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Some remarks on quadratic differentials on Klein surfaces

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Abstract. In this paper one proves some characterisations of quadratic differentials on Klein surfaces by symmetric quadratic differentials and families of meromorphic functions on the corresponding double covering.

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A dianalytic atlas $\mathcal{A}$ on a surface X is a family $\mathcal{A} = \{(\widetilde{U}_i, h_i, V_i)\}_{i\in I}$, where:

a) $(\widetilde{U}_i)_{i\in I}$ is an open cover of X,

b) for every $i \in I$, V_i is an open set in the complex plane $\mathbf{C}$,

c) $h_i : \widetilde{U}_i \longrightarrow h_i(\widetilde{U}_i) = V_i$ is an homeomorphism, for every $i \in I$,

d) if $i, j \in I$, then $\widetilde{U}_i \cap \widetilde{U}_j = \emptyset$ or $\widetilde{U}_i \cap \widetilde{U}_j \neq \emptyset$ and in this case $h_i \circ h_j^{-1} : h_j(\widetilde{U}_i \cap \widetilde{U}_j) \to h_i(\widetilde{U}_i \cap \widetilde{U}_j)$ is a dianalytic function on $h_j(\widetilde{U}_i \cap \widetilde{U}_j)$. The charts $(\widetilde{U}_i, h_i, V_i)$ and $(\widetilde{U}_j, h_j, V_j)$ are dianalytic compatible.

A Klein surface is a pair $(X, \mathcal{A})$, where X is a surface and $\mathcal{A}$ is a maximal dianalytic atlas on X, such that $\mathcal{A}$ does not contain any analytic subatlas.

Let $\mathcal{O}_2$ be a Riemann surface. A mapping $\mathbf{k} : \mathcal{O}_2 \to \mathcal{O}_2$ with property $\mathbf{k} \circ \mathbf{k} = Id$, where Id is the identity of $\mathcal{O}_2$, is an involution of $\mathcal{O}_2$.

A symmetric Riemann surface is a pair $(\mathcal{O}_2, \mathbf{k})$, consisting of a Riemann (orientable) surface $\mathcal{O}_2$ and an antianalytic involution, $\mathbf{k} : \mathcal{O}_2 \to \mathcal{O}_2$ having no fixed points.

Let X be a Klein surface. Then exists a Riemann surface $\widehat{X}$ and a covering mapping $\pi : \widehat{X} \to X$. For $\widehat{X}$ satisfies the universal property, then $\widehat{X}$ is the universal covering of X. $\widehat{X}$ is conformal equivalent with :

1) $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, if X is the real projective plane $\mathbf{P}^2$,

2) the complex plane $\mathbf{C}$ if X is the pointed real projective plane $\mathbf{P}^2 \backslash \{0\}$ or a Klein bottle,

3) $\{z \in \mathbf{C} \mid Imz > 0\}$ in the other cases.

Let $\mathcal{G}$ be the covering mappings group of π. Because X is nonorientable , $\mathcal{G}$ will contains either analytic automorphisms of $\widehat{X}$ or antianalytic automorphisms of $\widehat{X}$. Let $\mathcal{G}_1$ be the subgroup of analytic automorphisms of $\mathcal{G}$. Then $\mathcal{G}_1$ is a subgroup of $\mathcal{G}$ and for every $S \in \mathcal{G} \setminus \mathcal{G}_1$, $\mathcal{G} = \mathcal{G}_1 \cup S\mathcal{G}_1$, $\mathcal{G}_1 \cap S\mathcal{G}_1 = \emptyset$, where $S\mathcal{G}_1 = \{S \circ T \mid T \in \mathcal{G}_1\}$.

If $\widehat{P} \in \widehat{X}$, then we denote with $\widetilde{P}$ (respectively P) its $\mathcal{G}$- orbit (respectively its $\mathcal{G}_1$-orbit). Therefore

$$\widetilde{P} = \{G(\widehat{P}) \mid G \in \mathcal{G}\} \quad \text{and} \quad P = \{T(\widehat{P}) \mid T \in \mathcal{G}_1\}$$

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The quotient space $\widehat{X}/\mathcal{G} = \{\widetilde{P} \mid \widehat{P} \in \widehat{X}\}$ has a Klein surface structure and the projection mapping $\pi : \widehat{X} \to \widehat{X}/\mathcal{G}$, $\widehat{P} \stackrel{\pi}{\to} \widetilde{P}$ is a dianalytic covering mapping, by $\widetilde{P}$ definition.

Theorem 0.1. *Let X be a Klein surface, $\widehat{X}$ the universal covering of X, $\pi : \widehat{X} \to X$ the corresponding covering mapping and $\mathcal{G}$ the covering mappings group of π. If $\mathcal{O}_2 = \widehat{X}/\mathcal{G}_1$, then $\mathcal{O}_2$ has a Riemann surface structure.*

Proof. By construction $\mathcal{O}_2 = \{P \mid \widehat{P} \in \widehat{X}\}$, therefore $\mathcal{O}_2$ is a surface. The covering projection $p : \widehat{X} \to \mathcal{O}_2$, $p(\widehat{P}) = P$ is an analytic covering mapping. Because for every $S_1, S_2 \in \mathcal{G} \setminus \mathcal{G}_1$, $p(S_1(\widehat{P})) = p(S_2(\widehat{P}))$, for every $\widehat{P} \in \widehat{X}$, $S_1 \circ S_2^{-1}$ is analytic, it results that $\mathbf{k} : \mathcal{O}_2 \to \mathcal{O}_2$, $\mathbf{k}(P) = Q$, is well defined, namely doesn′t depend of S and $\widehat{P} \in p^{-1}(P)$ where $Q = \{T(\widehat{Q}) | T \in \mathcal{G}_1\}$, $\widehat{Q} = S(\widehat{P})$ and $S \in \mathcal{G} \setminus \mathcal{G}_1$. $\mathbf{k}(P)$ is the $\mathcal{G}_1$- orbit of $S(\widehat{P})$ and because S is antianalytic, the mapping $\mathbf{k}$ is antianalytic too. $\Big($the $\mathcal{G}_1$- orbit of $\widehat{Q}\Big) \cap \Big($the $\mathcal{G}_1$-orbit of $\widehat{P}\Big) = \emptyset$, for every $P \in \mathcal{O}_2$, means that the mapping $\mathbf{k}$ doesn′t have fixed points. Also, $\mathbf{k}$ is an involution because $(\mathbf{k} \circ \mathbf{k})(P) = \mathbf{k}(Q) =$ (the $\mathcal{G}_1-$orbit of $S_1(\widehat{Q})) =$ (the $\mathcal{G}_1$- orbit of $(S_1 \circ S)(\widehat{P})) =$ (the $\mathcal{G}_1$- orbit of $\widehat{P}) = P$, where $S_1 \circ S \in \mathcal{G}_1$, because S_1, $S \in \mathcal{G} \setminus \mathcal{G}_1$. Therefore, $\mathbf{k}$ is an antianalytic involution, without fixed points. If $q : \mathcal{O}_2 \to X$ is the covering projection, $q(P) = \widetilde{P}$, for every $P \in \mathcal{O}_2$, then $q = q \circ \mathbf{k}$ and the following diagram is comutative:

$$\begin{array}{ccc} \widehat{X} & \stackrel{S}{\to} & \widehat{X} \\ p\downarrow & & p\downarrow \\ \mathcal{O}_2 & \stackrel{\mathbf{k}}{\to} & \mathcal{O}_2 \\ q\downarrow & q\swarrow & \\ X & & \end{array}$$

Let $\mathcal{B}_1$ (respectively $\mathcal{B}_2$) be the maximal analytic atlases on $\mathcal{O}_2$. $\mathcal{B}_1$ determines the analytic structure on $\mathcal{O}_2$ and $\mathcal{B}_2$ the analytic structure on $\mathbf{k}(\mathcal{O}_2)$. $\mathbf{k}(\mathcal{O}_2)$ is thinking like the surface $\mathcal{O}_2$ endowed with its second orientation. Then $\mathbf{k} : \mathcal{O}_2 \to \mathbf{k}(\mathcal{O}_2)$ is an antianalytic isomorphism. So, $q : \mathcal{O}_2 \to X$ is a dianalytic mapping, which mixed the two structures of $\mathcal{O}_2$ and $\mathbf{k}(\mathcal{O}_2)$. Therefore $\mathcal{O}_2 = (\mathcal{O}_2, \mathcal{B}_1)$, $\mathbf{k}(\mathcal{O}_2) = (\mathcal{O}_2, \mathcal{B}_2)$. □

We denote by $\mathcal{H}$ the group consisting of $\mathbf{k}$ and the identity of $\mathcal{O}_2$, with respect to the usual composition of functions.

Theorem 0.2. *$\mathcal{O}_2/\mathcal{H}$ is dianalytic equivalent with X.*

Proof. Let $P \in \mathcal{O}_2$, its $\mathcal{H}$- orbit consists of two elements P and $\mathbf{k}(P)$. Therefore , $\widetilde{P} = P \cup \mathbf{k}(P)$ and the mapping $\{\ P\ ,\ \mathbf{k}(P)\ \} \to \widetilde{P}$ is a dianalytic isomorphism between $\mathcal{O}_2/\mathcal{H}$ and X. Then X can be identify with $\mathcal{O}_2/\mathcal{H}$. □

The diagram is comutative :

$$\begin{array}{ccccc} \widehat{X} & & & & \\ p\downarrow & \stackrel{\pi}{\searrow} & & & \\ \mathcal{O}_2 & \underset{q}{\longrightarrow} & X & \longleftrightarrow & \mathcal{O}_2/\mathcal{H} \end{array}$$

where we have denoted with $\longleftrightarrow$ the dianalytic equivalence.

Theorem 0.3. *Let $(\mathcal{O}_2, \mathbf{k})$ be a symmetric Riemann surface and $\mathcal{H}$ the two elements group generated by $\mathbf{k}$. Then the covering projection $q : \mathcal{O}_2 \to \mathcal{O}_2/\mathcal{H}$ induces a Klein surface structure on $\mathcal{O}_2/\mathcal{H}$.*

Let $\mathcal{O}_2$ be a Riemann surface with the analytic structure $\{(U_i, h_i, V_i)\}_{i\in I}$. A *meromorphic quadratic differential* Φ on $\mathcal{O}_2$ is a family of meromorphic functions $(\varphi_i)_{i\in I}$, in the local parameters $z_i = h_i(P)$, $i \in I$, for which the transformation law

$$\varphi_i(z_i)dz_i^2 = \varphi_j(z_j)dz_j^2, dz_j = \frac{dz_j}{dz_i}dz_i$$

holds for every $i, j \in I$, whenever z_i and z_j are parameters values which correspond to the same point P of $\mathcal{O}_2$.

Because for every parametric disk $\widetilde{U}$ of X , the preimage $q^{-1}(\widetilde{U}) = (U, \mathbf{k}(U))$, is natural to consider the restriction at $U \cup \mathbf{k}(U)$ in the local study of the meromorphic quadratic differentials on $\mathcal{O}_2$. But $\mathbf{k}$ is an involution without fixed points so we can consider $U \cap \mathbf{k}(U) = \emptyset$.

We denote with $Q^2(\mathcal{O}_2)$, respectively with $\overline{Q^2(\mathcal{O}_2)}$, the vectorial space of the meromorphic (respectively antimeromorphic) quadratic differentials on $(\mathcal{O}_2, \mathcal{B}_1)$.

Let V and $\mathbf{f}(V)$ the images through the corresponding charts of the parametric disks U, respectively $\mathbf{k}(U)$. Because $\mathbf{k}$ is an antianalytic involution it results that $\mathbf{f}$ is an antianalytic involution. We will use z, like local parameter on U and w, like local parameter on $\mathbf{k}(U)$.

Theorem 0.4. *There is an isomorphism $\mathbf{K}$, between $Q^2(\mathcal{O}_2)$ and $\overline{Q^2(\mathcal{O}_2)}$.*

Proof. $\Phi \in Q^2(\mathcal{O}_2)$ with the local representation :

$$\Phi^*/U \cup \mathbf{k}(U) = \begin{cases} \varphi(z)dz^2, & z \in V \\ \widehat{\varphi}(w)dw^2, & w \in \mathbf{f}(V) \end{cases}$$

where φ and $\widehat{\varphi}$ are meromorphic functions on V, respectively $\mathbf{f}(V)$. If φ is not holomorphic, namely it has at least a pole then $z \in V$ means z is not a pole of φ.

Then the symmetry $\mathbf{k}$ will induce the isomorphism $\mathbf{K} : Q^2(\mathcal{O}_2) \to \overline{Q^2(\mathcal{O}_2)}$:

$$\mathbf{K}(\Phi)^*/U \cup \mathbf{k}(U) = (\Phi \circ \mathbf{k})^*/U \cup \mathbf{k}(U) = \begin{cases} \widehat{\varphi}(\mathbf{f}(z))d\mathbf{f}(z)^2, & \text{if } z \in V \\ \varphi(\mathbf{f}(w))d\mathbf{f}(w)^2, & \text{if } w \in \mathbf{f}(V) \end{cases}$$

and because $\mathbf{f}$ is an antianalytic function

$$\mathbf{K}(\Phi)^*/U \cup \mathbf{k}(U) = \begin{cases} \widehat{\varphi}(\mathbf{f}(z))\left(\frac{\partial \mathbf{f}}{\partial \overline{z}}(z)\right)^2 d\overline{z}^2, & \text{if } z \in V \\ \varphi(\mathbf{f}(w))\left(\frac{\partial \mathbf{f}}{\partial \overline{w}}(w)\right)^2 d\overline{w}^2, & \text{if } w \in \mathbf{f}(V) \end{cases}.$$

We will use the following diagram :

$$\begin{array}{ccccccccc}
V & \overset{\widehat{h}_{T,\widehat{U}}}{\longleftarrow} & T(\widehat{U}) & \overset{T}{\longleftarrow} & \widehat{U} & & & & \\
& \overset{\varphi}{\searrow} & \downarrow & \overset{p/T(\widehat{U})}{\searrow} & p/\widehat{U}\downarrow & \overset{S}{\searrow} & & & \\
& & \mathbf{C} & \overset{\varphi\circ\mathbf{p}_{T,U}^{-1}}{\longleftarrow} & U & \longrightarrow & S(\widehat{U}) & \overset{\widehat{h}_{S,\widehat{U}}}{\longrightarrow} & \mathbf{f}(V) \\
& & & & \mathbf{k}\downarrow & \overset{p/S(\widehat{U})}{\swarrow} & \downarrow & \overset{\widehat{\varphi}}{\swarrow} & \\
& & & & \mathbf{k}(U) & \overset{\widehat{\varphi}\circ\mathbf{p}_{S,U}^{-1}}{\longrightarrow} & \mathbf{C} & &
\end{array}$$

Let $\Phi \in Q^2(\mathcal{O}_2)$ with the local representation $\varphi(z)dz^2$ on U. Then $\varphi\circ\, \mathbf{p}_{T,U}^{-1} : U \to \mathbf{C}$ is a meromorphic mapping on $(\mathcal{O}_2,\, \mathcal{B}_1)$. Also $\widehat{\varphi}\circ\mathbf{p}_{S,U}^{-1} : \mathbf{k}(U) \to \mathbf{C}$ is a meromorphic mapping on $(\mathcal{O}_2,\, \mathcal{B}_2)$.

Let $\mathbf{p}_{\mathbf{k},S(\widehat{U})} = \mathbf{p}_{S,U}^{-1} \circ \mathbf{k}/U$ and $(U, \mathbf{p}_{\mathbf{k},S(\widehat{U})}, V_{S,\widehat{U}} = \mathbf{f}(V))$ a mapping on $(\mathcal{O}_2, \mathcal{B}_2)$, where $S \in \mathcal{G}\setminus\mathcal{G}_1$. Then $(\widehat{\varphi}\circ\mathbf{p}_{S,U}^{-1}\circ\mathbf{k}) \circ \mathbf{p}_{\mathbf{k},S(\widehat{U})}^{-1} = \widehat{\varphi}$ and $\widehat{\varphi}\circ\mathbf{p}_{S,U}^{-1}\circ\mathbf{k}$ is a meromorphic mapping on $(\mathcal{O}_2, \mathcal{B}_2)$, namely $\widehat{\varphi} \circ \mathbf{f}$ is an antimeromorphic function in parameter z. But $\widehat{\varphi} \circ \mathbf{f}$ is the local representation of $\mathbf{K}(\Phi)$ in parameter z and by the defintion of Φ we have $\widehat{\varphi}(w)dw^2 = \widehat{\varphi}_0(w_0)dw_0^2$, for every w and w_0 parametrics values which correspond to the same point of $\mathcal{O}_2$ and for which the transition mapping is analytic, where $\widehat{\varphi}_0$ is the representation of Φ in the parameter w_0. We obtain $\mathbf{K}(\Phi) \in \overline{Q^2(\mathcal{O}_2)}$. Thus, $\mathbf{K}$ is well defined and by the definition we obtain that $\mathbf{K}$ is an isomorphism. $\square$

Let Δ be an open, $\mathbf{k}$-symmetric, subset of $\mathcal{O}_2$, namely an open subset which satisfies the condition $\mathbf{k}(\Delta) = \Delta$. Then $\Phi' \in Q^2(\mathcal{O}_2)\oplus \overline{Q^2(\mathcal{O}_2)}$ is called symmetric, respectively antisymmetric, quadratic differential , on Δ iff :

$$\Phi'/\Delta = (\Phi' \circ \mathbf{k})/\Delta, \text{ respectively } \Phi'/\Delta = -(\Phi' \circ \mathbf{k})/\Delta.$$

Φ' is a symmetric, respectively antisymmetric, quadratic differential on $\mathcal{O}_2$ iff $\Phi'/U \cup \mathbf{k}(U)$ a symmetric, respectively antisymmetric, quadratic differential on $U \cup \mathbf{k}(U)$, for every parametric disk U of $\mathcal{O}_2$.We denote with $Q^s(\mathcal{O}_2)$, respectively $Q^a(\mathcal{O}_2)$, the set of the symmetric quadratic differentials Φ_s, respectively antisymmetric Φ_a, on $\mathcal{O}_2$.

Let $\Phi' \in Q^2(\mathcal{O}_2)\oplus \overline{Q^2(\mathcal{O}_2)}$ with the following local representation:

$$(\Phi')^*/U \cup \mathbf{k}(U) = \begin{cases} \varphi_1(z)dz^2 + \varphi_2(z)d\overline{z}^2, & z \in V \\ \widehat{\varphi}_1(w)dw^2 + \widehat{\varphi}_2(w)d\overline{w}^2, & w \in \mathbf{f}(V) \end{cases}$$

where φ_1 and $\widehat{\varphi}_1$ are meromorphic functions and φ_2 , $\widehat{\varphi}_2$ are antimeromorphic functions.

Theorem 0.5. *a) Φ' is a symmetric quadratic differential on $\mathcal{O}_2$ iff $\varphi_1, \varphi_2, \widehat{\varphi}_1, \widehat{\varphi}_2$ satisfy the conditions:*

$$\begin{cases} \varphi_1(z) = \widehat{\varphi}_2(\mathbf{f}(z)) \left(\frac{\partial \overline{\mathbf{f}}}{\partial z}(z)\right)^2 \\ \varphi_2(z) = \widehat{\varphi}_1(\mathbf{f}(z)) \left(\frac{\partial \mathbf{f}}{\partial \overline{z}}(z)\right)^2 \end{cases} \text{, for every } z \in V$$

$$\begin{cases} \widehat{\varphi}_1(w) = \varphi_2(\mathbf{f}(w)) \left(\frac{\partial \overline{\mathbf{f}}}{\partial w}(w)\right)^2 \\ \widehat{\varphi}_2(w) = \varphi_1(\mathbf{f}(w)) \left(\frac{\partial \mathbf{f}}{\partial \overline{w}}(w)\right)^2 \end{cases} \text{, for every } w \in \mathbf{f}(V)$$

b) Φ' is an antisymmetric differential on $\mathcal{O}_2$ iff $\varphi_1, \varphi_2, \widehat{\varphi}_1, \widehat{\varphi}_2$ satisfy the conditions:

$$\begin{cases} \varphi_1(z) = -\widehat{\varphi}_2(\mathbf{f}(z)) \left(\frac{\partial \overline{\mathbf{f}}}{\partial z}(z)\right)^2 \\ \varphi_2(z) = -\widehat{\varphi}_1(\mathbf{f}(z)) \left(\frac{\partial \mathbf{f}}{\partial \overline{z}}(z)\right)^2 \end{cases} \text{, for every } z \in V$$

$$\begin{cases} \widehat{\varphi}_1(w) = -\varphi_2(\mathbf{f}(w)) \left(\frac{\partial \overline{\mathbf{f}}}{\partial w}(w)\right)^2 \\ \widehat{\varphi}_2(w) = -\varphi_1(\mathbf{f}(w)) \left(\frac{\partial \mathbf{f}}{\partial \overline{w}}(w)\right)^2 \end{cases} \text{, for every } w \in \mathbf{f}(V)$$

Proof. From definition of $\mathbf{K}$ it results:

$$(\Phi'\circ\mathbf{k})^*/U\cup\mathbf{k}(U) = \begin{cases} \widehat{\varphi}_2(\mathbf{f}(z))\left(\frac{\partial\overline{\mathbf{f}}}{\partial z}(z)\right)^2 dz^2 + \widehat{\varphi}_1(\mathbf{f}(z))\left(\frac{\partial\mathbf{f}}{\partial\overline{z}}(z)\right)^2 d\overline{z}^2, & z\in V \\ \varphi_2(\mathbf{f}(w))\left(\frac{\partial\overline{\mathbf{f}}}{\partial w}(w)\right)^2 dw^2 + \varphi_1(\mathbf{f}(w))\left(\frac{\partial\mathbf{f}}{\partial\overline{w}}(w)\right)^2 d\overline{w}^2, & w\in\mathbf{f}(V) \end{cases}.$$

The symmetry and antisymmetry conditions means :

$$(\Phi')^*/U\cup\mathbf{k}(U) = (\Phi'\circ\mathbf{k})^*/U\cup\mathbf{k}(U)$$

respectively,

$$(\Phi')^*/U\cup\mathbf{k}(U) = -(\Phi'\circ\mathbf{k})^*/U\cup\mathbf{k}(U)$$

and then the previous conditions. □

Remark 0.1. *Let D be an open subset of $\mathbf{C}$ and $f : D\to\mathbf{C}$ a C^2 - function. Then* $\overline{\frac{\partial f}{\partial z}} = \frac{\partial\overline{f}}{\partial\overline{z}}$.

Remark 0.2. *Let $\Phi_s\in Q^s(\mathcal{O}_2)$. Then, for every parametric disk U of $\mathcal{O}_2$, we have* $\Phi_s^*/U = \Phi_s^*/\mathbf{k}(U)$.

We will characterize the symmetric, respective antisymmetric, quadratic differentials on $\mathcal{O}_2$.

Theorem 0.6.

$$Q^s(\mathcal{O}_2) = \left\{\Phi+\Phi\circ\mathbf{k} \mid \Phi\in Q^2(\mathcal{O}_2)\right\}$$
$$Q^a(\mathcal{O}_2) = \left\{\Phi-\Phi\circ\mathbf{k} \mid \Phi\in Q^2(\mathcal{O}_2)\right\}.$$

Proof. It is enough to prove the first equality.

Let $\Phi'\in Q^s(\mathcal{O}_2)$. Then:

$$(\Phi')^*/U\cup\mathbf{k}(U) = \Phi^*/U\cup\mathbf{k}(U) + \mathbf{K}(\Phi)^*/U\cup\mathbf{k}(U), \text{ for every } U,$$

where $\Phi\in Q^2(\mathcal{O}_2)$ and U is a parametric disk of $\mathcal{O}_2$.

Conversely, if $\Phi' = \Phi+\Phi\circ\mathbf{k}$, with $\Phi\in Q^2(\mathcal{O}_2)$, because $\mathbf{k}$ is an involution on $\mathcal{O}_2$,

$$(\Phi'\circ\mathbf{k})^* = (\Phi\circ\mathbf{k})^* + [(\Phi\circ\mathbf{k})\circ\mathbf{k}]^* = (\Phi\circ\mathbf{k})^* + \Phi^* = (\Phi')^*$$

on $U\cup\mathbf{k}(U)$, for every parametric disk U of $\mathcal{O}_2$. □

Corollary 0.1.

$$Q^s(\mathcal{O}_2) = \left\{\Phi+\mathbf{K}(\Phi) \mid \Phi\in Q^2(\mathcal{O}_2)\right\}$$

and

$$Q^a(\mathcal{O}_2) = \left\{\Phi-\mathbf{K}(\Phi) \mid \Phi\in Q^2(\mathcal{O}_2)\right\}.$$

Let X be a Klein surface and $\mathcal{A} = \left\{(\widetilde{U}_i, h_i, V_i) \mid i\in I\right\}$ the corresponding atlas on X.

$\widetilde{\Phi}$ is a N- meromorphic quadratic differential, respectively holomorphic , on X iff:

a) For every $(\widetilde{U}, h, V)\in\mathcal{A}$, the local representation of $\widetilde{\Phi}$ on $\widetilde{U}$ is $\widetilde{\Phi}_1^*/\widetilde{U} + \widetilde{\Phi}_1^*/\widetilde{U}$, where $\widetilde{\Phi}_1$ is a meromorphic quadratic differential , respectively holomorphic on $\mathcal{O}_2$ and $\widetilde{\Phi}_2$ is an antimeromorphic, respectively antiholomorphic, quadratic differential on $\mathcal{O}_2$.

$\widetilde{\Phi}_1^*/\widetilde{U}$ and $\widetilde{\Phi}_2^*/\widetilde{U}$ are the local components of $\widetilde{\Phi}$ on chart $(\widetilde{U}, h, V)$.

b) If $(\widetilde{U}_1, h_1, V_1)$, $(\widetilde{U}_2, h_2, V_2) \in \mathcal{A}$ such that $\widetilde{U}_1 \cap \widetilde{U}_2 \neq \emptyset$ and $\widetilde{\Phi}^*/\widetilde{U}_1 = (\widetilde{\Phi}'_1)^*/\widetilde{U}_1 + (\widetilde{\Phi}'_2)^*/\widetilde{U}_1$, $\widetilde{\Phi}^*/\widetilde{U}_2 = (\widetilde{\Phi}''_1)^*/\widetilde{U}_2 + (\widetilde{\Phi}''_2)^*/\widetilde{U}_2$ are the corresponding local representations of $\widetilde{\Phi}$, then for every connected component $\widetilde{U}$ of $\widetilde{U}_1 \cap \widetilde{U}_2$ there are the following relations:

$$\text{b}_1)\ \begin{cases} (\widetilde{\Phi}'_1)^*/\widetilde{U} = (\widetilde{\Phi}''_1)^*/\widetilde{U} \\ (\widetilde{\Phi}'_2)^*/\widetilde{U} = (\widetilde{\Phi}''_2)^*/\widetilde{U} \end{cases} \text{, if } h_2 \circ h_1^{-1} \text{ is an analytic function on } h_1(\widetilde{U})$$

$$\text{b}_2)\ \begin{cases} (\widetilde{\Phi}'_1)^*/\widetilde{U} = (\widetilde{\Phi}''_2)^*/\widetilde{U} \\ (\widetilde{\Phi}'_2)^*/\widetilde{U} = (\widetilde{\Phi}''_1)^*/\widetilde{U} \end{cases} \text{, if } h_2 \circ h_1^{-1} \text{ is an antianalytic function on } h_1(\widetilde{U}).$$

Let $Q^2(X)$ be the vectorial space of N- meromorphic quadratic differentials on X, with respect to $\mathbf{C}$.

Theorem 0.7. *If $\mathcal{A} = \left\{(\widetilde{U}_i, h_i, V_i) \mid i \in I\right\}$ is a dianalytic atlas on X, the following sentences are equivalent :*

1. *Exists a $N-$ meromorphic quadratic differential $\widetilde{\Phi}$ on X.*
2. *Exists a family of mappings:*

$$\widetilde{\varphi}_A = \{\widetilde{\varphi}_i : \widetilde{U}_i \to \widehat{\mathbf{C}} \mid i \in I\} \textit{with :}$$

$(2')$ *For every $i \in I$, $\widetilde{\varphi}_i \circ h_i^{-1} : h_i(\widetilde{U}_i) \to \widehat{\mathbf{C}}$ is a meromorphic function.*

$(2'')$ *If $\widetilde{U}_{ij} \subseteq \widetilde{U}_i \cap \widetilde{U}_j$ is a connected component of $\widetilde{U}_i \cap \widetilde{U}_j$, then $\widetilde{\varphi}_i = \widetilde{\varphi}_j \left(\frac{\partial(h_j \circ h_i^{-1})}{\partial \widetilde{z}_i}\right)^2$, if $h_j \circ h_i^{-1}$ is a dianalytic function on $h_i(\widetilde{U}_{ij})$.*

Proof. Let $\widetilde{\Phi} \in Q^2(X)$ and Φ be the corresponding meromorphic quadratic differential on $\mathcal{O}_2$ associated with $\widetilde{\Phi}$ with the local representation $\Phi^*/U_i = \varphi_i(z_i) dz_i^2$ and $(\Phi \circ \mathbf{k})^*/U_i = \widehat{\varphi}_i(\mathbf{f}(z_i)) d\mathbf{f}(z_i)^2$, where $U_i = (q/U_i)^{-1}(\widetilde{U}_i)$ and $\mathbf{k}(U_i) = (q/\mathbf{k}(U_i))^{-1}(\widetilde{U}_i)$. If we consider a chart of X, we define $\widetilde{\varphi}_i/\widetilde{U}_i = \varphi_i/V_i \circ \mathbf{p}_{T,U_i}^{-1} \circ (q/U_i)^{-1}$ and then $\widetilde{\varphi}_i/\widetilde{U}_i \circ h_i^{-1} = \varphi_i/V_i$ is a meromorphic function, namely $\widetilde{\varphi}_i : \widetilde{U}_i \to \widehat{\mathbf{C}}$ is a meromorphic mapping on $\widetilde{U}_i$. For the second type of chart on X , we define $\widetilde{\varphi}_i/\widetilde{U}_i = \widehat{\varphi}_i/\mathbf{f}(V) \circ \mathbf{p}_{S,U_i}^{-1} \circ (q/\mathbf{k}(U_i))^{-1}$ and then $\widetilde{\varphi}_i/\widetilde{U}_i \circ g_i^{-1} = \widehat{\varphi}_i/\mathbf{f}(V_i)$ is a meromorphic function namely $\widetilde{\varphi}_i : \widetilde{U}_i \to \widehat{\mathbf{C}}$ is a meromorphic mapping on $\widetilde{U}_i$. The condition $(2'')$ is true.

Conversely, if $(\widetilde{\varphi}_i)_{i \in I}$, $\widetilde{\varphi}_i : \widetilde{U}_i \to \widehat{\mathbf{C}}$, for every $i \in I$, is a family of mappings with the properties $(2')$ and $(2'')$, then we define:

$$\widetilde{\Phi}^*/\widetilde{U}_i = \varphi_i(z_i) dz_i^2 + \widehat{\varphi}_i(\mathbf{f}(z_i)) d\mathbf{f}(z_i)^2,$$

where φ_i and $\widehat{\varphi}_i$ are the local representations of $\widetilde{\Phi}$ with respect to z_i, respectively $\mathbf{f}(z_i)$. □

Let $\widetilde{\Phi} \neq 0$ be a N- holomorphic quadratic differential on X . Then $\widetilde{\Phi}/\widetilde{U} = \Phi/U + (\Phi \circ \mathbf{k})/U$, where $q^{-1}(\widetilde{U}) = \{U, k(U)\}$, for every parametric disk $\widetilde{U}$ of X. The local representation for $\widetilde{\Phi}$ on $\widetilde{U}$ is

$$\widetilde{\Phi}/\widetilde{U} = \varphi(z) dz^2 + \widehat{\varphi}(\mathbf{f}(z)) d\mathbf{f}(z)^2,$$

where $\Phi/U = \varphi(z) dz^2$ is the local representation of Φ on U, $(\Phi \circ k)/U = \widehat{\varphi}(\mathbf{f}(z)) d\mathbf{f}(z)^2$ is the local representation of Φ on $k(U)$. Thus φ and $\widehat{\varphi}$ are holomorphic function in z, respective $\mathbf{f}(z)$.

Since there is a dianalytic isomorphism between $\mathcal{O}_2/\mathcal{H}$ and X and an arbitrary rectifiable curve on $\mathcal{O}_2$ which does not go through any critical point of Φ can be subdivided into intervals each one of which lies in a parametric disk, we identify a curve $\widetilde{\gamma}$ in a parametric disk $\widetilde{U}$ around a regular point $\widetilde{P}_0 \in X$ with the k - symmetric curve $q^{-1}(\widetilde{\gamma}) = \gamma \cup k(\gamma)$ in $U \cup k(U)$.

The N- holomorphic quadratic differential $\widetilde{\Phi}$ defines invariant N - length elements on X, $d\widetilde{S}(\widetilde{z}) = d\widetilde{s}(z) = d\widetilde{s}(\mathbf{f}(z))$, where

$$d\widetilde{s}(z) = \frac{1}{2}\left(\sqrt{|\varphi(z)|}\,|dz| \;\; + \sqrt{|\widehat{\varphi}(\mathbf{f}(z))|}\,|d\mathbf{f}(z)|\right)$$

is the k-symmetric element of Φ- length on $\mathcal{O}_2$ and area elements $d\widetilde{A}(\widetilde{z}) = d\widetilde{a}(z) = d\widetilde{a}(\mathbf{f}(z))$, where

$$d\widetilde{a}(z) = \frac{1}{2}\left(|\varphi(z)|\,dxdy + |\widehat{\varphi}(\mathbf{f}(z))|\,dudv\right)$$

is the k -symmetric element of Φ -area on $\mathcal{O}_2$, $z = x + iy$ and $\mathbf{f}(z) = u + iv$. This $\widetilde{\Phi}$-metric is euclidean except at the singularities and the trajectories are the geodesics of this metric.

Let $\widetilde{\gamma} : [0,1] \to X$ be a continuous differentiable arc on X, with $q^{-1}(\widetilde{\gamma}) = \{\gamma, k(\gamma)\}$.

The $\widetilde{\Phi}$ -length of $\widetilde{\gamma}$ is

$$l_{\widetilde{\Phi}}(\widetilde{\gamma}) = \int_{\widetilde{\gamma}} d\widetilde{S},$$

and the $\widetilde{\Phi}$ - distance of a pair of points $\widetilde{z}_1$, $\widetilde{z}_2$ is equal to

$$d_{\widetilde{\Phi}}\left[\widetilde{z}_1, \widetilde{z}_2\right] = \inf_{\{\widetilde{\gamma}\}} l_{\widetilde{\Phi}}(\widetilde{\gamma}),$$

where $\widetilde{\gamma}$ varies over all arcs connecting the two points. The $\widetilde{\Phi}$ -area of a Lebesque measurable set of X is the integral

$$A_{\widetilde{\Phi}}(\widetilde{E}) = \iint_{\widetilde{E}} d\widetilde{A},$$

and the total area of X in this metric is L^1- norm of $\widetilde{\Phi}$

$$A_{\widetilde{\Phi}}(X) = \left\|\widetilde{\Phi}\right\|_1 = \iint_X d\widetilde{A}.$$

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