-E). \end{aligned}$$

□

Now, we present the second elementary lemma.

Lemma 5.2. *For the sequence w_n induced by (30), we have*

$$\lim_{n\to+\infty}\int_B e^{\alpha_{N,\beta} w_n^{\gamma}}dx \geq \frac{\omega_{N-1}}{N}(N+1). \tag{31}$$

Proof. Let,

$$I_1 = \int_{e^{-\frac{n}{N}}\leq|x|\leq 1} e^{\alpha_{N,\beta} w_n^{\gamma}}dx = \int_{e^{-\frac{n}{N}}\leq|x|\leq 1} e^{N^{N'}\frac{(\log\frac{1}{|x|})^{N'}}{\|v_n\|^{\gamma} n^{\frac{1}{N-1}}}}dx$$

and

$$I_2 = \int_{0\leq|x|\leq e^{-\frac{n}{N}}} e^{\alpha_{N,\beta} w_n^{\gamma}}dx.$$

Then,

$$\lim_{n\to+\infty}\int_B e^{\alpha_{N,\beta}|w_n|^{\gamma}}dx = \lim_{n\to+\infty} I_1 + \lim_{n\to+\infty} I_2.$$

On one hand, from lemma 5.1 (ii) and ussing the fact that $nE \sim n^{N(1-\beta)+\beta}e^{-n}$.

$$I_2 = \int_{0\leq|x|\leq e^{-\frac{n}{N}}} e^{\alpha_{N,\beta} w_n^{\gamma}}dx \geq \int_{0\leq|x|\leq e^{-\frac{n}{N}}} e^{n(1-E)}dx \geq \frac{\omega_{N-1}}{N}e^{-nE} \to \frac{\omega_{N-1}}{N} \text{ as } n\to+\infty.$$

On the other hand,

$$I_1 \geq \int_{e^{-\frac{n}{N}}\leq|x|\leq 1} e^{N^{N'}\frac{(\log\frac{1}{|x|})^{N'}}{n^{\frac{1}{N-1}}}(1-E)}dx,$$

so,

$$I_1 \geq \omega_{N-1}\int_{e^{-\frac{n}{N}}}^{1} r^{N-1}e^{N^{N'}\frac{(\log\frac{1}{r})^{N'}}{n^{\frac{1}{N-1}}}(1-E)}dr.$$

We make the change of variable, $|x| = r = e^{-\frac{t}{N}}$. Then, we get

$$I_1 \geq \frac{\omega_{N-1}}{N}\int_0^n e^{\frac{t^{N'}(1-E)}{n^{\frac{1}{N-1}}}-t}dt.$$

For any $n > 1$, let

$$\varphi_n(t) := \frac{t^{N'}(1-E)}{n^{\frac{1}{N-1}}} - t, \ t \geq 0.$$

The interval $[0, n]$ is then divided as follows:

$$\Big[0, n\Big] = \Big[0, n^{\frac{1}{N}}\Big] \cup \Big[n^{\frac{1}{N}}, n - n^{\frac{1}{N}}\Big] \cup \Big[n - n^{\frac{1}{N}}, n\Big].$$

First, we consider the interval $\Big[0, n^{\frac{1}{N}}\Big]$. Using the fact that $1 - E \leq 1$, we get

$$\chi_{\Big[0,n^{\frac{1}{N}}\Big]} e^{\varphi_n(t)} \leq e^{1-t} \in L^1([0, +\infty)),$$

$$\chi_{\Big[0,n^{\frac{1}{N}}\Big]}(t) e^{\varphi_n(t)} \to e^{-t} \text{ for a.e } t \in [0, +\infty), \text{as } n \to +\infty,$$

then, using the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to+\infty} \int_0^{n^{\frac{1}{N}}} e^{\varphi_n(t)} dt = \lim_{n\to+\infty} \int_0^n \chi_{\Big[0,n^{\frac{1}{N}}\Big]} e^{\varphi_n(t)} dt = 1.$$

Now, we are going to study the limit of this integral on $\Big[n^{\frac{1}{N}}, n - n^{\frac{1}{N}}\Big]$ and $\Big[n - n^{\frac{1}{N}}, n\Big]$. So, we compute

$$\varphi_n\Big(n^{\frac{1}{N}}\Big) = 1 - E - n^{\frac{1}{N}} \leq 1 - n^{\frac{1}{N}}, \text{ for } n \text{ large enough}$$

and

$$\varphi_n\Big(n^{\frac{1}{N}}\Big) \leq -\frac{1}{N-1} n^{\frac{1}{N}} = -(N'-1) n^{\frac{1}{N}} \text{ for all } n \geq (\frac{N-1}{N-2})^{\frac{1}{N}}. \tag{32}$$

Also, for n large enough,

$$\begin{aligned}
\varphi_n\Big(n - n^{\frac{1}{N}}\Big) &= \frac{(n-n^{\frac{1}{N}})^{N'}(1-E)}{n^{\frac{1}{N-1}}} - n + n^{\frac{1}{N}} \\
&\leq \frac{n^{\frac{N}{N-1}}(1 - n^{\frac{-1}{N'}})^{N'}}{n^{\frac{1}{N-1}}} - n + n^{\frac{1}{N}} \\
&= n(1 - \frac{N'}{n^{\frac{1}{N'}}} + o(\frac{1}{n^{\frac{1}{N'}}})) - n + n^{\frac{1}{N}} \\
&= n(-\frac{N'}{n^{\frac{1}{N'}}} + \frac{1}{n^{\frac{1}{N'}}} + o(\frac{1}{n^{\frac{1}{N'}}})) \\
&= -N' n^{\frac{1}{N}} + o(\frac{1}{n^{\frac{1}{N}}})) \leq -(N'-1) n^{\frac{1}{N}} + o(\frac{1}{n^{\frac{1}{N}}})).
\end{aligned}$$

Therefore, for every $\varepsilon \in (0, 1)$ there exists $n_\varepsilon \geq 1$ such that

$$\varphi_n\Big(n - n^{\frac{1}{N}}\Big) \leq -(N'-1) n^{\frac{1}{N}}\big(1 - \varepsilon\big) \text{ for every } \ n \geq n_\varepsilon. \tag{33}$$

Let n fixed and large enough. A qualitative study conducted on φ_n in $[0,+\infty)$, shows that there exists a unique $s_n \in (0,n)$ such that the derivative $\varphi_n'(s_n) = 0$ and consequently

$$\int_{n^{\frac{1}{N}}}^{n-n^{\frac{1}{N}}} e^{\varphi_n(t)}dt \leq \left(n - 2n^{\frac{1}{N}}\right)e^{\max\left[\varphi_n\left(n^{\frac{1}{N}}\right),\varphi_n\left(n-n^{\frac{1}{N}}\right)\right]}.$$

In addition, from (31) and (32) with $\varepsilon < 1$, we obtain

$$\max\left[\varphi_n\left(n^{\frac{1}{N}}\right), \varphi_n\left(n - n^{\frac{1}{N}}\right)\right] \leq -\frac{1}{N-1}n^{\frac{1}{N}},$$

provided that n is large enough. Hence, there exists $\overline{n} \geq 1$ such that

$$\int_{n^{\frac{1}{N}}}^{n-n^{\frac{1}{N}}} e^{\varphi_n(t)}dt \leq \left(n - 2n^{\frac{1}{N}}\right)e^{-\frac{1}{N-1}n^{\frac{1}{N}}} \text{ for all } n \geq \overline{n}.$$

Therefore,

$$\lim_{n\to+\infty}\int_{n^{\frac{1}{n}}}^{n-n^{\frac{1}{n}}} e^{\left(\frac{t^{N'}}{n^{\frac{1}{N-1}}} - t\right)}dt = 0.$$

Finally, we will study the limit on the interval $\left[n - n^{\frac{1}{n}}, n\right]$. We mention that for a fixed $n \geq 1$ large enough, φ_n is a convex function on $\left[n - n^{\frac{1}{N}}, +\infty\right)$. Also, $\varphi_n(n) = n(1-E) - n \leq 0$. Then, we can get the following estimate

$$\varphi_n(t) \leq \varphi_n(t) - \varphi_n(n) \leq \frac{n-t}{n^{\frac{1}{N}}}\varphi_n\left(n - n^{\frac{1}{N}}\right),\ t \in \left[n - n^{\frac{1}{N}}, n\right].$$

On the another hand, in view of (32), if $\varepsilon \in (0,1)$ and $n \geq n_\varepsilon$, we have

$$\varphi_n(t) \leq (N'-1)\big(1-\varepsilon\big)(t-n),\ t \in \left[n - n^{\frac{1}{n}}, n\right]\cdot \tag{34}$$

Furtheremore, using the fact that φ_n is convex on $\left[n - n^{\frac{1}{N}}, +\infty\right)$ and $\varphi_n'(n) = N'(1-E) - 1$, we get

$$\varphi_n(t) \geq \varphi_n(n) + \varphi_n'(n)(t-n) \geq (N'(1-E)-1)(t-n),\ t \in [n - n^{\frac{1}{N}}, n]\cdot \tag{35}$$

Then by bringing together (34) and (35), we deduce

$$\lim_{n\to+\infty}\frac{1-e^{-n^{\frac{1}{N}}}}{(N'(1-E)-1)} \leq \lim_{n\to+\infty}\int_{n-n^{\frac{1}{N}}}^{n} e^{\varphi_n(t)}dt \leq \lim_{n\to+\infty}\frac{1-e^{-n^{\frac{1}{N}}}}{(N'-1)(1-\varepsilon)}\cdot$$

Using the fact that $\lim\limits_{n\to+\infty} E = 0$ and since ε is arbitrary , we get

$$\lim_{n\to+\infty}\int_{n-n^{\frac{1}{N}}}^{n} e^{\varphi_n(t)}dt = \frac{1}{N'-1} = N-1\cdot$$

The lemma follows. $\square$

5.2. The minimax value of the energy $\mathcal{E}$. Finally, we give the desired estimate.

Lemma 5.3. *Assume that (G_1), (G_2), (V_1) and (A_4) holds, then the minimax d defined by (29) verifies*

$$d < \frac{1}{N}G((\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}})\cdot$$

Proof. We have $v_n \geq 0$ and $\|v_n\| = 1$. Then from Lemma 4.3 $\mathcal{E}(tv_n) \to -\infty$ as $t \to +\infty$. As a consequence,

$$d \leq \max_{t\geq 0} \mathcal{E}(tv_n)\cdot$$

We argue by contradiction and suppose that for all $n \geq 1$,

$$\max_{t\geq 0} \mathcal{E}(tv_n) \geq \frac{1}{N}G((\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}})\cdot$$

Since $\mathcal{E}$ possesses the mountain pass geometry, for any $n \geq 1$, there exists $t_n > 0$ such that

$$\max_{t\geq 0} \mathcal{E}(tv_n) = \mathcal{E}(t_n v_n) \geq \frac{1}{N}G((\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}})\cdot$$

Using the fact that $F(x,t) \geq 0$ for all $(x,t) \in B \times \mathbb{R}$ we get

$$G(t_n^N) \geq G((\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}).$$

On one hand, the condition (G_1) implies that $G : [0,+\infty) \to [0,+\infty)$ is an increasing bijection. So

$$t_n^N \geq (\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}\cdot \tag{36}$$

On the other hand,

$$\frac{d}{dt}J(tv_n)\big|_{t=t_n} = g(t_n^N)t_n^{N-1} - \int_B f(x,t_n v_n)v_n dx = 0,$$

that is

$$g(t_n^N)t_n^N = \int_B f(x,t_n v_n)t_n v_n dx. \tag{37}$$

Now, we claim that the sequence (t_n) is bounded in $(0,+\infty)$.
Indeed, it follows from (A_4) that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$f(x,t)t \geq (\gamma_0 - \varepsilon)e^{\alpha_0 t^\gamma} \ \forall |t| \geq t_\varepsilon, \text{ uniformly in } x \in B. \tag{38}$$

By Lemma 5.1, if $|x| \leq e^{-\frac{n}{N}}$, then $\alpha_{N,\beta}w_n^\gamma(x) \geq n(1-E)$

$$\begin{aligned} g(t_n^N)t_n^N &\geq (\gamma_0 - \varepsilon)\int_{0\leq|x|\leq e^{-\frac{n}{N}}} e^{\alpha_0 t_n^\gamma w_n^\gamma}dx \\ &\geq (\gamma_0 - \varepsilon)\int_{0\leq|x|\leq e^{-\frac{n}{N}}} e^{\alpha_0 t_n^\gamma \frac{1}{\alpha_{N,\beta}}n(1-E)}dx \\ &= (\gamma_0 - \varepsilon)\frac{\omega_{N-1}}{N}e^{\alpha_0 t_n^\gamma \frac{1}{\alpha_{N,\beta}}n(1-E)-n}. \end{aligned} \tag{39}$$

Using the condition (G_2), we obtain

$$a_1 t_n^N + a_2 t_n^{N+N\sigma} \geq \frac{\omega_{N-1}}{N}(\gamma_0 - \varepsilon)e^{\alpha_0 t_n^\gamma \frac{1}{\alpha_{N,\beta}}n(1-E)-n}. \tag{40}$$

From (40), we obtain for n large enough

$$1 \geq \frac{\omega_{N-1}}{N}(\gamma_0 - \varepsilon)e^{\alpha_0 t_n^\gamma \frac{1}{\alpha_{N,\beta}}n(1-E)-n-\log a_1 t_n^N - \log a_2 t_n^{N+N\sigma}}.$$

Therefore (t_n) is bounded in $\mathbb{R}$. Also, we have

$$t_n^N \geq (\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}.$$

Now, suppose that

$$\lim_{n\to+\infty} t_n^N > (\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}.$$

For n large enough, $t_n^N > (\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}$ and in this case, the right hand side of the inequality (40) will gives the unboundedness of the sequence (t_n). Since (t_n) is bounded, we get

$$\lim_{n\to+\infty} t_n^N = (\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}. \tag{41}$$

We claim that the last equality leads to a contradiction with (A_4). For this purpose, the following sets should be used

$$\mathcal{A}_n = \{x \in B | t_n v_n \geq t_\varepsilon\} \text{ and } \mathcal{C}_n = B \setminus \mathcal{A}_n,$$

where t_ε is given in (38). We have

$$\begin{aligned} g(t_n^N)t_n^N &= \int_B f(x,t_n v_n)t_n v_n dx = \int_{\mathcal{A}_n} f(x,t_n v_n)t_n v_n dx + \int_{\mathcal{C}_n} f(x,t_n v_n)t_n v_n \\ &\geq (\gamma_0-\varepsilon)\int_{\mathcal{A}_n} e^{\alpha_0 t_n^\gamma w_n^\gamma} dx + \int_{\mathcal{C}_n} f(x,t_n v_n)t_n v_n dx \\ &= (\gamma_0-\varepsilon)\int_B e^{\alpha_0 t_n^\gamma w_n^\gamma} dx - (\gamma_0-\varepsilon)\int_{\mathcal{C}_n} e^{\alpha_0 t_n^\gamma w_n^\gamma} dx \\ &+ \int_{\mathcal{C}_n} f(x,t_n v_n)t_n v_n dx. \end{aligned}$$

Since $v_n \to 0$ a.e in B, $\chi_{\mathcal{C}_n} \to 1$ a.e in B, therefore using the dominated convergence theorem, we get

$$\begin{aligned} \lim_{n\to+\infty} g(t_n^N)t_n^N =& g((\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}})(\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}} \\ & \geq (\gamma_0-\varepsilon)\lim_{n\to+\infty}\int_B e^{\alpha_0 t_n^\gamma w_n^\gamma} dx - (\gamma_0-\varepsilon)\frac{\omega_{N-1}}{N}. \end{aligned}$$

By using (36) and the result of lemma 5.2, we obtain

$$g((\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}})(\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}} \geq \lim_{n\to+\infty}(\gamma_0-\varepsilon)\int_B e^{\alpha_{N,\beta} t_n^\gamma w_n^\gamma} dx - (\gamma_0-\varepsilon)\frac{\omega_{N-1}}{N} = (\gamma_0-\varepsilon)\omega_{N-1}.$$

Since, ε is arbitrary, we reach a contradiction with (A_4). The lemma is proved. $\square$

6. Proof of main results

First we begin by some crucial lemmas.
Now, we introduce the Nehari manifold associated to the functional $\mathcal{E}$, namely,

$$\mathcal{N} = \{u \in \mathbf{W} : \mathcal{E}'(u)u = 0, u \neq 0\},$$

and the number $c = \inf_{u\in\mathcal{N}} \mathcal{E}(u)$. We have the following lemmas.

Lemma 6.1. *Assume that the condition (A_3) hold, then for each $x \in B$,*

$$tf(x,t) - 2NF(x,t) \text{ is increasing for } t > 0.$$

In particular, $tf(x,t) - 2NF(x,t) \geq 0$ for all $(x,t) \in B \times [0,+\infty)$.

Proof. Assume that $0 < t < s$. For each $x \in B$, we have

$$\begin{aligned} tf(x,t) - 2NF(x,t) &= \frac{f(x,t)}{t^{2N-1}}t^{2N} - 2NF(x,s) + 2N\int_t^s f(x,\nu)d\nu \\ &< \frac{f(x,t)}{s^{2N-1}}t^{2N} - 2NF(x,s) + \frac{f(x,s)}{s^{2N-1}}(s^{2N} - t^{2N}) \\ &= sf(x,s) - 2NF(x,s)\cdot \end{aligned}$$

□

Lemma 6.2. *Let d the real defined by (29) and $c = \inf\limits_{u\in\mathcal{N}} \mathcal{E}(u)$. If (G_3), (V_1) and (A_3) are satisfied then $d \leq c$.*

Proof : Let $\bar{u} \in \mathcal{N}$, $\bar{u} > 0$ and consider the function $\psi : (0,+\infty) \to \mathbb{R}$ defined by $\psi(t) = \mathcal{E}(t\bar{u})$. ψ is differentiable and we have

$$\psi'(t) = \mathcal{E}'(t\bar{u})\bar{u} = g(t^N\|\bar{u}\|^N)t^{N-1}\|\bar{u}\|^N - \int_B f(x,t\bar{u})\bar{u}dx, \text{for all } t > 0.$$

Since $\bar{u} \in \mathcal{N}$, we have $\mathcal{E}'(\bar{u})\bar{u} = 0$ and therefore $g(\|\bar{u}\|^N)\|\bar{u}\|^N = \int_B f(x,\bar{u})\bar{u}dx$. Hence,

$$\psi'(t) = t^{2N-1}\|\bar{u}\|^{2N}\Big(\frac{g(t^N\|\bar{u}\|^N)\|}{t^N\|\bar{u}\|^N} - \frac{g(\|\bar{u}\|^N)}{\|\bar{u}\|^N}\Big) + t^{2N-1}\int_B \Big(\frac{f(x,\bar{u})}{\bar{u}^{2N-1}} - \frac{f(x,t\bar{u})}{(t\bar{u})^{2N-1}}\Big)\bar{u}^{2N}\, dx.$$

We have that $\psi'(1) = 0$. We have also by the conditions (G_3) and (A_3) that $\psi'(t) > 0$ for all $0 < t < 1$, $\psi'(t) \leq 0$ for all $t > 1$. It follows that

$$\mathcal{E}(\bar{u}) = \max_{t\geq 0} \mathcal{E}(t\bar{u}).$$

We define the function $\lambda : [0,1] \to \mathcal{E}$ such that $\lambda(t) = t\bar{t}\bar{u}$, with $\mathcal{E}(\bar{t}\bar{u}) < 0$. We have $\lambda \in \Lambda$, and hence

$$d \leq \max_{t\in[0,1]} \mathcal{E}(\lambda(t)) \leq \max_{t\geq 0} \mathcal{E}(t\bar{u}) = \mathcal{E}(\bar{u}).$$

Since $\bar{u} \in \mathcal{N}$ is arbitrary then $d \leq c$.

Proof of Theorem 1.2 and Theorem 1.3.

Proof. Proof of Theorem 1.3 .

Since $\mathcal{E}$ possesses the mountain pass geometry, there exists $u_n \in \mathbf{W}$ such that

$$\mathcal{E}(u_n) = \frac{1}{N}G(\|u_n\|^N) - \int_B F(x,u_n)dx \to d,\ n \to +\infty \tag{42}$$

and

$$\begin{aligned} |\mathcal{E}'(u_n)\varphi| = &\Big| g(\|u_n\|^N)\Big[\int_B \sigma(x)|\nabla u_n|^{N-2}\nabla u_n.\nabla\varphi + V(x)|u_n|^{N-2}u_n\varphi dx\Big] \\ &- \int_B f(x,u_n)\varphi dx\Big| \leq \epsilon_n\|\varphi\|, \end{aligned} \tag{43}$$

for all $\varphi \in \mathbf{W}$, where $\epsilon_n \to 0$, when $n \to +\infty$.
By (42), for all $\epsilon > 0$ there exists a constant $C > 0$

$$\frac{1}{N} G(\|u_n\|^N) \leq C + \int_B F(x, u_n)dx\cdot$$

From (11), for all $\epsilon > 0$, It follows that,

$$\frac{1}{N} G(\|u_n\|^N) \leq C + \int_{|u_n| \leq t_\epsilon} F(x, u_n)dx + \epsilon \int_B f(x, u_n)u_n dx\cdot$$

From (43) and (8) we get

$$\frac{1}{2N} g(\|u_n\|^N)\|u_n\|^N \leq \frac{1}{N} G(\|u_n\|^N) \leq C_1 + \epsilon\epsilon_n \|u_n\| + \epsilon g(\|u_n\|^N)\|u_n\|^N,$$

for some constant $C_1 > 0$. Using the condition (G_1), for all ϵ such that $0 < \epsilon < \frac{1}{2N}$, we get

$$g_0(\frac{1}{2N} - \epsilon)\|u_n\|^N \leq C_1 + \epsilon\epsilon_n \|u_n\|\cdot$$

We deduce that the sequence (u_n) is bounded in $\mathbf{W}$. As consequence, there exists $u \in \mathbf{W}$ such that, up to subsequence, $u_n \rightharpoonup u$ weakly in $\mathbf{W}$, $u_n \to u$ strongly in $L^q(B)$, for all $q \geq 1$.
In order to obtain a ground state solution for problem (1), it is enough to show that there is $u \in \mathcal{N}$ such that $\mathcal{E}(u) = d$. We have from (42) and (43), that

$$0 < \int_B f(x, u_n)u_n \leq C,$$

and

$$0 < \int_B F(x, u_n) \leq C.$$

Since by Lemma 3.2, we have

$$f(x, u_n) \to f(x, u) \text{ in } L^1(B) \; as \; n \to +\infty, \tag{44}$$

then, it follows from (A_2) and the generalized Lebesgue dominated convergence theorem that

$$F(x, u_n) \to F(x, u) \text{ in } L^1(B) \; as \; n \to +\infty. \tag{45}$$

So,

$$\lim_{n \to +\infty} G(\|u_n\|^N) = N(d + \int_B F(x, u)dx). \tag{46}$$

Also,

$$\left(|\nabla u_n|^{N-2}\nabla u_n\right) \text{ is bounded in } (L^{\frac{N}{N-1}}(B, \sigma))^N.$$

Then, up to subsequence, we can assume that

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u|^{N-2}\nabla u \text{ weakly in } \; (L^{\frac{N}{N-1}}(B, \sigma))^N\cdot \tag{47}$$

Next, we are going to make some claims.
Claim 1. $\nabla u_n(x) \to \nabla u(x)$ *a.e* $x \in B$.
Indeed, for any $\eta > 0$, let $\mathcal{A}_\eta = \{x \in B, |u_n - u| \geq \eta\}$. For all $t \in \mathbb{R}$, for all positive $c > 0$, we have

$$ct \leq e^t + c^2.$$

It follows that for $t = \alpha_{N,\beta}\big(\frac{|u_n-u|}{\|u_n-u\|}\big)^{\gamma}$, $c = \frac{1}{\alpha_{N,\beta}}\|u_n-u\|^{\gamma}$, we get

$$\begin{aligned} |u_n-u|^{\gamma} &\leq e^{\alpha_{N,\beta}\big(\frac{|u_n-u|}{\|u_n-u\|}\big)^{\gamma}} + \frac{1}{\alpha^2_{N,\beta}}\|u_n-u\|^{2\gamma} \\ &\leq e^{\alpha_{N,\beta}\big(\frac{|u_n-u|}{\|u_n-u\|}\big)^{N'}} + C_1(N), \end{aligned}$$

where $C_1(N)$ is a constant depending only on N and the upper bound of $\|u_n\|$. So, if we denote by $\mathcal{L}(\mathcal{A}_\eta)$ the Lebesgue measure of the set $\mathcal{A}_\eta$, we obtain

$$\begin{aligned} \mathcal{L}(\mathcal{A}_\eta) = \int_{\mathcal{A}_\eta} |u_n-u|^{\gamma}|u_n-u|^{-\gamma}dx &\leq e^{-\eta^{\gamma}}\int_{\mathcal{A}_\eta}\Big(e^{\alpha_{N,\beta}\big(\frac{|u_n-u|}{\|u_n-u\|}\big)^{N'}} + C_1(N)\Big)dx \\ &\leq e^{-\eta^{\gamma}}e^{C_1(N)}\int_B \exp\Big(\alpha_{N,\beta}\big(\frac{|u_n-u|}{\|u_n-u\|}\big)^{\gamma}\Big)dx \\ &\leq e^{-\eta^{\gamma}}C_2(N) \to 0 \text{ as } \eta \to +\infty, \end{aligned}$$

where $C_2(N)$ is a positive constant depending only on N and the upper bound of $\|u_n\|$. It follows that

$$\int_{\mathcal{A}_\eta}|\nabla u_n - \nabla u|dx \leq Ce^{-\frac{1}{2}\eta^{\gamma}}\Big(\int_B |\nabla u_n - \nabla u|^2\sigma(x)dx\Big)^{\frac{1}{2}} \to 0 \text{ as } \eta \to +\infty\cdot \tag{48}$$

We define for $\eta > 0$, the truncation function used in [6]

$$T_\eta(s) := \begin{cases} s & \text{si } |s| < \eta \\ \eta\frac{s}{|s|} & \text{si } |s| \geq \eta. \end{cases}$$

We take $\varphi = T_\eta(u_n - u) \in \mathbf{W}$ in (43) and since $\nabla\varphi = \chi_{\mathcal{A}_\eta}\nabla(u_n-u)$, we obtain

$$\begin{aligned} &(\Big|\int_{B\setminus\mathcal{A}_\eta}\sigma(x)\big(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\big).(\nabla u_n - \nabla u)dx \\ &\quad + \int_{B\setminus\mathcal{A}_\eta} V(x)(|u_n|^{N-2}u_n - |u|^{N-2}u)(u_n-u)\Big| \\ &\leq \Big|\int_{B\setminus\mathcal{A}_\eta}\sigma(x)|\nabla u|^{N-2}\nabla u.(\nabla u_n - \nabla u) + V(x)(|u|^{N-2}u)(u_n-u)dx\Big| \\ &\quad + \int_B f(x,u_n)T_\eta(u_n-u)dx + \varepsilon_n\|u_n-u\| \\ &\leq \Big|\int_B \sigma(x)|\nabla u|^{N-2}\nabla u.(\nabla u_n - \nabla u) + V(x)(|u|^{N-2}u)(u_n-u))dx\Big| \\ &\quad + \int_B f(x,u_n)T_\eta(u_n-u)dx + \varepsilon_n\|u_n-u\|\cdot \end{aligned}$$

Since $u_n \rightharpoonup u$ weakly , then

$$\int_B \big(\sigma(x)|\nabla u|^{N-2}\nabla u.(\nabla u_n - \nabla u) + V(x)(|u|^{N-2}u)(u_n-u)\big)dx \to 0\cdot$$

Moreover using (44) and the Lebesgue dominated convergence Theorem, we get

$$\int_B f(x,u_n)T_\eta(u_n-u)dx \to 0 \text{ as } n \to +\infty.$$

Using the well known inequality,

$$\langle |x|^{N-2}x - |y|^{N-2}y, x-y\rangle \geq 2^{2-N}|x-y|^N \ \forall\, x,y \in \mathbb{R}^N,\ N \geq 2, \tag{49}$$

$\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^N$, one has

$$\int_{B\setminus\mathcal{A}_\eta}(\sigma(x)|\nabla u_n-\nabla u|^N+V(x)|u_n-u|^N)dx\to 0\cdot$$

Since $V(x)>0$, then $\int_{B\setminus\mathcal{A}_\eta}\sigma(x)|\nabla u_n-\nabla u|^N dx\to 0$. Therefore,

$$\int_{B\setminus\mathcal{A}_\eta}|\nabla u_n-\nabla u|dx\le\Big(\int_{B\setminus\mathcal{A}_\eta}\sigma(x)|\nabla u_n-\nabla u|^N dx\Big)^{\frac{1}{N}}\mathcal{L}^{\frac{1}{N'}}(B\setminus\mathcal{A}_\eta)\to 0 \text{ as } n\to+\infty. \tag{50}$$

From (48) and (50), we deduce that

$$\int_B|\nabla u_n-\nabla u|dx\to 0 \text{ as } n\to+\infty.$$

Therefore, $\nabla u_n(x)\to\nabla u(x)$ *a.e* $x\in B$ and claim 1 is proved.

Claim 2. $u\neq 0$. Indeed, suppose that $u=0$. Then, $\displaystyle\int_B F(x,u_n)dx\ \to 0$ and consequently we get

$$\frac{1}{N}G(\|u_n\|^N)\to d<\frac{1}{N}G\Big(\big(\frac{\alpha_{N,\beta}}{\alpha_0}\big)^{\frac{N}{\gamma}}\Big). \tag{51}$$

By (43), we have

$$\Big|g(\|u_n\|^N)\|u_n\|^N-\int_B f(x,u_n)u_n dx\Big|\le C\epsilon_n.$$

First we claim that there exists $q>1$ such that

$$\int_B|f(x,u_n)|^q dx\le C. \tag{52}$$

So

$$g(\|u_n\|^N)\|u_n\|^N\le C\epsilon_n+\Big(\int_B|f(x,u_n)|^q\Big)^{\frac{1}{q}}dx\Big(\int_B|u_n|^{q'}\Big)^{\frac{1}{q'}}$$

where q' is the conjugate of q. Since (u_n) converge to $u=0$ in $L^{q'}(B)$

$$\lim_{n\to+\infty}g(\|u_n\|^N)\|u_n\|^N=0.$$

From the condition (G_1), we obtain

$$\lim_{n\to+\infty}\|u_n\|^N=0.$$

By Brezis-Lieb's Lemma [7], $u_n\to 0$ in $\mathbf{W}$. Therefore, $\mathcal{E}(u_n)\to 0$ which is in contradiction with $d>0$.

For the proof of the claim (52), since f has critical growth, for every $\epsilon>0$ and $q>1$ there exists $t_\epsilon>0$ and $C>0$ such that for all $|t|\ge t_\epsilon$, we have

$$|f(x,t)|^q\le Ce^{\alpha_0(\varepsilon+1)t^\gamma}.$$

Consequently,

$$\begin{aligned}\int_B|f(x,u_n)|^q dx &= \int_{\{|u_n|\le t_\varepsilon}|f(x,u_n)|^q dx+\int_{\{|u_n|>t_\varepsilon\}}|f(x,u_n)|^q dx\\ &\le \omega_{N-1}\max_{B\times[-t_\varepsilon,t_\varepsilon]}|f(x,t)|^q+C\int_B e^{\alpha_0(\varepsilon+1)|u_n|^\gamma})dx.\end{aligned}$$

Since $(G^{-1}(Nd))^{\frac{1}{N-1}} < \frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}$, there exists $\eta \in (0, \frac{1}{2})$ such that $(G^{-1}(Nd))^{\frac{1}{N-1}} = (1-2\eta)\alpha_{N,\beta}$. From (51), $\|u_n\|^\gamma \to (G^{-1}(Nd))^{\frac{1}{N-1}}$, so there exist $n_\eta \in \mathbb{N}$ such that $\alpha_0\|u_n\|^{N'} \leq (1-\eta)\alpha_{N,\beta}$, for all $n \geq n_\eta$. Therefore,

$$\alpha_0(1+\epsilon)\Big(\frac{|u_n|}{\|u_n\|}\Big)^\gamma \|u_n\|^\gamma \leq (1+\epsilon)(1-\eta)\Big(\frac{|u_n|}{\|u_n\|}\Big)^\gamma \alpha_{N,\beta}.$$

We choose $\epsilon > 0$ small enough to get

$$(1+\epsilon)(1-\eta) < 1,$$

hence the second integral is uniformly bounded in view of (7).

Claim 3. $g(\|u\|^N)\|u\|^N \geq \int_B f(x,u)udx$. Suppose that $g(\|u\|^N)\|u\|^N < \int_B f(x,u)udx$. Hence, $\mathcal{E}'(u)u < 0$. The function $\psi : t \to \psi(t) = \mathcal{E}'(tu)u$ is positive for t small enough. Indeed, from (10) and the critical (resp subcritical) growth of the nonlinearity f, for every $\epsilon > 0$, for every $q > N+1$, there exist positive constants C and c such that

$$|f(x,t)| \leq \epsilon |t|^{N-1} + Ct^q e^{c\ t^\gamma},\ \forall, (t,x) \in \mathbb{R} \times B.$$

Then using the condition (G_1), the last inequality and the Hölder inequality, we obtain

$$\begin{aligned}\psi(t) &= g(t^N\|u\|^N)t^{N-1}\|u\|^N - \int_B f(x,tu)udx \\ &\geq g_0 t^{N-1}\|u\|^N - \epsilon t^{N-1}\int_B u^{N-1}dx - C\Big(\int_B e^{cN\ t^\gamma u^\gamma})dx\Big)^{\frac{1}{N}}\Big(\int_B u^{N'q}dx\Big)^{\frac{1}{N'}}.\end{aligned}$$

In view of (7) the integral

$$\int_B e^{cN\ t^\gamma u^\gamma}dx \leq \int_B e^{cN\ t^\gamma \frac{u^\gamma}{\|u\|^\gamma}\|u\|^\gamma})dx \leq C,$$

provided $t \leq \frac{(\alpha_{N,\beta})^{\frac{1}{\gamma}}}{(Nc)^{\frac{1}{\gamma}}\|u\|}$. Using the radial Lemma 3.1 we get $\|u\|_{N'q} \leq C'\|u\|^q$. Then,

$$\begin{aligned}\psi(t) &\geq g_0 t^{N-1}\|u\|^N - C_1\epsilon t^{N-1}\|u\|^{N-1} - C_2\|u\|^q \\ &= \|u\|^{N-1}t^{N-1}\big[(g_0\|u\| - C_1\epsilon) - C_2 t^{q-(N-1)}\|u\|^{q-(N+1)}\big].\end{aligned}$$

We chose $\epsilon > 0$, such that $g_0\|u\| - C_1\epsilon > 0$ and since $q > N+1$, for small t, we get $\psi : t \to \psi(t) = \mathcal{E}'(tu)u > 0$. So there exists $\eta \in (0,1)$ such that $\psi(\eta u) = 0$. Therefore $\eta u \in \mathcal{N}$. Using (10), the result of Lemma 6.1, the semicontinuity of norm and Fatou's Lemma, we get

$$\begin{aligned}d \leq c \leq \mathcal{E}(\eta u) &= \mathcal{E}(\eta u) - \frac{1}{2N}\mathcal{E}'(\eta u)\eta u \\ &= \frac{1}{N}G(\|\eta u\|^N) - \frac{1}{2N}g(\|\eta u\|^N)\|\eta u\|^N + \frac{1}{2N}\int_B \big(f(x,\eta u)\eta u - 2NF(x,\eta u)\big)dx \\ &< \frac{1}{N}G(\|u\|^N) - \frac{1}{2N}g(\|u\|^N)\|u\|^N + \frac{1}{2N}\int_B \big(f(x,u)u - 2NF(x,u)\big)dx \\ &\leq \liminf_{n\to+\infty}\Big[\frac{1}{N}G(\|u_n\|^N) - \frac{1}{2N}g(\|u_n\|^N)\|u_n\|^N + \frac{1}{2N}\int_B \big(f(x,u_n)u_n - 2NF(x,u_n)\big)dx\Big] \\ &\leq \lim_{n\to+\infty}\big[\mathcal{E}(u_n) - \frac{1}{2N}\mathcal{E}'(u_n)u_n\big] = d\end{aligned}$$

which is absurd and the claim is well established.

Claim 4. $u > 0$. Indeed, since (u_n) is bounded, up to a subsequence, $\|u_n\| \to \rho > 0$. In addition, $\mathcal{J}'(u_n) \to 0$ leads to

$$g(\rho^N)\Big[\int_B \sigma(x)|\nabla u|^{N-2}\nabla u.\nabla\varphi + V(x)|u|^{N-2}u\varphi dx\Big] = \int_B f(x,u)\varphi dx, \ \forall\varphi \in \mathbf{W}.$$

By taking $\varphi = u^-$, with $w^\pm = \max(\pm w, 0)$, we get $\|u^-\|^N = 0$ and so $u = u^+ \geq 0$. By taking $\varphi = u^-$, with $w^\pm = \max(\pm w, 0)$, we get $\|u^-\|^N = 0$ and so $u = u^+ \geq 0$. Since the nonlinearity has critical growth at $+\infty$ and from Trudinger-Moser inequality (7), $f(.,u) \in L^p(B)$, for all $p \geq 1$. So, by elliptic regularity $u \in W^{2,p}(B,\sigma)$, for all $p \geq 1$. Therefore, by Sobolev embedding $u \in C^{1,\gamma}(\overline{B})$.
Let define $B_0 = \{x \in B : u(x) = 0\}$. The set $B_0 = \emptyset$. Indeed, suppose by contradiction that $B_0 \neq \emptyset$. Since $f(x,u) \geq 0$, by Harnark inequality we can deduce that B_0 is an open and closed set of B. In virtue of the connectedness of B, we reach a contradiction. Hence Claim 4 is proved.
Now, we affirm that $\mathcal{E}(u) = d$. Indeed, on one hand, by claim 2, (8) and Lemma 6.1 we obtain

$$\mathcal{E}(u) \geq \frac{1}{N}G(\|u\|^N) - \frac{1}{2N}g(\|u\|^N)\|u\|^N + \frac{1}{2N}\int_B [f(x,u)u - 2NF(x,u)]dx \geq 0. \quad (53)$$

On the other side, by the semicontinuity of the norm and (45), we get

$$\mathcal{E}(u) \leq \frac{1}{N}\liminf_{n\to\to\infty} G(\|u_n\|^N) - \int_B F(x,u)dx = d.$$

We argue by contradiction and we suppose that

$$\mathcal{E}(u) < d. \quad (54)$$

Then

$$\|u\|^N < \rho^N. \quad (55)$$

In addition,

$$\frac{1}{N}G(\rho^N) = \frac{1}{N}\lim_{n\to+\infty} G(\|u_n\|^N) = \big(d + \int_B F(x,u)dx\big), \quad (56)$$

which means that

$$\rho^N = G^{-1}(\big(N(d + \int_B F(x,u)dx)\big).$$

Set

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \frac{u}{\rho}.$$

We have $\|v_n\| = 1$, $v_n \rightharpoonup v$ in $\mathbf{W}$, $v \not\equiv 0$ and $\|v\| < 1$. So, by Lemma **??**, we get

$$\sup_n \int_B e^{p\ \alpha_{N,\beta}|v_n|^\gamma} dx < +\infty,$$

provided $1 < p < \dfrac{1}{(1 - \|v\|^N)^{\frac{\gamma}{N}}}$.
From (53), (56) and the following equality

$$Nd - N\mathcal{E}(u) = G(\rho^N) - G(\|u\|^N),$$

we get

$$G(\rho^N) \le Nd + G(\|u\|^N) < G((\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}})).$$

Now, using the condition (G_1) one has

$$\rho^N < G^{-1}\big(G((\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}})) + G(\|u\|^N)\big) \le (\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}} + \|u\|^N. \tag{57}$$

Since

$$\rho^\gamma = \big(\frac{\rho^N - \|u\|^N}{(1-\|v\|^N}\big)^{\frac{1}{(N-1)(1-\beta)}},$$

we deduce from (57) that

$$\rho^\gamma < \frac{\frac{\alpha_{N,\beta}}{\alpha_0}}{(1-\|v\|^N)^{\frac{\gamma}{N}}}. \tag{58}$$

On one hand, we have this estimate $\displaystyle\int_B |f(x,u_n)|^q dx < C$. Indeed, For $\epsilon > 0$,

$$\begin{aligned}
\int_B |f(x,u_n)|^q dx &= \int_{\{|u_n|\le t_\epsilon\}} |f(x,u_n)|^q dx + \int_{\{|u_n|>t_\epsilon\}} |f(x,u_n)|^q dx \\
&\le \omega_{N-1} \max_{B\times[-t_\epsilon,t_\epsilon]} |f(x,t)|^q + C\int_B e^{\alpha_0(\varepsilon+1)|u_n|^\gamma}\big)dx. \\
&\le C_\epsilon + C\int_B e^{\alpha_0(1+\varepsilon)\|u_n\|^\gamma|v_n|^\gamma}\big)dx \le C
\end{aligned}$$

provided $\alpha_0(1+\varepsilon)\|u_n\|^\gamma \le p\ \alpha_{N,\beta}$ and $1 < p < U(v) = (1-\|v\|^N)^{\frac{-\gamma}{N}}$.
From (58), there exists $\delta \in (0,\frac{1}{2})$ such that $\rho^\gamma = (1-2\delta)\Big(\frac{(\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}}{1-\|v\|^N}\Big)^{\frac{1}{(N-1)(1-\beta)}}$.
Since $\lim\limits_{n\to+\infty}\|u_n\|^\gamma = \rho^\gamma$ then, for n large enough

$$\alpha_0(1+\epsilon)\|u_n\|^\gamma \le (1+\epsilon)(1-\delta)\ \alpha_{N,\beta}\Big(\frac{1}{1-\|v\|^N}\Big)^{\frac{\gamma}{N}}.$$

We choose $\epsilon > 0$ small enough such that $(1+\epsilon)(1-\delta) < 1$ which implies that

$$\alpha_0(1+\epsilon)\|u_n\|^\gamma < \alpha_{N,\beta}\Big(\frac{1}{1-\|v\|^N}\Big)^{\frac{\gamma}{N}}.$$

So, the sequence $(f(x,u_n))$ is bounded in L^q, $q>1$. Using the Hölder inequality, we deduce that

$$\begin{aligned}
\Big|\int_B f(x,u_n)(u_n-u)dx\Big| &\le \big(\int_B |f(x,u_n)|^q dx\big)^{\frac{1}{q}}\big(\int_B |u_n-u|^{q'}\big)^{\frac{1}{q'}}dx \\
&\le C\big(\int_B |u_n-u|^{q'}\big)^{\frac{1}{q'}}dx \to 0 \ as\ n\to+\infty
\end{aligned} \tag{59}$$

where $\frac{1}{q}+\frac{1}{q'}=1$.
Since $\mathcal{E}'(u_n)(u_n-u) = o_n(1)$, it follows that

$$g(\|u_n\|^N)\int_B (\sigma(x)|\nabla u_n|^{N-2}\nabla u_n.(\nabla u_n - \nabla u) + V(x)|u_n|^{N-2}u_n(u_n-u)dx) \to 0.$$

On the other side,

$$g(\|u_n\|^N)\int_B (\sigma(x)|\nabla u_n|^{N-2}\nabla u_n.(\nabla u_n - \nabla u) + V(x)|u_n|^{N-2}u_n(u_n-u)dx)$$

$$= g(\|u_n\|^N)\|u_n\|^N - g(\|u_n\|^N)\int_B (\sigma(x)|\nabla u_n|^{N-2}\nabla u_n.\nabla u + V(x)|u_n|^{N-2}u_n u)dx\cdot$$

Passing to the limit in the last equality, using the result of claim 1 and (59), we get

$$g(\rho^N)\rho^N - g(\rho^N)\|u\|^N = 0\cdot$$

Therefore $\|u\| = \rho$ and $\|u_n\| \to \|u\|$. This is in contradiction with (55). It follows that $\mathcal{E}(u) = d$. Also, by (44),(47) and claim 1

$$g(\|u\|^N)\int_B (\sigma(x)|\nabla u|^{N-2}\nabla u\nabla\varphi + V(x)|u|^{N-2}u\varphi)dx = \int_B f(x,u)\varphi dx,\ \forall\varphi\in\mathbf{W}.$$

So u is a ground state solution of the problem (1). The proof of Theorem 1.3 is achieved.

Proof of Theorem 1.2. In the sub-critical case, we do not have a problem of compactness. Indeed, Up a subsequence (u_n), there exists $M>0$, such that $\|u_n\|\leq M$. By the subcritical case of f at $+\infty$, there exist $\alpha\leq\frac{\alpha_{N,\beta}}{2M^\gamma}$ and positive constants C_M, such that

$$f(x,s)\leq C_M e^{\alpha|s|^\gamma},\forall(x,s)\in B\times(0,+\infty).$$

Using the Hölder inequality

$$\begin{aligned}
|\int_B f(x,u_n)(u_n-u)dx| &\leq \int_B |f(x,u_n)(u_n-u)|dx\\
&\leq \Big(\int_B |f(x,u_n)|^2dx\Big)^{\frac12}\Big(\int_B |u_n-u|^2dx\Big)^{\frac12}\\
&\leq C\Big(\int_B e^{2\alpha|u_n|^\gamma}dx\Big)^{\frac12}\|u_n-u\|_2\\
&\leq C\Big(\int_B e^{2\alpha\frac{|u_n|^\gamma}{\|u_n\|^\gamma}\|u_n\|^\gamma}dx\Big)^{\frac12}\|u_n-u\|_2\\
&\leq C\|u_n-u\|_2\to 0 \text{ as } n\to+\infty.
\end{aligned}$$

In (43), we take $\varphi = u_n - u$, then

$$\begin{aligned}
&|\int_B \sigma(x)\big(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\big).(\nabla u_n-\nabla u)dx\\
&\quad+\textstyle\int_B V(x)(|u_n|^{N-2}u_n-|u|^{N-2}u)(u_n-u)\big|\\
&\leq|\int_B \sigma(x)|\nabla u|^{N-2}\nabla u.(\nabla u_n-\nabla u)\\
&\quad+V(x)(|u|^{N-2}u)(u_n-u)dx\big|+\textstyle\int_B f(x,u_n)(u_n-u)dx+\varepsilon_n\|u_n-u\|\\
&\leq|\int_B \sigma(x)|\nabla u|^{N-2}\nabla u.(\nabla u_n-\nabla u)\\
&\quad+V(x)(|u|^{N-2}u)(u_n-u))dx\big|+\textstyle\int_B f(x,u_n)(u_n-u)dx+\varepsilon_n\|u_n-u\|.
\end{aligned}$$

Using the fact $u_n\rightharpoonup u$ weakly , we get

$$\int_B \sigma(x)|\nabla u|^{N-2}\nabla u.(\nabla u_n-\nabla u)+V(x)(|u|^{N-2}u)(u_n-u)dx\to 0.$$

Therefore, by (49),

$$2^{2-N}\|u_n-u\|^N\leq|\int_B f(x,u_n)(u_n-u)dx|+o_n(1)\to 0 \text{ as } n\to+\infty.$$

As a consequence, $\mathcal{E}(u)=d$. Also, again passing to the limit in (43), we get that u is a solution of (1). This completes the proof. □

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