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Characterization of continuous pseudoconvex functions' extrema

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Abstract. We characterize both the minima and maxima of continuous pseudoconvex functions using respectively variational inequalities and normal cones.

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1. Introduction

Generalized convexity is becoming a usual ingredient in recent results of Optimization and Mathematical Programming. In this note, we characterize the extrema of continuous pseudoconvex functions. Let $f\colon X \to \mathbb{R}$ be a continuous and pseudoconvex function on a Banach space X. Let $C \subset X$ be convex and $\bar{x} \in C$.

In §2, we give a necessary and sufficient condition in terms of variational inequalities for $\bar{x}$ to be a minimum of f on C. Note that C needs not be a neighborhood of $\bar{x}$. In §3, we give an other necessary and sufficient condition for $\bar{x}$ to be a maximum of f on C using a relation between the normal cone of C at $\bar{x}$ and the subdifferential of f at $\bar{x}$. Our results are closely related to (extending in some sense) those of [7] and [9].

A *differentiable function* was called pseudoconvex in [2] if for every x, y the inequality

$$\langle df(x), y - x\rangle \geq 0 \quad \text{ensures} \quad f(y) \geq f(x).$$

This notion was then extended to *less smooth functions* using the concepts of subdifferential and generalized directional derivatives (see for example [3] and [10]).

We will adopt throughout the text the following definition.

Definition 1.1. *A function $f\colon X \to \mathbb{R}\cup\{+\infty\}$ is said to be pseudoconvex if for every $x,\ y \in \mathrm{dom} f = \{x \in X;\ f(x) < +\infty\}$ and $x^* \in \partial f(x)$,*

$$\langle x^*, y - x\rangle \geq 0 \quad \textit{implies} \quad f(y) \geq f(x).$$

We have the following useful properties of continuous pseudoconvex functions.

Proposition 1.1. *If f is a continuous pseudoconvex function, then*
- *f is quasiconvex,*
- *∂f is quasimonotone, and*
- *there is $x^* \in \partial f(x)$ such that,*

$$\langle x^*, y - x\rangle > 0 \quad \textit{implies} \quad f(y) > f(x).$$

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For a proof see either [5] or [1].

The notation $\partial f(x)$ stands in all the text for the Clarke-Rockafellar subdifferential of f at x:

$$\partial f(x) = \{x^* \in X^*;\ \langle x^*, v\rangle \le f^{\nearrow}(x; v),\ \forall v \in X\}$$

where X^* is the dual space of X and the directional derivative of f at x along the direction v, when f is continuous, is defined by

$$f^{\nearrow}(x; v) = \sup_{\varepsilon>0} \limsup_{\substack{y \to_f x \\ t \to 0}} \inf_{u\in B(v,\varepsilon)} \frac{f(y + tu) - f(y)}{t}$$

where $y \to_f x$ means that $y \to x$ and $f(y) \to f(x)$.

The reader is referred to Clarke's book [4] for a beautiful introduction of these notions.

2. Characterization of minima using variational inequalities

Consider the following problem

$$(\mathcal{P}) \begin{cases} \text{minimize } f(x) \\ x \in C \end{cases}$$

where $f\colon X \to \mathbb{R}$ is continuous and pseudoconvex on the Banach space X and C is a convex subset of X.

Using Definition 1, we can check easily that a local minimum $\bar{x}$ (in some neighborhood of $\bar{x}$) is in fact a global minimum of f on all X. We will show that $\bar{x}$ is a minimum of f on some convex set C, not necessarily a neighborhood of $\bar{x}$, and hence $\bar{x}$ may be not a local minimum, if and only if a certain variational inequality that will be denoted (D) holds.

We say that ∂f satisfies the variational inequality (D) at $\bar{x}$ if

$$\forall x \in C,\ \langle x^*, x - \bar{x}\rangle \ge 0,\ \text{for all } x^* \in \partial f(x).$$

Theorem 2.1. *Let X be a Banach space, $f\colon X \to \mathbb{R}$ continuous and pseudoconvex. Consider the following assertions:*

- *A vector $\bar{x} \in C$ is a solution of $(\mathcal{P})$.*
- *The subdifferential ∂f satisfies (D) at $\bar{x}$.*

Then, (1) *implies* (2). *And* (2) *implies* (1) *if $\partial f(x) \neq \varnothing$, for all $x \in C$.*

Proof. **(1) implies (2)**
Suppose that $\bar{x}$ is a solution of $(\mathcal{P})$. By Proposition 1 (3), if $f(\bar{x}) \le f(x)$ for some $x \in C$ we have

$$\langle x^*, \bar{x} - x\rangle \le 0,\ \forall x^* \in \partial f(x).$$

So, we have (D).

(2) implies (1) when $\boldsymbol{\partial f(x) \neq \varnothing}$, $\boldsymbol{\forall x \in C}$
Consider $x \in C$ such that $x \neq \bar{x}$ and take y in the open segment $(\bar{x}, x) = \{z \in C;\ z = t\bar{x} + (1 - t)x,\ 0 < t < 1\}$ Then,

$$\langle y^*, \bar{x} - y\rangle \le 0,\ \forall y^* \in \partial f(y).$$

Hence, because y lies between $\bar{x}$ and x,

$$\langle y^*, x - y\rangle \ge 0,\ \forall y^* \in \partial f(y).$$

As $\partial f(y) \neq \varnothing$ and f is pseudoconvex, we get

$$f(y) \leq f(x), \ \forall y \in (\bar{x}, x).$$

But f is continuous, so $f(\bar{x}) \leq f(x)$. □

Remark 2.1. *The authors of [7] characterized local minima of quasiconvex functions using the variational inequality* (D)*. They supposed the set* C *to be either a convex neighborhood of* $\bar{x}$ *or* C *is the whole space* X*. While we do not require here our set* C *to be a neighborhood of* $\bar{x}$*.*

3. Characterization of maxima using normal cones

Consider the following problem

$$(\mathcal{Q}) \quad \begin{cases} \text{maximize } f(x) \\ \quad x \in C \end{cases}$$

where $f\colon X \to \mathbb{R}$ is continuous and pseudoconvex on the Banach space X and C is a convex subset of X as in §1.

Denote by

$$C_f(z) = \{x \in C;\ f(x) = f(z)\},$$

and the normal cone $N_C(x)$ to a convex set C at a point x by

$$N_C(x) = \{x^* \in X^*;\ \forall y \in C,\ \langle x^*, y - x\rangle \leq 0\}.$$

Before stating our main result of this section, we shall point out that if f is pseudoconvex, $-f$ is not pseudoconvex in general. So, the results of the former section do not apply when considering a maximum.

Theorem 3.1. *Let* X *be a Banach space,* $f\colon X \to \mathbb{R}$ *continuous and pseudoconvex. Consider* $\bar{x} \in C$ *such that* $\inf_C f < f(\bar{x})$*. Consider also the following assertions:*

- *The vector* $\bar{x} \in C$ *is a solution of* $(\mathcal{Q})$*.*
- *For any* $x \in C_f(\bar{x})$*, we have*

$$\partial f(x) \subset N_C(x).$$

Then, (1) *implies* (2)*. And* (2) *implies* (1) *if* $\partial f(x) \neq \varnothing$*, for all* $x \in C_f(\bar{x})$*.*

Proof. **(1) implies (2)**
Suppose that $\bar{x}$ is a solution of $(\mathcal{Q})$. Then, for any $y \in C$ and any $x \in C_f(\bar{x})$

$$f(y) \leq f(x).$$

By Proposition 1 (3), we get

$$\partial f(x) \subset N_C(x),\ \forall x \in C_f(\bar{x}).$$

(2) implies (1) when $\partial \mathbf{f}(\mathbf{x}) \neq \varnothing$, $\forall \mathbf{x} \in \mathbf{C_f}(\mathbf{x})$
Suppose by contradiction that there is some $\bar{z} \in C$ such that $f(\bar{z}) > f(\bar{x})$. Then, since $f(\bar{x}) > \inf_C f$, we can find $z \in C$ such that

$$f(z) < f(\bar{x}).$$

By the usual mean value theorem for continuous functions, there is $x_0 \in C_f(\bar{x}) \cap (z, \bar{z})$. By (2), for all $x_0^* \in \partial f(x_0)$,

$$\langle x_0^*, z - z_0\rangle \leq 0,$$

and

$$\langle x_0^*, \bar{z} - z_0\rangle \leq 0.$$

So that $\langle x_0^*, z - z_0 \rangle = 0$. Since $\partial f(x_0) \neq \varnothing$ and f is pseudoconvex, we get the contradiction $f(x_0) \leq f(z)$. □

Remark 3.1. *This result extends a previous one by Huriart-Urruty and Ledayev [9] where continuous convex functions were considered.*

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