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Asymptotically ρ-almost periodic type functions in general metric

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Abstract. In this paper, we analyze various classes of multi-dimensional asymptotically ρ-almost periodic type functions in general metric. We clarify the main structural properties for the introduced classes of asymptotically ρ-almost periodic type functions, and provide some applications of our results to the abstract Volterra integro-differential equations.

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1. Introduction and preliminaries

As is well known, the notion of almost periodicity was introduced by the Danish mathematician H. Bohr around 1924-1926 and later generalized by many other authors (see the research monographs [2], [4], [7], [8], [9], [10], [19], [20] and [22] for further information concerning almost periodic functions and their applications). Suppose that $(X,\|\cdot\|)$ is a complex Banach space and $F:\mathbb{R}^n\rightarrow X$ is a continuous function ($n\in\mathbb{N}$). Then it is said that the function $F(\cdot)$ is almost periodic if and only if for each $\epsilon>0$ there exists $l>0$ such that for each $\mathbf{t}_0\in\mathbb{R}^n$ there exists $\tau\in B(\mathbf{t}_0,l)\equiv\{\mathbf{t}\in\mathbb{R}^n : |\mathbf{t}-\mathbf{t}_0|\leq l\}$ such that

$$\big\|F(\mathbf{t}+\tau)-F(\mathbf{t})\big\|\leq\epsilon,\quad \mathbf{t}\in\mathbb{R}^n;$$

here, $|\cdot-\cdot|$ denotes the Euclidean distance in $\mathbb{R}^n$. Any trigonometric polynomial in $\mathbb{R}^n$ is almost periodic, and we know that a continuous function $F(\cdot)$ is almost periodic if and only if there exists a sequence of trigonometric polynomials in $\mathbb{R}^n$ which converges uniformly to $F(\cdot)$.

In M. Fečkan et al [6], we have initiated the study of multi-dimensional ρ-almost periodic type functions. In our previous research studies, we have also analyzed the Stepanov and Weyl classes of multi-dimensional ρ-almost periodic type functions ([16]). Further on, in our recent research study [13], we have initiated the study of multi-dimensional ρ-almost periodic functions in general metric. The Stepanov class of metrical multi-dimensional ρ-almost periodic functions and the Weyl class of

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metrical multi-dimensional ρ-almost periodic functions have recently been analyzed in [14] and [15], respectively.

The main aim of this research paper is to continue the research studies [13, 14, 15] by investigating various classes of multi-dimensional asymptotically ρ-almost periodic type functions in general metric. More precisely, we analyze here the following classes of multi-dimensional asymptotically ρ-almost periodic type functions:

(i) the class of metrically $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, ρ)-periodic functions;
(ii) the class of metrically $(S, \mathcal{B})$-asymptotically $(\omega_j, \rho_j, \mathbb{D}_j)_{j\in\mathbb{N}_n}$-periodic functions;
(iii) the class of metrically ρ-slowly oscillating type functions;
(iv) the class of $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic functions of type 1;
(v) the class of $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic functions;
(vi) the class of $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-uniformly recurrent functions.

The organization of paper can be briefly described as follows. Section 2 investigates the notion of metrical $(S, \mathbb{D}, \mathcal{B})$-asymptotical (ω, ρ)-periodicity and the notion of metrical $(S, \mathcal{B})$-asymptotical $(\omega_j, \rho_j, \mathbb{D}_j)_{j\in\mathbb{N}_n}$-periodicity. Section 3 is devoted to the study of metrically ρ-slowly oscillating type functions in $\mathbb{R}^n$, and Section 4 is devoted to the study of metrically quasi-asymptotically ρ-almost periodic type functions. We present some applications of our theoretical results to the abstract Volterra integro-differential equations in the final section of paper.

For more details about binary relations, we refer the reader to the classical university textbooks; see also [6]. We will always assume henceforth that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces, $n \in \mathbb{N}$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\mathcal{B}$ is a non-empty collection of non-empty subsets of X satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. By $L(X, Y)$ we denote the Banach space of all bounded linear operators from X into Y; $L(X, X) \equiv L(X)$ and I denotes the identity operator on Y. The Lebesgue measure in $\mathbb{R}^n$ is denoted by $m(\cdot)$; we define the function $\check{f}(\cdot) := f(-\cdot)$, whenever it make sense. If A and B are non-empty sets, then we define $\mathrm{B}^{\mathrm{A}} := \{f | f : \mathrm{A} \to \mathrm{B}\}$. Set $\mathbb{N}_n := \{1, 2, ..., n\}$. The basic facts about Lebesgue spaces with variable exponent $L^{p(x)}$ can be obtained by consulting the monograph [5] by L. Diening et al. Suppose now that the set I is Lebesgue measurable as well as that $\nu : I \to (0, \infty)$ is a Lebesgue measurable function. We will work with the following Banach space

$$L^{p(\mathbf{t})}_{\nu}(I : Y) := \{u : I \to Y \ ; \ u(\cdot) \text{ is measurable and } ||u||_{p(\mathbf{t})} < \infty\},$$

where $p \in \mathcal{P}(I)$, the space of of all Lebesgue measurable functions $p : I \to [1, \infty]$, and

$$\|u\|_{p(\mathbf{t})} := \|u(\mathbf{t})\nu(\mathbf{t})\|_{L^{p(\mathbf{t})}(I:Y)}.$$

If $\nu : I \to (0, \infty)$ is any function such that the function $1/\nu(\cdot)$ is locally bounded, then we will also work with the Banach space $C_{0,\nu}(I : Y)$ consisting of all continuous functions $u : I \to Y$ satisfying that $\lim_{|\mathbf{t}|\to\infty, \mathbf{t}\in I} \|u(\mathbf{t})\|_Y \nu(\mathbf{t}) = 0$. Equipped with the norm $\|\cdot\| := \sup_{\mathbf{t}\in I} \|\cdot(\mathbf{t})\nu(\mathbf{t})\|_Y$, $C_{0,\nu}(I : Y)$ is a Banach space.

2. Metrical $(S, \mathbb{D}, \mathcal{B})$-asymptotical (ω, ρ)-periodicity and metrical $(S, \mathcal{B})$-asymptotical $(\omega_j, \rho_j, \mathbb{D}_j)_{j\in\mathbb{N}_n}$-periodicity

This section investigates the notions of metrical $(S, \mathbb{D}, \mathcal{B})$-asymptotical (ω, ρ)-periodicity and metrical $(S, \mathcal{B})$-asymptotical $(\omega_j, \rho_j, \mathbb{D}_j)_{j\in\mathbb{N}_n}$-periodicity.

In the following two definitions, we extend the notion introduced in [10, Definition 7.3.1, Definition 7.3.2] and [16, Definition 2.1, Definition 2.2] (in case of consideration of the first-mentioned definition, set only, for every set $B \in \mathcal{B}$, $P_B := C_0(\mathbb{D} \times B : Y) := \{F : \mathbb{D} \times B \to Y \ ; \ \lim_{|\mathbf{t}| \to +\infty, \mathbf{t} \in \mathbb{D}} \sup_{x \in B} \|F(\mathbf{t}; x)\|_Y = 0\}$ and $d_B(F, G) := \sup_{x \in B} \sup_{\mathbf{t} \in \mathbb{D}} \|F(\mathbf{t}; x)\|_Y$ for all $F,\ G \in P_B$; similarly we can analyze the notion from the second-mentioned definition):

Definition 2.1. Let ρ be a binary relation on Y, $\omega \in \mathbb{R}^n \setminus \{0\}$, $\omega + I \subseteq I$, $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. Suppose, further, that for each set $B \in \mathcal{B}$, $\mathcal{P}_B = (P_B, d_B)$ is a metric space of functions from $[0, \infty)^{\mathbb{D} \times B}$ containing the zero function. A function $F : I \times X \to Y$ is said to be $(S, \mathbb{D}, \mathcal{B}, \mathcal{P}_{\mathcal{B}})$-asymptotically (ω, ρ)-periodic if and only if for each $B \in \mathcal{B}$ there exists a function $G : \mathbb{D} \times B \to Y$ such that $G(\cdot; \cdot) \in \rho(F(\cdot; \cdot))$ and

$$\big\|F(\cdot + \omega; \cdot) - G(\cdot; \cdot)\big\|_Y \in P_B.$$

Definition 2.2. Let ρ_j be a binary relation on Y, $\omega_j \in \mathbb{R} \setminus \{0\}$, $\omega_j e_j + I \subseteq I$, $\mathbb{D}_j \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}_j$ be unbounded ($1 \leq j \leq n$). Suppose, further, that for each $j \in \mathbb{N}_n$ and for each set $B \in \mathcal{B}$, $\mathcal{P}^j_B = (P^j_B, d^j_B)$ is a metric space of functions from $[0, \infty)^{\mathbb{D}_j \times B}$ containing the zero function. A function $F : I \times X \to Y$ is said to be $(S, \mathcal{B})$-asymptotically $(\omega_j, \rho_j, \mathbb{D}_j, \mathcal{P}^j_{\mathcal{B}})_{j \in \mathbb{N}_n}$-periodic if and only if for each $j \in \mathbb{N}_n$ and for each $B \in \mathcal{B}$ there exists a function $G_j : \mathbb{D}_j \times B \to Y$ such that $G_j(\cdot; \cdot) \in \rho_j(F(\cdot; \cdot))$ and

$$\big\|F(\cdot + \omega; \cdot) - G_j(\cdot; \cdot)\big\|_Y \in P^j_B.$$

Before proceeding further, we would like to note that, in the case of consideration of functions of the form $F : I \to Y$, we have $X = \{0\}$ and $\mathcal{B} = \{X\}$, so that we actually work with the metric space $\mathcal{P}_B = \mathcal{P} = (P_B, d_B) = (P, d)$ consisting of certain functions from $[0, \infty)^{\mathbb{D}}$ and functions $G : \mathbb{D} \to Y$; see e.g., Definition 2.1.

The following result extends the statement of [16, Proposition 2.5] and particularly shows that the notion introduced in Definition 2.1 is more general, in a certain sense, than the notion introduced in Definition 2.2:

Proposition 2.1. *Let $\omega_j \in \mathbb{R} \setminus \{0\}$, $T_j \in L(X)$, $\omega_j e_j + I \subseteq I$, $\mathbb{D}_j \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}_j$ be unbounded ($1 \leq j \leq n$). If $F : I \times X \to X$ is $(S, \mathcal{B})$-asymptotically $(\omega_j, T_j, \mathbb{D}_j, \mathcal{P}^j_{\mathcal{B}})_{j \in \mathbb{N}_n}$-periodic and the set $\mathbb{D}$, consisting of all tuples $\mathbf{t} \in \mathbb{D}_n$ such that $\mathbf{t} + \sum_{i=j+1}^n \omega_i e_i \in \mathbb{D}_j$ for all $j \in \mathbb{N}_{n-1}$, is unbounded in $\mathbb{R}^n$, then the function $F(\cdot; \cdot)$ is $(S, \mathbb{D}, \mathcal{B})$-asymptotically $(\omega, T, \mathcal{P})$-periodic, where $\omega := \sum_{j=1}^n \omega_j e_j$ and $T := \prod_{j=1}^n T_j$, provided that for each set $B \in \mathcal{B}$ there exists a finite real constant $c_B > 0$ such that:*

$$\begin{aligned} \big\|F(\cdot; \cdot)\big\|_P \leq c_B \Bigg[&\bigg\| F\bigg(\cdot + \sum_{i=1}^n \omega_i e_i; \cdot\bigg) - T_1 F\bigg(\cdot + \sum_{i=2}^n \omega_i e_i; \cdot\bigg) \bigg\|_{P^1_B} \\ &+ \bigg\| F\bigg(\cdot + \sum_{i=2}^n \omega_i e_i; \cdot\bigg) - T_2 F\bigg(\cdot + \sum_{i=3}^n \omega_i e_i; \cdot\bigg) \bigg\|_{P^2_B} + \dots + \bigg\| F\big(\cdot + \omega_n e_n; \cdot\big) - T_n F(\cdot; \cdot) \bigg\|_{P^n_B} \Bigg]. \end{aligned}$$

Proof. The proof is an almost direct consequence of the corresponding definitions, the prescribed assumption and the following calculation ($\mathbf{t} \in \mathbb{D}$):

$$\begin{aligned}
&\big\|F(\mathbf{t}+\omega;x)-TF(\mathbf{t};x)\big\| = \Big\|F\big(t_1+\omega_1,...,t_n+\omega_n;x\big)-T_1...T_nF\big(t_1,...,t_n;x\big)\Big\| \\
&\leq \Big\|F\big(t_1+\omega_1,t_2+\omega_2,...,t_n+\omega_n;x\big)-T_1F\big(t_1,t_2+\omega_2,...,t_n+\omega_n;x\big)\Big\| \\
&+\|T_1\|\cdot\Big\|F\big(t_1,t_2+\omega_2,...,t_n+\omega_n;x\big)-T_2...T_nF\big(t_1,...,t_n;x\big)\Big\| \\
&\leq \Big\|F\big(t_1+\omega_1,t_2+\omega_2,...,t_n+\omega_n;x\big)-T_1F\big(t_1,t_2+\omega_2,...,t_n+\omega_n;x\big)\Big\| \\
&+\|T_1\|\cdot\Bigg[\Big\|F\big(t_1,t_2+\omega_2,...,t_n+\omega_n;x\big)-T_2F\big(t_1,t_2,...,t_n+\omega_n;x\big)\Big\| \\
&+\|T_2\|\cdot\Big\|F\big(t_1,t_2,...,t_n+\omega_n;x\big)-T_3...T_nF\big(t_1,t_2,...,t_n;x\big)\Big\|\Bigg] \\
&\leq ... .
\end{aligned}$$

□

In the following proposition, we examine the convolution invariance of function spaces introduced in this section:

Proposition 2.2. *Suppose that $h \in L^1(\mathbb{R}^n)$ and $F : \mathbb{R}^n \times X \to Y$ is a function which satisfies that for each set $B \in \mathcal{B}$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B}\|F(\mathbf{t},x)\|_Y < +\infty$. Suppose, further, that $\rho = A$ is a closed linear operator on Y satisfying that:*

(D) *For each $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$, the function $\mathbf{s} \mapsto AF(\mathbf{t}-\mathbf{s};x)$, $\mathbf{s} \in \mathbb{R}^n$ is essentially bounded; for each $B \in \mathcal{B}$, the function $\mathbf{s} \mapsto \sup_{x\in B}\|AF(\mathbf{s};x)\|_Y$, $\mathbf{s} \in \mathbb{R}^n$ is bounded.*

Then the function

$$(h * F)(\mathbf{t};x) := \int_{\mathbb{R}^n} h(\sigma)F(\mathbf{t}-\sigma;x)\,d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X$$

is well defined and for each set $B \in \mathcal{B}$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B}\|(h*F)(\mathbf{t};x)\|_Y < +\infty$. Furthermore, the following holds:

(i) *Suppose that $\mathbb{D} = \mathbb{R}^n$, $\nu \in L^\infty(\mathbb{R}^n : (0,\infty))$ and there exists a function $\varphi : \mathbb{R}^n \to (0,\infty)$ such that $\nu(x) \leq \nu(y)\varphi(x-y)$ for all $x,\ y \in \mathbb{R}^n$ and $h\varphi \in L^1(\mathbb{R}^n)$. Suppose also that, for every set $B \in \mathcal{B}$, we have $P_B = C_{0,\nu}(\mathbb{R}^n \times B : (0,\infty))$ or $P_B = L^\infty_\nu(\mathbb{R}^n \times B : (0,\infty))$. If the function $F(\cdot;\cdot)$ is $(S,\mathbb{R}^n,\mathcal{B})$-asymptotically $(\omega, A, \mathcal{P}_{\mathcal{B}})$-periodic, then the function $(h*F)(\cdot;\cdot)$ is likewise $(S,\mathbb{R}^n,\mathcal{B})$-asymptotically $(\omega, A, \mathcal{P}_{\mathcal{B}})$-periodic.*

(ii) *Suppose that $\mathbb{D}_j = \mathbb{R}^n$ and the function $\nu_j \in L^\infty(\mathbb{R}^n : (0,\infty))$ satisfies that there exists a function $\varphi_j : \mathbb{R}^n \to (0,\infty)$ such that $\nu_j(x) \leq \nu_j(y)\varphi_j(x-y)$ for all $x,\ y \in \mathbb{R}^n$ and $h\varphi_j \in L^1(\mathbb{R}^n)$, for all $j \in \mathbb{N}_n$. Suppose also that, for every set $B \in \mathcal{B}$ and $j \in \mathbb{N}_n$, we have $P^j_B = C_{0,\nu_j}(\mathbb{R}^n \times B : (0,\infty))$ or $P^j_B = L^\infty_{\nu_j}(\mathbb{R}^n \times B : (0,\infty))$. If for each $j \in \mathbb{N}_n$ condition* (D) *holds with the closed linear operator A replaced therein with the closed linear operator A_j, and the function $F(\cdot;\cdot)$ is $(S,\mathcal{B})$-asymptotically $(\omega_j, A_j, \mathbb{R}^n, \mathcal{P}^j_{\mathcal{B}})_{j\in\mathbb{N}_n}$-periodic, then the function $(h*F)(\cdot;\cdot)$ is likewise $(S,\mathcal{B})$-asymptotically $(\omega_j, A_j, \mathbb{R}^n, \mathcal{P}^j_{\mathcal{B}})_{j\in\mathbb{N}_n}$-periodic.*

Proof. The proof is very similar to the proof of [12, Theorem 2.6], and we will only provide the main details of proof for the issue (i). It can be easily proved that the function $(h*F)(\cdot;\cdot)$ is well defined as well as that $\sup_{\mathbf{t}\in\mathbb{R}^n,x\in B}\|(h*F)(\mathbf{t};x)\|_Y<+\infty$ for all $B\in\mathcal{B}$. Let a real number $\epsilon>0$ and a set $B\in\mathcal{B}$ be fixed. Then there exists a sufficiently large real number $M_1>0$ such that $\|F(\mathbf{t}+\omega;x)-AF(\mathbf{t};x)\|_Y\nu(\mathbf{t})<\epsilon$, provided $|\mathbf{t}|>M_1$ and $x\in B$. Since A is closed and condition (D) holds, for every $\mathbf{t}\in\mathbb{R}^n$ and $x\in B$, the value of $G_x(\mathbf{t}):=A((h*F)(\mathbf{t};x))=\int_{\mathbb{R}^n}h(\mathbf{s})A(F(\mathbf{t}-\mathbf{s};x))\,d\mathbf{s}$ is well defined. Due to the second part of condition (D), the essential boundedness of function $\nu(\cdot)$ and the fact that $h\varphi\in L^1(\mathbb{R}^n)$, we know that there exists a finite constant $c_B\geq 1$ such that

$$\begin{aligned}
&\Big\|(h*F)(\mathbf{t}+\omega;x)-G_x(\mathbf{t})\Big\|_Y\nu(\mathbf{t})\\
&\leq\int_{\mathbb{R}^n}|h(\sigma)|\cdot\big\|F(\mathbf{t}+\omega-\sigma;x)-AF(\mathbf{t}-\sigma;x)\big\|_Y\nu(\mathbf{t})\,d\sigma\\
&=\int_{|\sigma|\leq M_1}|h(\mathbf{t}-\sigma)|\cdot\big\|F(\sigma+\omega;x)-AF(\sigma;x)\big\|_Y\nu(\mathbf{t})\,d\sigma\\
&+\int_{|\sigma|\geq M_1}|h(\mathbf{t}-\sigma)|\cdot\big\|F(\sigma+\omega;x)-AF(\sigma;x)\big\|_Y\nu(\mathbf{t})\,d\sigma\\
&\leq\int_{|\sigma|\leq M_1}|h(\mathbf{t}-\sigma)|\cdot\big\|F(\sigma+\omega;x)-AF(\sigma;x)\big\|_Y\nu(\mathbf{t})\,d\sigma\\
&+\int_{|\sigma|\geq M_1}|h(\mathbf{t}-\sigma)|\varphi(\mathbf{t}-\sigma)\cdot\big\|F(\sigma+\omega;x)-AF(\sigma;x)\big\|_Y\nu(\sigma)\,d\sigma\\
&\leq c_B\|\nu\|_\infty\int_{|\sigma|\leq M_1}|h(\mathbf{t}-\sigma)|\,d\sigma+\epsilon\|\varphi h\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

To complete the proof, it suffices to observe that there exists a finite real number $M_2>0$ such that $\int_{|\sigma|\geq M_2}|h(\sigma)|\,d\sigma<\epsilon$. □

We would like to note that an analogue of Proposition 2.2, with the same choice of metric spaces, can be formulated for metrically quasi-asymptotically ρ-almost periodic type functions in $\mathbb{R}^n$ and metrically ρ-slowly oscillating type functions in $\mathbb{R}^n$. Furthermore, the composition principle clarified in [16, Theorem 4.5] can be straightforwardly reformulated for metrically ρ-slowly oscillating type functions in $\mathbb{R}^n$, providing the same choice of metric spaces as in Proposition 2.2.

3. Metrically ρ-slowly oscillating type functions in $\mathbb{R}^n$

In the following two definitions, we extend the notion recently introduced in [16, Definition 4.1, Definition 4.2]:

Definition 3.1. Let ρ be a binary relation on Y, $\mathbb{D}\subseteq I\subseteq\mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. Suppose, further, that for each set $B\in\mathcal{B}$, $\mathcal{P}_B=(P_B,d_B)$ is a metric space of functions from $[0,\infty)^{\mathbb{D}\times B}$ containing the zero function. Set

$$A_I:=\big\{\omega\in\mathbb{R}^n\setminus\{0\}:\omega+I\subseteq I\big\}.$$

A function $F:I\times X\to Y$ is said to be $(\mathbb{D},\mathcal{B},\rho,\mathcal{P}_{\mathcal{B}})$-slowly oscillating if and only if for each $\omega\in A_I$ the function $F(\cdot;\cdot)$ is $(S,\mathbb{D},\mathcal{B},\mathcal{P}_{\mathcal{B}})$-asymptotically (ω,ρ)-periodic.

Definition 3.2. Let ρ_j be a binary relation on Y, $\mathbb{D}_j \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}_j$ be unbounded ($1 \leq j \leq n$). Suppose, further, that for each $j \in \mathbb{N}_n$ and for each set $B \in \mathcal{B}$, $\mathcal{P}_B^j = (P_B^j, d_B^j)$ is a metric space of functions from $[0,\infty)^{\mathbb{D}_j \times B}$ containing the zero function. Set

$$B_I := \big\{(\omega_1, ..., \omega_n) \in (\mathbb{R} \setminus \{0\})^n : \omega_j e_j + I \subseteq I \text{ for all } j \in \mathbb{N}_n\big\}.$$

A function $F : I \times X \to Y$ is said to be $(\mathcal{B}, (\omega_j, \rho_j, \mathbb{D}_j, \mathcal{P}_{\mathcal{B}}^j)_{j\in\mathbb{N}_n})$-slowly oscillating if and only if for each tuple $(\omega_1, ..., \omega_n) \in B_I$ the function $F(\cdot;\cdot)$ is $(S, \mathcal{B})$-asymptotically $(\omega_j, \rho_j, \mathbb{D}_j, \mathcal{P}_{\mathcal{B}}^j)_{j\in\mathbb{N}_n}$-periodic.

There is no need to say that the terms "$(S, \mathbb{D}, \mathcal{B}, \mathcal{P}_{\mathcal{B}})$-asymptotically (ω, ρ)-periodic" and "$(\mathbb{D}, \mathcal{B}, \rho, \mathcal{P}_{\mathcal{B}})$-slowly oscillating", as well as their relatives from Definition 2.2 and Definition 3.2, are not ideal but suitable in the situation in which we use the weighted C_0-spaces of functions.

It is also worth noting that, in the usual approach developed by D. Sarason [21] for the functions of form $F : [0,\infty) \to \mathbb{C}$, the boundedness and continuity of function $F(\cdot)$ are assumed a priori; we do not use these assumptions here. Further on, in the usual approach, any slowly oscillating function $F : [0,\infty) \to \mathbb{C}$ is uniformly continuous. We have expanded this result in [17, Proposition 2.3]; it is clear that we cannot expect the uniform continuity of functions introduced in Definition 3.1 and Definition 3.2.

We continue by providing two illustrative examples:

Example 3.1. Let $X := c_0(\mathbb{C})$ be the Banach space of all numerical sequences tending to zero, equipped with the sup-norm. Consider the function

$$f(t) := \left(\frac{4k^2t^2}{(t^2+k^2)^2}\right)_{k\in\mathbb{N}}, \ t \geq 0.$$

Then we know (see e.g., [11, Example 2.6]) that the function $f(\cdot)$ is uniformly continuous, the range of $f(\cdot)$ is not relatively compact in X and, for every positive real number $\tau > 0$, we have

$$\|f(t+\tau) - f(t)\| \leq \frac{1}{t^4} + 4\frac{\tau^2}{t^2}, \quad t > 0.$$

This implies that the function $f(\cdot)$ is $\mathcal{P}$-slowly oscillating with $P = C_{0,\nu}([0,\infty) : X)$ and $\nu(t) = (1+t)^\zeta$, where $\zeta \in (0,2)$.

Example 3.2. Suppose that $F : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and there exist finite real numbers $\sigma \in \mathbb{R}$ and $M \geq 1$ such that $|\nabla F(x)| \leq M|x|^\sigma$, $|x| \geq 1$. Suppose, further, that $\nu : \mathbb{R}^n \to (0,\infty)$ is any continuous function such that $\lim_{|x|\to+\infty} |x|^\sigma \nu(x) = 0$. Applying the Lagrange mean value theorem, we get that, for every $x,\ \tau \in \mathbb{R}^n \setminus \{0\}$, there exists a number $c \in (0,1)$ such that:

$$\begin{aligned}|f(x+\tau) - f(x)|\nu(x) &\leq |\nabla f(x+(1-c)\tau)| \cdot |\tau| \cdot \nu(x) \\ &\leq M|\tau| \cdot |x+(1-c)\tau|^\sigma \cdot \nu(x) = M|\tau| \cdot \frac{|x+(1-c)\tau|^\sigma}{|x|^\sigma}\Big[|x|^\sigma \nu(x)\Big].\end{aligned}$$

Along with an elementary argumentation, this computation shows that, for every $\tau \in \mathbb{R}^n$, we have: $\lim_{|x|\to+\infty} |f(x+\tau) - f(x)|\nu(x) = 0$. Hence, the function $F(\cdot)$ is $\mathcal{P}$-slowly oscillating with $P = C_{0,\nu}(\mathbb{R}^n : \mathbb{R})$ and the metric d induced by the norm of this Banach space.

In [17, Proposition 2.2], we have recently observed that it is not so logical to study the class of $(\mathbb{D},\mathcal{B},c)$-slowly oscillating functions by replacing the term $\|F(\mathbf{t}+\omega;x)-F(\mathbf{t};x)\|_Y$ in the usual definition by the term $\|F(\mathbf{t}+\omega;x)-cF(\mathbf{t};x)\|_Y$, where $c\in\mathbb{C}\setminus\{0\}$. This result can be further generalized as follows:

Proposition 3.1. *Let $c\in\mathbb{C}\setminus\{0\}$, $\emptyset\neq I\subseteq\mathbb{R}^n$, $\mathbb{D}\subseteq I\subseteq\mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. Suppose that $\nu: I\to(0,\infty)$ is a function which satisfies that there exists a function $\varphi: A_I\to(0,\infty)$ such that $\nu(\mathbf{t})\leq\nu(\mathbf{t}+\omega)\varphi(\omega)$ for all $\mathbf{t}\in\mathbb{D}$ and $\omega\in A_I$, as well as that $A_I\subseteq 2A_I$ and $\omega'+\mathbb{D}\subseteq\mathbb{D}$ for all $\omega'\in A_I/2$. Let $P_B:=C_{0,\mathbb{D},\nu}(I\times B:(0,\infty))$ for all $B\in\mathcal{B}$. Then the following holds:*

(i) *If a function $F: I\times X\to Y$ is $(\mathbb{D},\mathcal{B},\mathrm{cI},\mathcal{P}_{\mathcal{B}})$-slowly oscillating, then for each set $B\in\mathcal{B}$ we have*

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}}\sup_{x\in B}\big\|F(\mathbf{t};x)\big\|_Y\nu(\mathbf{t})=0. \tag{1}$$

(ii) *If, in addition to the above, we have $\omega+\mathbb{D}\subseteq\mathbb{D}$ for all $\omega\in A_I$, then a function $F: I\times X\to Y$ is $(\mathbb{D},\mathcal{B},\mathrm{cI},\mathcal{P}_{\mathcal{B}})$-slowly oscillating if and only if for each set $B\in\mathcal{B}$ we have (1).*

Proof. In order to prove (i), suppose that $\omega'\in A_I$ and $B\in\mathcal{B}$; then there exists $\omega\in A_I$ such that $\omega'=2\omega$. We have ($\mathbf{t}\in I$; $x\in B$):

$$\begin{aligned}\Big[F\big(\mathbf{t}+\omega';x\big)-c^2F(\mathbf{t};x)\Big]\nu(\mathbf{t})&=\Big[F\big(\mathbf{t}+2\omega;x\big)-c^2F(\mathbf{t};x)\Big]\nu(\mathbf{t})\\&=\big[F\big(\mathbf{t}+2\omega;x\big)-cF\big(\mathbf{t}+\omega;x\big)\big]\nu(\mathbf{t})+c\big[F\big(\mathbf{t}+\omega;x\big)-cF\big(\mathbf{t};x\big)\big]\nu(\mathbf{t}),\end{aligned}$$

which implies

$$\begin{aligned}\Big\|F\big(\mathbf{t}+\omega';x\big)-c^2F(\mathbf{t};x)\Big\|_Y\nu(\mathbf{t})&\leq\big\|F\big(\mathbf{t}+2\omega;x\big)-cF\big(\mathbf{t}+\omega;x\big)\big\|_Y\nu(\mathbf{t}+\omega)\varphi(\omega)\\&\quad+|c|\cdot\big\|F\big(\mathbf{t}+\omega;x\big)\cdot F\big(\mathbf{t};x\big)\big\|_Y\nu(\mathbf{t}).\end{aligned}$$

Our assumption $(A_I/2)+\mathbb{D}\subseteq\mathbb{D}$ implies $\mathbf{t}+\omega\in\mathbb{D}$, $\mathbf{t}\in\mathbb{D}$ and

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}}\big\|F\big(\mathbf{t}+\omega';x\big)-c^2F(\mathbf{t};x)\big\|_Y\nu(\mathbf{t})=0,\quad\text{uniformly in } x\in B.$$

Subtracting the term in the above equality and the corresponding term from the definition of $(\mathbb{D},\mathcal{B},\mathrm{cI},\mathcal{P}_{\mathcal{B}})$-slowly oscillating property, with the number ω replaced therein with the number ω', we obtain

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}}\big\|\big(c^2-c\big)\cdot F(\mathbf{t};x)\big\|_Y\nu(\mathbf{t})=0,\quad\text{uniformly in } x\in B.$$

This immediately implies (1) since $c\neq 1$. In order to prove (ii), it suffices to apply (i) and observe that the assumption $\omega+\mathbb{D}\subseteq\mathbb{D}$ for all $\omega\in A_I$ implies

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}}\big\|F(\mathbf{t}+\omega;x)\big\|_Y\nu(\mathbf{t})=0,\quad\text{uniformly in } x\in B,$$

since $\nu(\mathbf{t})\leq\nu(\mathbf{t}+\omega)\varphi(\omega)$ for all $\mathbf{t}\in\mathbb{D}$ and $\omega\in A_I$. □

4. Metrically quasi-asymptotically ρ-almost periodic type functions

In this section, we investigate various classes of metrical quasi-asymptotically ρ-almost periodic type functions. We will always assume the validity of the following condition:

(QAAP-1) Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$, $\emptyset \neq I' \subseteq \mathbb{R}^n$, $\emptyset \neq I \subseteq \mathbb{R}^n$ and $I + I' \subseteq I$. If $\tau \in I'$ and $M > 0$, then we define

$$\mathbb{D}_M := \{\mathbf{t} \in \mathbb{D} : |\mathbf{t}| \geq M\}$$

and

$$I_{\tau,M} := \{\mathbf{t} \in I : \mathbf{t},\ \mathbf{t} + \tau \in \mathbb{D}_M\};$$

further on, we assume that $\mathcal{P}_{\tau,M} = (P_{\tau,M}, d_{\tau,M})$ is a metric space, where $P_{\tau,M} \subseteq Y^{I_{\tau,M}}$ and $P_{\tau,M}$ contains the zero function. We set $\|f\|_{P_{\tau,M}} := d_{\tau,M}(0, f)$ for all $f \in P_{\tau,M}$.

Now we are able to introduce the following notion:

Definition 4.1. Suppose that (QAAP-1) holds as well as that $F : I \times X \to Y$ is a given function. Then we say that:

(i) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic of type 1 if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exist finite real numbers $l > 0$ and $M > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that, for every $x \in B$, there exists a function $G_x \in Y^{I_{\tau,M}}$ such that $G_x(\mathbf{t}) \in \rho(F(\mathbf{t}; x))$ for all $\mathbf{t} \in I_{\tau,M}$, $x \in B$ and

$$\sup_{x \in B} \big\| F(\cdot + \tau; x) - G_x(\cdot) \big\|_{P_{\tau,M}} \leq \epsilon.$$

(ii) $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that there exists a finite real number $M \equiv M(\epsilon, \tau) > 0$ such that, for every $x \in B$, there exists a function $G_x \in Y^{I_{\tau,M}}$ such that $G_x(\mathbf{t}) \in \rho(F(\mathbf{t}; x))$ for all $\mathbf{t} \in I_{\tau,M}$, $x \in B$ and

$$\sup_{x \in B} \big\| F(\cdot + \tau; x) - G_x(\cdot) \big\|_{P_{\tau,M}} \leq \epsilon.$$

(iii) $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-uniformly recurrent (or, equivalently, $\mathbb{D}$-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-uniformly recurrent of type 1) if and only if for every $B \in \mathcal{B}$ there exist a sequence (τ_k) in I' and a sequence (M_k) in $(0, \infty)$ such that $\lim_{k\to+\infty} |\tau_k| = \lim_{k\to+\infty} M_k = +\infty$ and, for every $x \in B$, there exists a function $G_x \in Y^{I_{\tau_k, M_k}}$ such that $G_x(\mathbf{t}) \in \rho(F(\mathbf{t}; x))$ for all $\mathbf{t} \in I_{\tau,M}$, $x \in B$ and

$$\lim_{k\to+\infty} \sup_{x \in B} \big\| F(\cdot + \tau_k; x) - G_x(\cdot) \big\|_{P_{\tau_k, M_k}} = 0.$$

If $I' = I$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, \rho, \mathcal{P})$-almost periodic of type 1 ($\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, \rho, \mathcal{P})$-almost periodic, $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, \rho, \mathcal{P})$-uniformly recurrent); furthermore, if $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically $(I', \rho, \mathcal{P})$-almost periodic of type 1 ($\mathbb{D}$-quasi-asymptotically $(I', \rho, \mathcal{P})$-almost periodic, $\mathbb{D}$-quasi-asymptotically $(I', \rho, \mathcal{P})$-uniformly recurrent). If $I' = I$ and $X \in \mathcal{B}$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically $(\rho, \mathcal{P})$-almost periodic of type 1 ($\mathbb{D}$-quasi-asymptotically $(\rho, \mathcal{P})$-almost periodic, $\mathbb{D}$-quasi-asymptotically $(\rho, \mathcal{P})$-uniformly recurrent). We remove the prefix "$\mathbb{D}$-" in the case that $\mathbb{D} = I$, remove the prefix "$(\mathcal{B},)$" in the case that $X \in \mathcal{B}$ and remove the prefix "ρ-" if $\rho = \mathrm{I}$, the identity operator on Y.

It is worth noting that we do not assume the continuity of a function $F(\cdot;\cdot)$ here. The notion of $\mathbb{D}$-quasi-asymptotical Bohr $(\mathcal{B}, I', \rho)$-almost periodicity and the notion of $\mathbb{D}$-quasi-asymptotical $(\mathcal{B}, I', \rho)$-uniform recurrence, introduced and analyzed in [16], are obtained by plugging that $P_{\tau,M} = L^{\infty}(I_{\tau,M} : Y)$ for all $\tau \in I'$ and $M > 0$; similarly, the notion of $\mathbb{D}$-asymptotical Bohr $(\mathcal{B}, I', \rho)$-almost periodicity of type 1 and the notion of $\mathbb{D}$-asymptotical $(\mathcal{B}, I', \rho)$-uniform recurrence of type 1, introduced and analyzed in [6], are obtained in the same way; see also [10, Definition 6.1.33, Definition 7.1.23, Definition 7.3.14].

Remark 4.1. The notion introduced in the former part of this paper can be understood in a more general setting. By that, we primarily mean that all considered metric spaces can be pseudometric spaces as well as that X and Y can be general non-empty sets, only. Let us consider in more detail part (i) of Definition 4.1; then it suffices to assume that I, I', X and Y are non-empty sets, (I', d') is a pseudometric space [then for each $\mathbf{t}_0 \in I'$ the inclusion $\tau \in B(\mathbf{t}_0, l) \cap I'$ means $\tau \in I'$ and $d'(\mathbf{t}_0, \tau) \leq l$], the operation $\oplus : I \times I' \to I$ is defined [then $\cdot + \tau$ means $\cdot \oplus \tau$], $\mathcal{P}_{\tau,M} = (P_{\tau,M}, d_{\tau,M})$ is a pseudometric space, where $P \subseteq Y^{I_{\tau,M}}$ contains a zero function, and $(Y, \ominus)$ is a grupoid [then $F(\cdot + \tau; x) - G_x(\cdot) \in P_{\tau,M}$ means $F(\cdot \oplus \tau; x) \ominus G_x(\cdot) \in P_{\tau,M}$], for any $\tau \in I'$ and $M > 0$.

Suppose that (QAAP-1) holds and $F : I \times X \to Y$ is a given function. Then it is clear that the following holds:

(i) If $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic of type 1, then $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic.

(ii) If $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic, then $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-uniformly recurrent.

For simplicity, we will not consider here the Stepanov generalizations of the notion introduced in Definition 4.1; see [10, Subsection 6.2.3, Subsection 6.2.5, Subsection 7.3.4] for some results obtained in the case that $\rho = \mathrm{I}$. Recall also that the notion of Stepanov quasi-asymptotical almost periodicity intermediates the concepts Stepanov asymptotical almost periodicity and Weyl almost periodicity, considered in the general approach of A. S. Kovanko ([18]). Concerning this issue, we would like to note that an analogue of [11, Proposition 2.12] and its multi-dimensional extensions can be proved in metrical framework, provided that the metric spaces under our consideration are weighted L^p-spaces. Details can be left to the interested readers. Furthermore, in [16, Subsection 3.1], we have recently analyzed various classes of remotely ρ-almost periodic type functions. The notion of metrical remote ρ-almost periodicity can be introduced and analyzed, as well; for simplicity, we will skip all related details concerning this topic here.

We continue by providing an illustrative example:

Example 4.1. Suppose that $I = \mathbb{R}$, $\nu : \mathbb{R} \to (0, \infty)$ is any function satisfying that the function $1/\nu(\cdot)$ is locally bounded, and $f(\cdot)$ is any scalar-valued continuous function such that $f(t) = 1$ for all $t \geq 0$ and $f(t) = 0$ for all $t \leq -1$. Then $f(\cdot)$ is not equi-Weyl-p-almost periodic for any finite exponent $p \geq 1$ (cf. [9] for the notion) but $f(\cdot)$ is quasi-asymptotically $\mathcal{P}$-almost periodic, where $P_{\tau,M} = C_{0,\nu}(\mathbb{R} : \mathbb{R})$ for every $\tau \in \mathbb{R}$ and $M > 0$.

Denote by $A_{X,Y}$ any of the function spaces introduced in the former part of this paper. If $F(\cdot;\cdot)$ belongs to $A_{X,Y}$, $c_1 \in \mathbb{R} \setminus \{0\}$, $\tau \in \mathbb{R}^n$, c, $c_2 \in \mathbb{C} \setminus \{0\}$ and $x_0 \in X$, then it is not difficult to clarify certain sufficient conditions ensuring that the function $cF(\cdot;\cdot)$, $\check{F}(\cdot;\cdot)$, $F(c_1\cdot; c_2\cdot)$, $\|F(\cdot;\cdot)\|_Y$ or $F(\cdot + \tau; \cdot + x_0)$ also belongs to $A_{X,Y}$. In some cases, it is almost trivial to say when $A_{X,Y}$ will be a vector space. Concerning the uniformly convergent sequences of functions belonging to some of the above-introduced function spaces, we must impose some restrictive conditions on the metric spaces under our considerations in order to obtain any relevant. For example, suppose that $(F_k(\cdot;\cdot) : I \times X \to Y)$ is a sequence of functions and there exists a function $F : I \times X \to Y$ such that $\lim_{k\to+\infty} F_k(\mathbf{t}; x) = F(\mathbf{t}; x)$, uniformly on $I \times B$ for each set B of collection $\mathcal{B}$. Concerning the binary relation ρ on Y, we assume that $D(\rho)$ is closed, ρ is single-valued on $R(F)$ and continuous on $D(\rho)$ in the usual sense ([6]); we assume the same conditions for the sequence $(\rho_j)_{j\in\mathbb{N}_n}$ of binary relations on Y, if considered. Then we have the following:

(i) Suppose that for each positive integer $k \in \mathbb{N}$ the function $F_k(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic of type 1, respectively, $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic [$\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-uniformly recurrent]. Suppose, further, that for each $\tau \in I'$ and $M > 0$ we have that $P_{\tau,M} = C_{0,\nu}(I_{\tau,M} : Y)$ or $P_{\tau,M} = L^\infty_\nu(I_{\tau,M} : Y)$, where $\nu \in L^\infty(I : (0,\infty))$. Then $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic of type 1, respectively, $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic [$\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-uniformly recurrent].

(ii) Suppose that for each positive integer $k \in \mathbb{N}$ the function $F_k(\cdot;\cdot)$ is $(S, \mathbb{D}, \mathcal{B}, \mathcal{P}_{\mathcal{B}})$-asymptotically (ω, ρ)-periodic [$(\mathbb{D}, \mathcal{B}, \rho, \mathcal{P}_{\mathcal{B}})$-slowly oscillating]. Suppose, further, that for each $B \in \mathcal{B}$ we have that $P_B = C_{0,\nu}(\mathbb{D} \times B : [0,\infty))$ or $P_B = L^\infty_\nu(\mathbb{D} \times B : [0,\infty))$, where $\nu \in L^\infty(I : (0,\infty))$. Then $F(\cdot;\cdot)$ is $(S, \mathbb{D}, \mathcal{B}, \mathcal{P}_{\mathcal{B}})$-asymptotically (ω, ρ)-periodic [$(\mathbb{D}, \mathcal{B}, \rho, \mathcal{P}_{\mathcal{B}})$-slowly oscillating]. Here, $C_{0,\nu}(\mathbb{D} \times B : Y) := \{F : \mathbb{D} \times B \to Y \ ; \ \lim_{|\mathbf{t}|\to+\infty, \mathbf{t}\in\mathbb{D}} \sup_{x\in B} \|F(\mathbf{t}; x)\|_Y \nu(\mathbf{t}) = 0\}$ and $d_B(F, G) := \sup_{x\in B} \sup_{\mathbf{t}\in\mathbb{D}} \|F(\mathbf{t}; x)\|_Y \nu(\mathbf{t})$ for all $F,\ G \in C_{0,\nu}(\mathbb{D} \times B : Y)$; we define $L^\infty_\nu(\mathbb{D} \times B : Y)$ similarly.

(iii) Suppose that for each positive integer $k \in \mathbb{N}$ the function $F_k(\cdot;\cdot)$ is $(S, \mathcal{B})$-asymptotically $(\omega_j, \rho_j, \mathbb{D}_j, \mathcal{P}^j_{\mathcal{B}})_{j\in\mathbb{N}_n}$-periodic [$(\mathcal{B}, (\omega_j, \rho_j, \mathbb{D}_j, \mathcal{P}^j_{\mathcal{B}})_{j\in\mathbb{N}_n})$-slowly oscillating]. Suppose, further, that for each $B \in \mathcal{B}$ we have that $P_B = C_{0,\nu}(\mathbb{D}\times B : [0,\infty))$ or $P_B = L^\infty_\nu(\mathbb{D} \times B : [0,\infty))$, where $\nu \in L^\infty(I : (0,\infty))$. Then $F(\cdot;\cdot)$ is $(S, \mathcal{B})$-asymptotically $(\omega_j, \rho_j, \mathbb{D}_j, \mathcal{P}^j_{\mathcal{B}})_{j\in\mathbb{N}_n}$-periodic [$(\mathcal{B}, (\omega_j, \rho_j, \mathbb{D}_j, \mathcal{P}^j_{\mathcal{B}})_{j\in\mathbb{N}_n})$-slowly oscillating].

The following result generalizes [12, Proposition 3.4(i)] and [16, Proposition 3.2]:

Proposition 4.1. *Let $\omega \in I \setminus \{0\}$, $T \in L(X)$, $\|T\| \le 1$, $\omega + I \subseteq I$, $\omega + \mathbb{D} \subseteq \mathbb{D}$ and $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$. Set $I' := \omega \cdot \mathbb{N}$. Suppose that $\nu : I \to (0,\infty)$, $P_B = C_{0,\nu}(\mathbb{D}\times : [0,\infty))$ ($P_B = L^\infty_\nu(\mathbb{D}\times : [0,\infty))$) for any $B \in \mathcal{B}$, and $P_{\tau,M} = C_{0,\nu}(I_{\tau,M} : Y)$ ($P_{\tau,M} = L^\infty_\nu(I_{\tau,M} : Y)$) for any $\tau \in I'$ and $M > 0$. If a function $F : I \times X \to Y$ is $(S, \mathbb{D}, \mathcal{B}, \mathcal{P}_{\mathcal{B}})$-asymptotically (ω, ρ)-periodic, then the function $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic, provided that for each $k \in \mathbb{N}$ we have*

$$\sup_{\mathbf{t}\in\mathbb{D}} \sum_{j=0}^{k-1} \frac{\nu(\mathbf{t})}{\nu(\mathbf{t} + j\omega)} < +\infty. \tag{2}$$

Proof. The proof essentially follows from the argumentation contained in the proof of [12, Proposition 3.4(i)] and our assumption (2). We will only note here that the assumption $\mathbf{t} \in \mathbb{D}$ implies $\mathbf{t} + k\omega \in \mathbb{D}$ for each fixed integer $k \in \mathbb{N}$, so that for each

$x \in B$ we have:

$$\begin{aligned}
&\left\|F(\mathbf{t}+k\omega;x)-TF(\mathbf{t};x)\right\|_Y\nu(\mathbf{t})\\
&\leq \sum_{j=0}^{k-1}\|T\|^{k-1-j}\Big\|F(\mathbf{t}+(j+1)\omega;x)-TF(\mathbf{t}+j\omega;x)\Big\|_Y\nu(\mathbf{t})\\
&\leq \sum_{j=0}^{k-1}\Big\|F(\mathbf{t}+(j+1)\omega;x)-TF(\mathbf{t}+j\omega;x)\Big\|_Y\nu(\mathbf{t}+j\omega)\frac{\nu(\mathbf{t})}{\nu(\mathbf{t}+j\omega)}.
\end{aligned}$$

□

Concerning the invariance of metrical c-quasi-asymptotical almost periodicity under the actions of convolution products, where $c \in \mathbb{C} \setminus \{0\}$, we will only state the following slight extensions of [11, Proposition 3.1, Proposition 3.2] without proofs and say that these results can be also established in the multi-dimensional setting (we can also consider the Banach space $L^\infty_\nu(I:(0,\infty))$ here):

Proposition 4.2. *Suppose that* $c \in \mathbb{C} \setminus \{0\}$, $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family and* $\int_0^\infty \|R(s)\|\,ds < \infty$.

(i) *Suppose that* $\nu : [0,\infty) \to (0,\infty)$ *is an essentially bounded function which satisfies that the function* $1/\nu(\cdot)$ *is locally bounded, as well as that there exists a function* $\varphi : [0,\infty) \to (0,\infty)$ *such that* $\nu(t) \leq \nu(t-s)\varphi(s)$ *for all* $t,\ s \geq 0$ *and* $\int_0^\infty \varphi(s)\|R(s)\|\,ds < \infty$. *If* $f : [0,\infty) \to X$ *is bounded and* $(c\mathrm{I},\mathcal{P})$*-quasi-asymoptotically almost periodic, then the function* $F(\cdot)$*, defined through*

$$F(t) := \int_0^t R(t-s)f(s)\,ds,\ \ t \geq 0,$$

is likewise bounded and $(c\mathrm{I},\mathcal{P})$*-quasi-asymoptotically almost periodic.*

(ii) *Suppose that* $\nu : \mathbb{R} \to (0,\infty)$ *is an essentially bounded function which satisfies that the function* $1/\nu(\cdot)$ *is locally bounded, as well as that there exists a function* $\varphi : [0,\infty) \to (0,\infty)$ *such that* $\nu(t) \leq \nu(t-s)\varphi(s)$ *for all* $t \in \mathbb{R}$, $s \geq 0$ *and* $\int_0^\infty \varphi(s)\|R(s)\|\,ds < \infty$. *If* $f : \mathbb{R} \to X$ *is bounded and* $(c\mathrm{I},\mathcal{P})$*-quasi-asymoptotically almost periodic, then the function* $\mathbf{F}(t)$*, defined through*

$$\mathbf{F}(t) := \int_{-\infty}^t R(t-s)f(s)\,ds, \quad t \in \mathbb{R},$$

is likewise bounded and $(c\mathrm{I},\mathcal{P})$*-quasi-asymoptotically almost periodic.*

It would be very tempting to extend the statements of [3, Theorem 2.34] and [6, Theorem 2.27] for some special kinds of metrically (asymptotically) almost periodic type functions (see also [10, Theorem 7.1.25] for the case in which $\rho = c\mathrm{I}$ for some $c \in \mathbb{C} \setminus \{0\}$). The same observation can be given for the statements of [10, Theorem 6.1.40, Proposition 7.3.15, Proposition 7.3.17, Proposition 7.3.18].

5. Applications to the abstract Volterra integro-differential equations

In this section, we will briefly explain how the obtained results can be applied in the analysis of existence and uniqueness of asymptotically ρ-almost periodic type solutions for some classes of the abstract Volterra integro-differential equations.

1. It is clear that Proposition 2.2 can be applied in the analysis of inhomogeneous heat equation in $\mathbb{R}^n$, and to the Poisson semigroup in $\mathbb{R}^n$; let us only note here that the use of multivalued binary relations is inevitable here and that a fairly complete analysis of this problematic cannot be obtained by assuming that ρ is a function, only ([6, 16]).

In this part, we will apply Proposition 2.2 in the analysis of certain classes of the abstract ill-posed Cauchy problems; we will consider the integrated semigroups here (see [10] and [17] for more details about this kind of applications). Suppose that $k \in \mathbb{N}$, $a_\alpha \in \mathbb{C}$, $0 \leq |\alpha| \leq k$, $a_\alpha \neq 0$ for some α with $|\alpha| = k$, $P(x) = \sum_{|\alpha| \leq k} a_\alpha i^{|\alpha|} x^\alpha$, $x \in \mathbb{R}^n$, $P(\cdot)$ is an elliptic polynomial, i.e., there exist $C > 0$ and $L > 0$ such that $|P(x)| \geq C|x|^k$, $|x| \geq L$, $\omega := \sup_{x \in \mathbb{R}^n} \Re(P(x)) < \infty$, and $X := BUC(\mathbb{R}^n)$, the space of bounded uniformly continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ equipped with the sup-norm. Define

$$P(D) := \sum_{|\alpha| \leq k} a_\alpha f^{(\alpha)} \text{ and } Dom(P(D)) := \{ f \in X : P(D)f \in X \text{ distributionally} \},$$

and assume that $n_X > n/2$. It is well known that the operator $P(D)$ generates an exponentially bounded r-times integrated semigroup $(S_r(t))_{t \geq 0}$ in X for any $r > n_X$ (see [1], [9] and references cited therein for more details about fractionally integrated semigroups). Furthermore, it is well known that for each $t \geq 0$ there exists a function $f_t \in L^1(\mathbb{R}^n)$ such that

$$\big[S_r(t)f\big](x) := \big(f_t * f\big)(x), \quad x \in \mathbb{R}^n, \ f \in X.$$

Let a number $t_0 \geq 0$ be fixed. Assume that $A : BUC(\mathbb{R}^n) \to BUC(\mathbb{R}^n)$ is given by $(Af)(x) := m(x)f(x)$, $x \in \mathbb{R}^n$, $f \in BUC(\mathbb{R}^n)$, where $m \in L^\infty(\mathbb{R}^n)$. Let $\nu \in L^\infty(\mathbb{R}^n : (0, \infty))$ and let there exist a function $\varphi : \mathbb{R}^n \to (0, \infty)$ such that $\nu(x) \leq \nu(y)\varphi(x - y)$ for all x, $y \in \mathbb{R}^n$ and $h\varphi \in L^1(\mathbb{R}^n)$. Assume also that, for every set $B \in \mathcal{B}$, we have $P_B = L^\infty_\nu(\mathbb{R}^n \times B : (0, \infty))$ and the function $f(\cdot)$ is $(S, \mathbb{R}^n, \mathcal{B})$-asymptotically $(\omega, A, \mathcal{P}_\mathcal{B})$-periodic. Applying Proposition 2.2, we get that the function $[S_r(t)f](\cdot)$ is likewise $(S, \mathbb{R}^n, \mathcal{B})$-asymptotically $(\omega, A, \mathcal{P}_\mathcal{B})$-periodic. We can simply incorporate the obtained result in the analysis of corresponding ill-posed Cauchy problems.

2. Because of a great similarity with our previous research studies (see e.g., [10]), we will only note here that the function spaces introduced in this paper can be important in the qualitative analysis of solutions of the inhomogeneous wave equation in $\mathbb{R}^3$, $\mathbb{R}^2$ and $\mathbb{R}$, which are given by the famous Kirchhoff formula, the Poisson formula and the d'Alembert formula, respectively.

3. It is clear that Proposition 4.2 can be applied in the analysis of the existence and uniqueness of metrically c-quasi-asymptotically almost periodic solutions for various classes of abstract (degenerate) inhomogeneous Cauchy problems ([9, 10]); for example, we can apply this result in the qualitative analysis of the following fractional equation with higher order differential operators in the Hölder space $X = C^\alpha(\overline{\Omega})$:

$$\begin{cases} \mathbf{D}_t^\gamma u(t,x) = -\sum\limits_{|\beta| \leq 2m} a_\beta(t,x) D^\beta u(t,x) - \sigma u(t,x) + f(t,x), \ t \geq 0, \ x \in \Omega; \\ u(0,x) = u_0(x), \quad x \in \Omega; \end{cases}$$

see [9, Example 3.10.4] for the notion and more details. Since the composition principle clarified in [16, Theorem 4.5] can be reformulated for metrically ρ-slowly oscillating type functions in $\mathbb{R}^n$, providing the same special choices of metric spaces, we are in a position to analyze the metrically $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, ρ)-periodic solutions, e.g., for the class of semilinear Hammerstein integral equations of convolution type in $\mathbb{R}^n$ ([3, 16]). We can also analyze metrically $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, ρ)-periodic solutions for certain classes of the abstract semilinear fractional Cauchy problems ([9, 10]). Details can be left to the interested readers.

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