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Heteroclinic solutions for damped p-Laplacian difference equations

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Abstract. In this paper, we investigate the existence of heteroclinic solutions for a class of p-Laplacian difference equations with a parameter. The proof of the main theorem is variational and based on the use of the Mountain Pass Theorem. Our results successfully improve recent ones in the literature and partially answer an open problem proposed by Cabada and Tersian in [7].

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1. Introduction

In this work, we explore the existence of solutions for a class of damped difference equations with p-Laplacian of the type

$$\begin{cases} \Delta\big(\phi_p(\Delta u(n-1))\big) + c\,\phi_p(\Delta u(n)) + \lambda f(n,u(n)) = 0,\ n \in \mathbb{Z}^+, \\ u(0)=0, \quad u(+\infty)=1 \end{cases} \tag{1}$$

where $p>1$ and $c>0$ are fixed numbers, $\lambda>0$ is a parameter, $\phi_p(t)=|t|^{p-2}t$ for all $t\in\mathbb{R}$ and $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ is continuous with respect to the second variable and satisfies specific growth conditions. Moreover, Δ is the forward difference operator defined as $\Delta u(n-1)=u(n)-u(n-1)$ and $\mathbb{Z}^+=\{1,2,3,...\}$.

Let $v=\big\{v(n)|n\in\mathbb{Z}^+\cup\{0\}\big\}$ be a solution of the following problem

$$\begin{cases} \Delta\big(\phi_p(\Delta v(n-1))\big) + c\,\phi_p(\Delta v(n)) + \lambda g(n,v(n)) = 0,\ n \in \mathbb{Z}^+, \\ v(0)=-1, \quad v(+\infty)=0, \end{cases} \tag{2}$$

where $g(n,v)=f(n,v+1)$ for $n\in\mathbb{Z}^+$ and $v\in\mathbb{R}$.
Set $u=\big\{u(n)|n\in\mathbb{Z}^+\cup\{0\}\big\}$, where $u(n)=v(n)+1,\ n\in\mathbb{Z}^+\cup\{0\}$. Then u is a solution of (1). Moreover, suppose $c=0$ and f is odd with respect to both variables, that is :
(F) $f(-n,-u)=-f(n,u)$, for all $n\in\mathbb{Z}$ and $u\in\mathbb{R}$,
and let $u=\big\{u(n)|n\in\mathbb{Z}\big\}$ be defined by

$$u(n)=\begin{cases} v(n)+1, & \text{for } n\in\mathbb{Z}^+, \\ 0, & \text{for } n=0, \\ -v(-n)-1, & \text{for } n\in\mathbb{Z}^-. \end{cases}$$

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Then, a straightforward calculation proves that u is a solution of the problem

$$\begin{cases} \Delta\big(\phi_p(\Delta u(n-1))\big) + \lambda f(n, u(n)) = 0, \ n \in \mathbb{Z}, \\ u(-\infty) = -1, \quad u(+\infty) = 1. \end{cases} \tag{3}$$

Hence, during this paper, we are concerned about the existence of solutions for (2).

If $p = 2$, problem (1) is reduced to the damped second order problem

$$\begin{cases} \Delta^2 u(n-1) + c\Delta u(n-1) + \lambda f(n, u(n)) = 0, \ n \in \mathbb{Z}^+, \\ u(0) = 0, \quad u(+\infty) = 1. \end{cases} \tag{4}$$

Nonlinear difference equations and mainly discrete p-Laplacian problems is a very vast field. It has been dealt with for more than a decade with various boundary value conditions. Particularly, existence of homoclinic orbits were considered in many works (see for example [11, 12, 15, 17, 25]). Solutions of problem (3) are known as heteroclinic orbits. The study of this type of solutions is scarce in the literature compared to the continuous case. In fact, the continuity of the solutions is a crucial argument in their study and therefore it can not be applied directly to discrete systems. We mention here the first work [27] which attacked to heteroclinic orbits for the discrete equations

$$\Delta^2 u(n-1) + A \sin(u(n)) = 0, \ n \in \mathbb{Z},$$

where A is a positive constant. The authors chose to follow certain effective techniques, first presented by Rabinowitz in [23]. Their results are generalized in [26] to a class of discrete Hamiltonian systems.

In 2017, the authors of [24] followed the same approach to study the existence and multiplicity of heteroclinics for the equation

$$\Delta^2 u(n-1) + p(n) f(u(n)) = 0, \ n \in \mathbb{Z}.$$

Since the nonlinearity is non autonomous (by the presence of the sequence $p(n)$), this work seems to be not accurate. Nevertheless, we can deduce that this approach still useless in case of equations presenting autonomous term or in case where the sequence $p(n)$ is even. Note that in the above works, the periodicity of the nonlinearity is fundamental.

To the best of our knowledge, heteroclinic solutions for discrete p-Laplacian problems were first considered by Cabada and Tersian in [7]. They proved the existence of at least one solution of the problem (3) for all $\lambda > 2/p$. Also, contrary to the above works, the assumption of the periodicity of the nonlinearity is omitted and replaced by a symmetry one. Their proof follows from the construction of a sequence of solutions of a suitable related Dirichlet problems combined with an argument of troncature. They left open the question of existence or nonexistence of heteroclinic solutions to (3) if $\lambda \leq 2/p$.

Newly, in [18], Kuang and Guo partially answered the problem proposed by Cabada and Tersian using variational methods under an assumption of Ambrosetti-Rabinowitz-type on the growth of the nonlinearity.

In this paper, we continue treating problem (3) where the parameter λ may be less than $2/p$ by applying variational methods under assumptions weaker than those used in [18].

On the other hand, the damped second order problem (4), which is related in autonomous case to Fisher-Kolmogorov's equation, were considered in [6] (see also [4]). The authors of [6] obtained the existence of decreasing and heteroclinic type solutions using monotonicity and continuity arguments. Here, we consider the more general problem (1). Under less restrictive assumptions on the nonlinearity and the damping coefficient c, we derive the existence of at least one solution to (1) via Mountain Pass Theorem.

Throughout this work, we suppose that

(A) $a(n) > 0$ for all $n \in \mathbb{Z}^+$ and $\lim_{n\to+\infty} a(n) = +\infty$.

(G) For all $n \in \mathbb{Z}$, $g(n,v) = -a(n)\phi_p(v) + w(n,v)$, where $w(n,v)$ is continuous with respect to the second variable and $W(n,v) = \displaystyle\int_0^v w(n,t)dt$ for all $v \in \mathbb{R}$,

(H_1) $\displaystyle\lim_{v\to 0} \frac{w(n,v)}{|v|^{p-1}} = 0$ uniformly for all $n \in \mathbb{Z}^+$,

(H_2) there exists $\mu > p$ such that

$$\mu W(n,v) \leqslant w(n,v)v,\ \forall v \in \mathbb{R},\ \forall n \in \mathbb{Z}^+,$$

(H_3) $W(n_0, \pm v_0) > 0$ for some $n_0 \in \mathbb{Z}^+,\ v_0 \in \mathbb{R}$.

Our main results are the following:

Theorem 1.1. *Suppose that (A), (G), $(H_1) - (H_3)$ are satisfied. Then there exists $\overline{\lambda} > 0$ such that for any $\lambda > \overline{\lambda}$, equation (2) has at least a nontrivial solution.*

In the particular case where $c = 0$ and f is an odd function with respect to both variables, one gets the existence of heteroclinic solutions for (3). Precisely, we have

Corollary 1.2. *Suppose that $(A), (G), (F)$, $(H_1) - (H_3)$ are satisfied. Then, there exists $\overline{\lambda} > 0$ such that for any $\lambda > \overline{\lambda}$, problem (3) admits at least a nontrivial solution.*

The remainder of the paper is organized as follows: In Section 2, we present some preliminary results. In Section 3 we give the proof of Theorem 1.1 and Corollary 1.2.

2. Preliminary results

Let $S = \{v = \{v(n)\} | v(n) \in \mathbb{R},\ n \in \mathbb{Z}^+\}$ be a vector space with $av + bx = \{av(n) + bx(n)\}$ for $v, x \in S$ and $a, b \in \mathbb{R}$. Also, let

$$l_c^p(\mathbb{Z}^+) = \left\{ v \in S | \sum_{n=1}^{+\infty} (1+c)^n |v(n)|^p < \infty \right\},$$

and

$$H = \left\{ v \in S | \sum_{n=1}^{+\infty} [(1+c)^n a(n) |v(n)|^p] < \infty \right\}.$$

Define the norms of $l^p_c(\mathbb{Z}^+)$ and H respectively, as follows

$$|v|_p = \left(\sum_{n=1}^{+\infty}(1+c)^n|v(n)|^p\right)^{\frac{1}{p}}, \quad \|v\| = \left(\sum_{n=1}^{+\infty}[(1+c)^n a(n)|v(n)|^p]\right)^{\frac{1}{p}}.$$

Evidently, $(l^p_c(\mathbb{Z}^+), |.|_p)$ is a reflexive Banach space and the embedding $l^p_c(\mathbb{Z}^+) \hookrightarrow l^p(\mathbb{Z}^+)$ is compact where $l^p(\mathbb{Z}^+) = \left\{v \in S|\sum_{n=1}^{+\infty}|v(n)|^p < \infty\right\}$ (see Proposition 3 in [15]). The following lemma displays the main properties of the spaces defined above.

Lemma 2.1. *Suppose that* (A) *holds. Then, for all* $1 < p < \infty$, $(H, \|.\|)$ *is a reflexive and separable Banach space and the embedding* $H \hookrightarrow l^p_c(\mathbb{Z}^+)$ *is compact.*

Proof. Similarly to Proposition 3 in [15], one obtains that $(H, \|.\|)$ is a reflexive Banach space. Let $M = \inf_{n\in\mathbb{Z}^+} a(n)$, from the assumption (A), it's clear that $M > 0$. Moreover, we have

$$|v|_p \leqslant M^{-\frac{1}{p}}\|v\|, \ \forall v \in H. \tag{5}$$

Then the embedding $H \hookrightarrow l^p_c$ is continuous. It remains to prove that this embedding is compact. Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence in H. Assume, without loss of generality, that $v_k \rightharpoonup 0$ in H. Then, there is $N > 0$ such that

$$\|v_k\| \leqslant N, \quad \forall k \in \mathbb{N}.$$

Using condition (A), for $\varepsilon > 0$, we can find $n^* \in \mathbb{Z}^+$ such that

$$a(n) \geqslant \frac{2}{\varepsilon}N^p, \quad \forall n > n^*.$$

hence, one obtains

$$\sum_{n>n^*}(1+c)^n|v_k(n)|^p \leqslant \frac{\varepsilon}{2N^p}\sum_{n>n^*}(1+c)^n a(n)|v_k(n)|^p \leqslant \frac{\varepsilon}{2}. \tag{6}$$

On the other hand, using condition (A) and $(1+c)^n > 0$ for all $n \in \mathbb{Z}^+$, we can say that $v_k \rightharpoonup 0$ in H_{n^*}, where $H_{n^*} = \{v \in H|\, v(n) \in \mathbb{R},\, n \in \mathbb{N}[1, n^*]\}$. Since H_{n^*} is finite dimensional space, thus $v_k \to 0$ in H_{n^*}. As l^p_{c,n^*} is also finite dimensional space, so there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{n=1}^{n^*}(1+c)^n|v_k(n)|^p \leqslant \frac{\varepsilon}{2}, \quad \forall k \geqslant k_0. \tag{7}$$

From (6) and (7), one have, for all $k \geqslant k_0$,

$$\begin{aligned}\sum_{n=1}^{+\infty}(1+c)^n|v_k(n)|^p &= \sum_{n=1}^{n^*}(1+c)^n|v_k(n)|^p + \sum_{n=n^*+1}^{+\infty}(1+c)^n|v_k(n)|^p \\ &\leqslant \varepsilon.\end{aligned}$$

Since ε is arbitrary, one gets $v_k \to 0$ in l^p_c. □

For all $v \in H$ such that $v(0) = -1$, consider the functionals Ψ_1 and Ψ_2 defined by

$$\Psi_1(v) = \frac{1}{p}\sum_{n=1}^{+\infty}(1+c)^n|\Delta v(n-1)|^p + \frac{\lambda}{p}\|v\|^p - \frac{1+c}{p},$$

$$\Psi_2(v) = \lambda \sum_{n=1}^{+\infty} (1+c)^n W(n, v(n)),$$

and

$$\Psi(v) = \Psi_1(v) - \Psi_2(v).$$

Lemma 2.2. [15] *If V is a compact subset of l_c^p, then for all $\varepsilon > 0$ there is $n' \in \mathbb{N}$ such that*

$$\left[\sum_{n>n'} (1+c)^n |v(n)|^p\right]^{\frac{1}{p}} \leqslant \varepsilon, \quad \forall v \in V.$$

Remark 2.1. Similarly to Lemma 2.2, suppose that $V \subset l^p$ is a compact subset. Thus for every $\varepsilon > 0$ there is $n' > 0$ such that

$$\left[\sum_{|n|>n'} |v(n)|^p\right]^{\frac{1}{p}} \leqslant \varepsilon, \quad \forall v \in V.$$

Proposition 2.3. [15] *If condition (A) is satisfied then $\Psi_1 \in C^1(H)$ and for all $v, x \in H$,*

$$\langle \Psi_1'(v), x\rangle = \sum_{n=1}^{+\infty} (1+c)^n [\phi_p(\Delta v(n-1))\Delta x(n-1) + \lambda a(n)\phi_p(v(n))x(n)].$$

Analogously to Proposition 2.6 in [9], we have

Proposition 2.4. *If (H_1) holds, then $\Psi_2 \in C^1(l_c^p(\mathbb{Z}^+))$ with*

$$\langle \Psi_2'(v), x\rangle = \lambda \sum_{n=1}^{+\infty} (1+c)^n w(n, v(n))x(n), \ \forall v, x \in l_c^p(\mathbb{Z}^+).$$

Remark 2.2. From Lemma 2.1 and Proposition 2.4, one gets $\Psi_2 \in C^1(l_c^p(\mathbb{Z}^+))$ and the embedding $H \hookrightarrow l_c^p(\mathbb{Z}^+)$ is continuous. So we obtain that $\Psi_2 \in C^1(H)$.

Similarly to Proposition 2.8 in [9], one easily gets the following result, which implies that a nonzero critical point of the functional Ψ defined on H is a nontrivial solution of (2) (see also [18]).

Proposition 2.5. [18] *If (A) and (H_1) hold, then $\Psi \in C^1(H)$ and any critical point $v \in H$ of Ψ is a solution of (2) with*

$$\begin{aligned}\langle \Psi'(v), x\rangle \quad &= \quad -\sum_{n=1}^{+\infty} (1+c)^n \left[\Delta(\phi_p(\Delta v(n-1)) + c\phi_p(\Delta v(n)) - \lambda a(n)\phi_p(v(n))\right.\\ &\qquad \left.+\lambda l(n, v(n))\right] x(n), \quad \forall v, x \in H.\end{aligned}$$

Proposition 2.6. *Assume that (A), (H_1) and (H_2) hold. Then the functional Ψ satisfies the Palais-Smale condition.*

Proof. Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence in H such that $\{\Psi(v_k)\}_{k\in\mathbb{N}}$ is bounded and $\Psi'(v_k) \to 0$ as $k \to +\infty$. Let B a positive constant such that $|\Psi(v_k)| \leqslant B$ for all $k \in \mathbb{N}$. We will show that $\{v_k\}_{k\in\mathbb{N}}$ possesses a convergent subsequence. By (H_2), it yields

$$\begin{aligned} B+\|v_k\| &\geqslant \Psi(v_k)-\frac{1}{\mu}\Psi'(v_k)v_k \\ &= \sum_{n=1}^{+\infty}(1+c)^n[(\frac{1}{p}-\frac{1}{\mu})|\Delta v_k(n-1)|^p+\frac{\lambda}{\mu}w(n,v_k(n))v_k(n) \\ &\quad -\lambda W(n,v_k(n))]+\lambda(\frac{1}{p}-\frac{1}{\mu})\|v_k\|^p-\frac{1+c}{p}, \\ &\geqslant \lambda(\frac{1}{p}-\frac{1}{\mu})\|v_k\|^p-\frac{1+c}{p}. \end{aligned}$$

As $\mu>p>1$, it follows that the sequence $\{v_k\}_{k\in\mathbb{N}}$ is bounded in H. Using Lemma 2.1, passing to a subsequence still denoted by $\{v_k\}_{k\in\mathbb{N}}$, one gets $v_k \rightharpoonup v$ in H and $v_k\to v$ in $l_c^p(\mathbb{Z}^+)$. Thus, for all $\varepsilon>0$, there exists $k_1\in\mathbb{N}$ such that

$$|v_k-v|_p\leqslant\frac{\varepsilon}{2},\quad \forall k\geqslant k_1. \tag{8}$$

On the other hand, by (H_1), there exists $\beta>0$ such that

$$|w(n,t)|\leqslant|t|^{p-1},\ \forall n\in\mathbb{Z}^+,\ |t|\leqslant\beta. \tag{9}$$

Also, we have

$$\|v_k-v\|_p\leqslant|v_k-v|_p.$$

Hence, using Remark 2.1, there is an integer $n_1>0$ with $|v_k(n)|\leqslant\beta$ for every $k\in\mathbb{N}$ and $|v(n)|\leqslant\beta$ for every $n\in\mathbb{N}$, $n>n_1$. Then, using (9), we get

$$|w(n,v_k(n))|\leqslant|v_k(n)|^{p-1},\ \forall k\in\mathbb{N},\ \ |w(n,v(n))|\leqslant|v(n)|^{p-1}. \tag{10}$$

Moreover, since $\{v_k\}_{k\in\mathbb{N}}\cup\{v\}$ is a compact subset of $l_c^p(\mathbb{Z}^+)$, Lemma 2.2 implies that there is $n_2>0$, $n_2\geqslant n_1$, such that

$$\sum_{n>n_2}(1+c)^n|v_k(n)|^p\leqslant 1,\ \forall k\in\mathbb{N},\ \ \sum_{n>n_2}(1+c)^n|v(n)|^p\leqslant 1. \tag{11}$$

Combining Minkowski's inequality with (8), (10) and (11), we get

$$\left|\sum_{n>n_2}(1+c)^n[w(n,v_k(n))-w(n,v(n))](v_k(n)-v(n))\right| \quad\leqslant\quad \varepsilon,\ \forall\ k\geqslant k_1$$

which yields

$$\lim_{k\to+\infty}\sum_{n>n_1}(1+c)^n[w(n,v_k(n))-w(n,v(n))](v_k(n)-v(n))=0. \tag{12}$$

Besides, by continuity of the finite sum, the uniform continuity of $w(n,v)$ in v and $v_k\to v$ in $l_c^p(\mathbb{Z}^+)$, it yields

$$\lim_{k\to+\infty}\sum_{n=1}^{n_1}(1+c)^n[w(n,v_k(n))-w(n,v(n))](v_k(n)-v(n))=0. \tag{13}$$

As a result, (12) and (13) give us

$$\lim_{k\to+\infty}\sum_{n=1}^{+\infty}(1+c)^n[w(n,v_k(n))-w(n,v(n))](v_k(n)-v(n))=0. \tag{14}$$

Let's recall the following well-known inequalities [20]:
There exists $d > 0$ *such that*

$$(\phi_p(a) - \phi_p(b))(a - b) \geqslant d|a - b|^p, \ \forall a, b \in \mathbb{R}, \ for\ p \geqslant 2, \tag{15}$$

$$(\phi_p(a) - \phi_p(b))(a - b) \geqslant d(|a| + |b|)^{p-2}|a - b|^2, \ \forall a, b \in \mathbb{R}, \ for\ 1 < p < 2.$$

Considering the case $p \geqslant 2$. It yields from (15) that

$$d|\Delta v_k(n-1) - \Delta v(n-1)|^p \leqslant [\phi_p(\Delta v_k(n-1)) - \phi_p(\Delta v(n-1))](\Delta v_k(n-1) - \Delta v(n-1)),$$

$$d|v_k(n) - v(n)|^p \leqslant [\phi_p(v_k(n)) - \phi_p(v(n))](v_k(n) - v(n)).$$

Summarizing what proved above, one obtains

$$\begin{aligned} & d\lambda\|v_k - v\|^p + d\sum_{n=1}^{+\infty}(1+c)^n|\Delta v_k(n-1) - \Delta v(n-1)|^p \qquad (16) \\ \leqslant \ & \lambda\sum_{n=1}^{+\infty}(1+c)^n[w(n, v_k(n)) - w(n, v(n))](v_k(n) - v(n)) + \langle \Psi'(v_k) - \Psi'(v), v_k - v\rangle. \end{aligned}$$

From $\Psi'(v_k) \to 0$ as $k \to +\infty$, the boundedness of the sequence $\{v_k\}_{k\in\mathbb{N}}$ and the fact that $v_k \rightharpoonup v$ in H, it follows that

$$\langle \Psi'(v_k) - \Psi'(v), v_k - v\rangle \to 0 \ as\ k \to \infty. \tag{17}$$

Moreover, by (14), (16) and (17), one gets

$$d\lambda\|v_k - v\|^p + d\sum_{n=1}^{+\infty}(1+c)^n|\Delta v_k(n-1) - \Delta v(n-1)|^p \to 0 \ as\ k \to \infty.$$

Since $\sum_{n=1}^{+\infty}(1+c)^n|\Delta v_k(n-1) - \Delta v(n-1)|^p \geqslant 0$, it yields $\|v_k - v\| \to 0$ as $k \to \infty$.
The proof is completed. □

Lemma 2.7. *Assume that* (H_2) *holds. Then for all* $(n, x) \in \mathbb{N}^* \times \mathbb{R}$, $s^{-\mu}W(n, sx)$ *is increasing on* $]0, +\infty[$.

In order to prove our main result, we need the following theorem introduced in [22]. Let B_ρ denote a closed ball of radius ρ about 0.

Theorem 2.8. ***(Mountain Pass Theorem*** [22]***)***
Let E *be a real Banach space and* $I \in C^1(E, \mathbb{R})$ *satisfying the Palais-Smale condition. Suppose* $I(0) = 0$ *and*
(I_1) *there are constants* $\rho, \alpha > 0$ *such that* $I|_{\partial B_\rho} \geqslant \alpha$;
(I_2) *there is an* $e \in E \setminus B_\rho$ *such that* $I(e) \leqslant 0$.
Then I *possesses a critical value* $c \geqslant \alpha$ *given by*

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0, g(1) = e\}.$$

3. Proof of Theorem 1.1

At first, we show that Ψ satisfies condition (I_1) of Theorem 2.8. It follows from (H_2) that there exists $\delta > 0$ such that

$$|W(n,v)| \leqslant \frac{M}{2p}|v|^p, \ \forall n \in \mathbb{Z}^+, \ \forall |v| \leqslant \delta. \tag{18}$$

Let $\delta > 0$ be such that (18) satisfied and $v \in H$ be such that

$$\|v\| \leqslant M^{\frac{1}{p}}\delta.$$

Moreover, it is easy to check from (A) that

$$\|v\|_p \leqslant M^{-\frac{1}{p}}\|v\|, \ \forall v \in H,$$

where $\|.\|_p$ is the norm in the space $l^p(\mathbb{Z}^+)$. Combining the previous inequality with $\|v\|_\infty \leqslant \|v\|_p$ where $\|.\|_\infty$ is the norm of $l^\infty(\mathbb{Z}^+)$, we obtain

$$\|v\|_\infty \leqslant M^{-\frac{1}{p}}\|v\|.$$

Thus, one gets

$$|v(n)| \leqslant \delta, \ \forall n \in \mathbb{Z}^+. \tag{19}$$

From (5) together with (18) and (19), we obtain

$$|\sum_{n=1}^{+\infty}(1+c)^n W(n,v(n))| \leqslant \frac{1}{2p}\|v\|^p. \tag{20}$$

Set $\rho = \delta M^{1/p}$. By (20), one gets for all $v \in \partial B_\rho \cap H$,

$$\begin{aligned}\Psi(v) &= \frac{1}{p}(\sum_{n=1}^{+\infty}(1+c)^n|\Delta v_k(n-1)|^p + \lambda\|v\|^p) - \lambda\sum_{n=1}^{+\infty}(1+c)^n W(n,v(n)) - \frac{1+c}{p},\\ &\geqslant \frac{\lambda}{2p}\|v\|^p - \frac{1+c}{p}.\end{aligned}$$

Let $\lambda > 2(1+c)/\delta^p M := \overline{\lambda}$. Then

$$\Psi(v) \geqslant \alpha := \frac{\lambda}{2p}\delta^p M - \frac{1+c}{p} > 0.$$

As a result Ψ satisfies (I_1) of Theorem 2.8.
Finally, we prove that Ψ satisfies (I_2) of Theorem 2.8. Without loss of generality, we may assume that $v_0 = 1$ in (H_3) and let $\beta_0 := \min\{W(n_0, \pm 1)\}$. Using a similar result as Lemma 2.7 with $s_1 = \frac{1}{|v|}$ and $s_2 = 1$ for $|v| \geqslant 1$, it is easy to check that

$$W(n_0, v) \geqslant \beta_0|v|^\mu, \ \forall |v| \geqslant 1. \tag{21}$$

Now, let $e \in H$ be such that

$$e(n) = \begin{cases} \sigma, & for \ \ n = n_0, \\ 0, & for \ \ n \in \{i \in \mathbb{Z}^+ | i \neq n_0\}.\end{cases}$$

From (21), we have

$$\begin{aligned}\Psi(e) =& \frac{1}{p}\sum_{n=1}^{+\infty}|\Delta e(n-1)|^p + \frac{\lambda}{p}\|e\|^p - \lambda(1+c)^{n_0}W(n_0,\sigma) - \frac{1+c}{p},\\ =& \frac{1}{p}(1+c)^{n_0}\sigma^p - \frac{1}{p}(1+c)^{n_0+1}\sigma^p + \frac{\lambda}{p}(1+c)^{n_0}a(n_0)\sigma^p - \lambda(1+c)^{n_0}W(n_0,\sigma)\\ &- \frac{1+c}{p} \leqslant (1+c)^{n_0}\Big(\frac{1+\lambda a(n_0)}{p}\sigma^p - \lambda\beta_0\sigma^\mu\Big) - \frac{1+c}{p}.\end{aligned}$$

As $\mu > p$, we can choose $\sigma > 1$ such that $\|e\| > \rho$ where ρ is defined above and $\Psi(e) \leqslant \Psi(0) = 0$.

Hence all assumptions of Theorem 2.8 are satisfied and therefore Ψ possesses a critical value which yields a nontrivial solution of (2) and the proof is completed. $\square$

Remark 3.1. Choosing M large enough, one obtains $\overline{\lambda} < \frac{2}{p}$. Consequently, in Corollary 1.2, we give a partial answer for the open problem proposed by Cabada and Tersian in [7] for the case where $\overline{\lambda} < \lambda \leqslant \frac{2}{p}$ by weakening the conditions chosen by Kuang and Guo in [18].

Example 3.1. Let $p = 3, c = 1, \mu = 4, a(n) = 20\ n$ and W be defined by

$$W(n,v) = (\frac{1}{|n|+1} - 1)v^4 + |v|^5, \ \forall\ (n,t) \in \mathbb{Z}\times\mathbb{R}.$$

A straightforward calculation proves that inequality (18) holds true with $\delta = 0.8$ and then $\overline{\lambda} = \frac{4}{(0.8)^3 20} < 2/p$. Hence Theorem 1.1 and Corollary 1.2 extend the results in [6, 7, 18] for instance.

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