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Integral inequalities for mappings whose derivatives are $(h, m, s)-$convex modified of second type via Katugampola integrals

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Abstract. In this paper, using the definition of functions (h, m, s)-convex modified of second type, various extensions of the classic Hermite-Hadamard Inequality are obtained using Katugampola integrals. In addition, we show that several results known are particular cases of ours.

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1. Introduction

The notion of convex function has been the object of attention of many researchers in recent years, due to its multiple applications and links with various mathematical areas. Readers interested in the aforementioned development, can consult [32], where a panorama, practically complete, of these branches is presented.

A function $\psi : I = [a, b] \to \mathbb{R}$, is said to be convex if $\psi\big(\tau\xi + (1-\tau)\varsigma\big) \leq \tau\ \psi(\xi) + (1-\tau)\psi(\varsigma)$ holds $\forall\ \xi, \varsigma \in I$,$\tau \in [0, 1]$. And they say that the function ψ is concave on $[a, b]$ if the inequality is the opposite.

Definition 1.1. Let $h : [0, 1] \to [0, 1]$ be a nonnegative function, $h \neq 0$ and $\psi : I = [0, +\infty) \to [0, +\infty)$. If inequality

$$\psi\left(\tau\xi + m(1-\tau)\varsigma\right) \leq h^s(\tau)\psi(\xi) + m(1-h^s(\tau))\psi\left(\varsigma\right) \tag{1}$$

is fulfilled for all $\xi, \varsigma \in I$ and $\tau \in [0, 1]$, where $m \in [0, 1]$, $s \in [-1, 1]$. Then is said function ψ is a $(h, m, s)-$convex modified of first type on I.

Definition 1.2. Let $h : [0, 1] \to [0, 1]$ nonnegative functions, $h \neq 0$ and $\psi : I = [0, +\infty) \to [0, +\infty)$. If inequality

$$\psi\left(\tau\xi + m(1-\tau)\varsigma\right) \leq h^s(\tau)\psi(\xi) + m(1-h(\tau))^s\psi\left(\varsigma\right) \tag{2}$$

is fulfilled for all $\xi, \varsigma \in I$ and $\tau \in [0, 1]$, where $m \in [0, 1]$, $s \in [-1, 1]$. Then is said function ψ is a $(h, m, s)-$convex modified of second type on I.

Remark 1.1. From Definitions 1.1 and 1.2 we can define $N^s_{h,m}[a, b]$, where $a, b \in [0, +\infty)$, as the set of functions $(h, m, s)-$convex modified, for which $\psi(a) \geq 0$, characterized by the triple $(h(\tau), m, s)$. Note that if:
(1) $(h(\tau), 0, 0)$ we have the increasing functions ([8]).

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(2) $(\tau, 0, s)$ we have the $s-$starshaped functions ([8]).
(3) $(\tau, 0, 1)$ we have the starshaped functions ([8]).
(4) $(\tau, 1, 1)$ then ψ is a convex function on $[0, +\infty)$ ([8]).
(5) $(\tau, 1, -1)$ then ψ is a Godunova-Levin convex function on $[0, +\infty)$ ([16])
(6) $(\tau, m, 1)$ then ψ is a $m-$convex function on $[0, +\infty)$ ([49]).
(7) $(\tau, 1, s)$ $s \in (0, 1]$ then ψ is a $s-$convex function on $[0, +\infty)$ ([9, 22]).
(8) $(\tau, 1, s)$ $s \in [-1, 1]$ then ψ is a extended $s-$convex function on $[0, +\infty)$ ([55]).
(9) $(\tau, m, s), s \in [-1, 1]$ then ψ is a extended (s, m)-convex function on $[0, +\infty)$ ([57]).
(10) $(\tau^\alpha, 1, s)$ with $\alpha \in (0, 1]$, then ψ is a $(\alpha, s)-$convex function on $[0, +\infty)$ ([56]).
(11) $(\tau^a, m, 1)$ with $\alpha \in (0, 1]$, then ψ is a $(\alpha, m)-$convex function on $[0, +\infty)$ ([29]).
(12) (τ^α, m, s) with $\alpha \in (0, 1]$, then ψ is a $s - (\alpha, m)-$convex function on $[0, +\infty)$ ([31, 56]).
(13) $(h(\tau), m, 1)$ then we have a variant of a function (h, m)-convex on $[0, +\infty)$ ([35]).

One of the most important inequalities, for convex functions, is the called Hermite–Hadamard inequality:

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \psi(\xi)d\xi \leq \frac{\psi(a)+\psi(b)}{2} \tag{3}$$

valid for any function ψ convex on the interval $[a, b]$. This inequality was published by Hermite ([19]) in 1883 and, independently, by Hadamard in 1893 ([18]). It gives an estimation of the mean value of a convex function, and it is important to note that it also provides a refinement to the Jensen inequality. Several results can be consulted in [3–10, 12–14, 17, 20, 28, 33] and references therein for more information and other extensions of the Hermite–Hadamard inequality. All through the work we utilize the functions Γ (see [52]) and Γ_k (see [11]):

$$\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau}\, d\tau, \quad \Re(z) > 0,$$
$$\Gamma_k(z) = \int_0^\infty \tau^{z-1} e^{-\tau^k/k}\, d\tau, k > 0.$$

Unmistakably if $k \to 1$ we have $\Gamma_k(z) \to \Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$.

To encourage comprehension of the subject, we present the definition of Riemann-Liouville fractional integral (with $0 \leq a < \tau < \nu_2 \leq \infty$). The first is the classic Riemann-Liouville fractional integrals.

Definition 1.3. Let $\psi \in L_1[a, b]$. Then the Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ are defined by (right and left respectively):

$$I^\alpha_{\nu_1+}\psi(\xi) = \frac{1}{\Gamma(\alpha)}\int_{\nu_1}^{\xi} (\xi-\tau)^{\alpha-1}\psi(\tau)\, d\tau, \quad \xi > \nu_1$$
$$I^\alpha_{\nu_2-}\psi(\xi) = \frac{1}{\Gamma(\alpha)}\int_{\xi}^{\nu_2} (\tau-\xi)^{\alpha-1}\psi(\tau)\, d\tau, \quad \xi < \nu_2.$$

Next we present Katugampola fractional integral, which will be the basis of our work (see [27]).

Definition 1.4. Let $\psi \in L_1(\nu_1, \nu_2)$ the left-sided ${}^\rho I^\alpha_{\nu_1+}$ and the right-sided ${}^\rho I^\alpha_{\nu_2-}$ Katugampola fractional integral is defined by

$$\left({}^\rho\mathcal{I}^\alpha_{\nu_1+}\psi\right)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\nu_1}^{x} \frac{\tau^{\rho-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau,$$

$$\left({}^\rho\mathcal{I}^\alpha_{\nu_2-}\psi\right)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{x}^{\nu_2} \frac{\tau^{\rho-1}}{(\tau^\rho - x^\rho)^{1-\alpha}} f(\tau) d\tau,$$

with $0 \leq \nu_1 < x < \nu_2 \leq \infty$, where α is the order of the fractional integral $\alpha \in C$, $Re(\alpha) > 0$.

Remark 1.2. If $\rho = 1$ then the Katugampola fractional integrals become Riemann–Liouville fractional integrals. If additionally $\alpha = 1$, then we have the classic Riemann integral.

In this paper, we obtain several integral inequalities of the Hermite – Hadamard type for differentiable $(h, m, s)-$convex second–order modified functions. To obtain the inequalities, we use Katugampola fractional integral, previously defined.

2. Hermite – Hadamard type inequalities for $(h, m, s)-$ convex modified functions of second type

We have the first result:

Theorem 2.1. *Let $\psi : [0, +\infty) \to \mathbb{R}$ and $\psi \in N^s_{h,m}[\nu_1, m\nu_2]$, with $m \in (0, 1]$. If $0 \leq \nu_1 < m\nu_2 < +\infty$, $\psi \in L^1[\nu_1, m\nu_2]$ and $h \in L^1[0, 1]$, then for all integers $\ell \geq 0$ we have the following inequality:*

$$\begin{aligned}
\psi\left(\frac{\nu_1^\rho + \nu_2^\rho}{2}\right) &\leq h^s\left(\frac{1}{2}\right) \frac{(\ell+1)^\alpha \Gamma(\alpha+1)\rho^\alpha}{(\nu_2^\rho - \nu_1^\rho)} \; {}_\rho I^\alpha_{\nu_2-}\psi\left(\frac{\nu_1 + \ell\nu_2}{\ell+1}\right) \\
&\quad + \left(1 - h\left(\frac{1}{2}\right)\right)^s \frac{(\ell+1)^\alpha \Gamma(\alpha+1)\rho^\alpha}{(\nu_2^\rho - \nu_1^\rho)} \; {}_\rho I^\alpha_{\nu_1+}\psi\left(\frac{\ell\nu_1 + \nu_2}{\ell+1}\right) \qquad (4) \\
&\leq \rho\alpha\left[h^s\left(\frac{1}{2}\right)\psi(\nu_1^\rho) + \left(1 - h\left(\frac{1}{2}\right)\right)^s \psi(\nu_2^\rho)\right] \int_0^1 \tau^{\rho\alpha-1} h^s\left(\frac{\tau^\rho}{\ell+1}\right) d\tau \\
&\quad + m\rho\alpha\left[h^s\left(\frac{1}{2}\right)\psi\left(\frac{\nu_2^\rho}{m}\right) + \left(1 - h\left(\frac{1}{2}\right)\right)^s \psi\left(\frac{\nu_1^\rho}{m}\right)\right] \\
&\quad \times \int_0^1 \tau^{\rho\alpha-1}\left(1 - h\left(1 - \frac{\tau^\rho}{\ell+1}\right)\right)^s d\tau.
\end{aligned}$$

Proof. For $x, y \in [0, +\infty)$, $\tau = \frac{1}{2}$ and $m = 1$, we have

$$\psi\left(\frac{x^\rho + y^\rho}{2}\right) \leq h^s\left(\frac{1}{2}\right)\psi(x^\rho) + \left(1 - h\left(\frac{1}{2}\right)\right)^s \psi(y^\rho).$$

If we choose $x^\rho = \frac{\tau^\rho}{\ell+1}\nu_1^\rho + \left(1 - \frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho$ and $y^\rho = \frac{\tau^\rho}{\ell+1}\nu_2^\rho + \left(1 - \frac{\tau^\rho}{\ell+1}\right)\nu_1^\rho$, with $\tau \in [0, 1]$, we get

$$\begin{aligned}
\psi\left(\frac{\nu_1^\rho + \nu_2^\rho}{2}\right) &\leq h^s\left(\frac{1}{2}\right)\psi\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho + \left(1 - \frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right) \qquad (5) \\
&\quad + \left(1 - h\left(\frac{1}{2}\right)\right)^s \psi\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho + \left(1 - \frac{\tau^\rho}{\ell+1}\right)\nu_1^\rho\right).
\end{aligned}$$

Multiplying both members of the previous inequality by $\tau^{\rho\alpha-1}$, integrating with respect to τ from 0 to 1 we obtain

$$\begin{aligned}\frac{1}{\rho\alpha}\psi\left(\frac{\nu_1^\rho+\nu_2^\rho}{2}\right) \leq h^s\left(\frac{1}{2}\right)\frac{(\ell+1)^\alpha\Gamma(\alpha)\rho^{\alpha-1}}{(\nu_2^\rho-\nu_1^\rho)}\ {}_\rho I^\alpha_{\nu_2-}\psi\left(\frac{\nu_1+\ell\nu_2}{\ell+1}\right) \\ +\left(1-h\left(\frac{1}{2}\right)\right)^s\frac{(\ell+1)^\alpha\Gamma(\alpha)\rho^{\alpha-1}}{(\nu_2^\rho-\nu_1^\rho)}\ {}_\rho I^\alpha_{\nu_1+}\psi\left(\frac{\ell\nu_1+\nu_2}{\ell+1}\right).\end{aligned}$$

the first inequality of (4).

From right member of (5) we obtain

$$\begin{aligned}&h^s\left(\frac{1}{2}\right)\psi\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho+\left(1-\frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right)\\
&\quad+\left(1-h\left(\frac{1}{2}\right)\right)^s\psi\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho+\left(1-\frac{\tau^\rho}{\ell+1}\right)\nu_1^\rho\right)\\
&=h^s\left(\frac{1}{2}\right)\psi\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho+m\left(1-\frac{\tau^\rho}{\ell+1}\right)\frac{\nu_2^\rho}{m}\right)\\
&\quad+\left(1-h\left(\frac{1}{2}\right)\right)^s\psi\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho+m\left(1-\frac{\tau^\rho}{\ell+1}\right)\frac{\nu_1^\rho}{m}\right)\\
&\leq h^s\left(\frac{1}{2}\right)\left[h^s\left(\frac{\tau^\rho}{\ell+1}\right)\psi(\nu_1^\rho)+m\left(1-h\left(1-\frac{\tau^\rho}{\ell+1}\right)\right)^s\psi\left(\frac{\nu_2^\rho}{m}\right)\right]\\
&\quad+\left(1-h\left(\frac{1}{2}\right)\right)^s\left[h^s\left(\frac{\tau^\rho}{\ell+1}\right)\psi(\nu_2^\rho)+m\left(1-h\left(1-\frac{\tau^\rho}{\ell+1}\right)\right)^s\psi\left(\frac{\nu_1^\rho}{m}\right)\right].\end{aligned}$$

Multiplying this by $\tau^{\rho\alpha-1}$ and integrating with respect to τ, between 0 and 1, allows us to get the right member of (4). In this way the proof is completed. □

Remark 2.1. If in the previous Theorem we consider convex functions, i.e. $s=m=1$ and $h(\tau)=\tau$ and we put $\rho=\alpha=1$, with $\ell=0$, from (4) we obtain the classic Hermite-Hadamard inequality (3).

Remark 2.2. In the framework of convex functions, $\ell=0$ we have Theorem 4 of [41] for Riemann-Liouville integrals.

Remark 2.3. If we consider $s-$convex functions, $\ell=0$ we have Theorem 2.1 of [28] (also see Theorem 2.1 of [10]). Also with $\ell=1$, we obtain Theorem 1 of [58].

Remark 2.4. Readers can obtain, without much difficulty, Theorem 2.1 of [15], Theorem 5 of [53] and Theorem 3.1 of [21] obtained for the $k-$integral of [30], with $\ell=0$.

The following result will be basic from now on.

Lemma 2.2. *Let ψ be a real function defined on some interval $[\nu_1,\nu_2]\subset\mathbb{R}$, differentiable on (ν_1,ν_2). If $\psi'\in L_1(\nu_1,\nu_2)$, and $w(\tau)$ a differentiable function on $[\nu_1,\nu_2]$,* *then for all integers $\ell \geq 0$ we have the following equality:*

$$\begin{aligned}
&\left\{-w(1)\left(\psi(\frac{\nu_1+\ell\nu_2}{\ell+1})+\psi(\frac{\ell\nu_1+\nu_2}{\ell+1})\right)+w(0)\left(\psi(\nu_1)+\psi(\nu_2)\right)\right\} \qquad (6)\\
&\quad+\frac{\ell+1}{\nu_2-\nu_1}\left(\int_{\nu_1}^{\frac{\ell\nu_1+\nu_2}{\ell+1}} w'\left[\frac{u-\nu_1}{\frac{\nu_2-\nu_1}{\ell+1}}\right]\psi(u)du+\int_{\frac{\nu_1+\ell\nu_2}{\ell+1}}^{\nu_2} w'\left[\frac{\nu_2-u}{\frac{\nu_2-\nu_1}{\ell+1}}\right]\psi(u)du\right)\\
&=\frac{\nu_2-\nu_1}{\ell+1}\int_0^1 w(\tau)\left[\psi'\left(\frac{\tau}{\ell+1}\nu_1+\left(1-\frac{\tau}{\ell+1}\right)\nu_2\right)\right.\\
&\quad\left.-\psi'\left(\frac{\tau}{\ell+1}\nu_2+\left(1-\frac{\tau}{\ell+1}\right)\nu_1\right)d\tau\right].
\end{aligned}$$

Proof. First note that

$$\begin{aligned}
&\int_0^1 w(\tau)\left[\psi'\left(\frac{\tau\nu_1}{\ell+1}+\left(1-\frac{\tau}{\ell+1}\right)\nu_2\right)-\psi'\left(\frac{\tau\nu_2}{\ell+1}+\left(1-\frac{\tau}{\ell+1}\right)\nu_1\right)\right]d\tau\\
&=\int_0^1 w(\tau)\psi'\left(\frac{\tau\nu_1}{\ell+1}+\left(1-\frac{\tau}{\ell+1}\right)\nu_2\right)d\tau\\
&-\int_0^1 w(\tau)\psi'\left(\frac{\tau\nu_2}{\ell+1}+\left(1-\frac{\tau}{\ell+1}\right)\nu_1\right)d\tau\\
&= I_1 - I_2.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
I_1 =&\frac{\ell+1}{\nu_2-\nu_1}\left[-w(1)\psi\left(\frac{\nu_1+\ell\nu_2}{\ell+1}\right)+w(0)\psi(\nu_2)\right]\\
&+\frac{(\ell+1)^2}{(\nu_2-\nu_1)^2}\int_{\nu_1}^{\frac{\ell\nu_1+\nu_2}{\ell+1}} w'\left[\frac{u-\nu_1}{\frac{\nu_2-\nu_1}{\ell+1}}\right]\psi(u)du,
\end{aligned}$$

since

$$\int_0^1 w'(\tau)\psi\left(\frac{\tau}{\ell+1}\nu_1+\left(1-\frac{\tau}{\ell+1}\right)\nu_2\right)d\tau=\frac{\ell+1}{\nu_2-\nu_1}\int_{\nu_1}^{\frac{\ell\nu_1+\nu_2}{\ell+1}} w'\left[\frac{u-\nu_1}{\frac{\nu_2-\nu_1}{\ell+1}}\right]\psi(u)du.$$

Analogously

$$\begin{aligned}
I_2 =&\frac{\ell+1}{\nu_2-\nu_1}\left[w(1)\psi\left(\frac{\ell\nu_1+\nu_2}{\ell+1}\right)-w(0)\psi(\nu_1)\right]\\
&-\frac{(\ell+1)^2}{(\nu_2-\nu_1)^2}\int_{\frac{\nu_1+\ell\nu_2}{\ell+1}}^{\nu_2} w'\left[\frac{\nu_2-u}{\frac{\nu_2-\nu_1}{\ell+1}}\right]\psi(u)du.
\end{aligned}$$

From $I_1 - I_2$, and grouping appropriately, we have the required inequality. □

Let's analyze some consequences of this result.

Remark 2.5. In [25], starting from Lemma 2.1 of [27] (see Remark 2.8 below), through a change of variables obtains, for convex functions and with $w(\tau) = \tau$, the integral of the right member of (6), we must point out that the + sign in said work is incorrect. In this way, it is clear that equation (5) of [25] is obtained from the previous Lemma for $\ell = 0$, $w(\tau) = \tau$.

We want to point out various results that can be obtained from the previous Lemma, which shows the scope and generality of said result.

Remark 2.6. Putting $w(\tau) = \tau^{\alpha}$ and considering convex functions and $\ell = 1$ we obtain the Lemma 3 of [44].

Remark 2.7. If $w(\tau) = \tau$, $\ell = 0$ we obtain a new result for classic integral.

Remark 2.8. Let's consider $\ell = 0$.
For various choices of the weight $w'(\tau)$ and taking, not only the right member of (6) but only one of the integrals, they can be obtained without difficulty a variant of Lemma 1 of [4], Lemma 2.1 of [12] (also see Lemma 2.1 of [24]), Lemma 2.3 of [15], Lemma 1 of [23], Lemma 1 of [26], Lemma 2.1 of [27], Lemma 1 of [34], Lemma 1 of [37], Lemma 3.1 of [46] and Lemma 2 of [43] (see also [38]) are obtained.

Remark 2.9. Also, the reader will be able to verify, without much difficulty, that under different variants of the weight $w(\tau)$ we can obtain Lemma 2 of [39], Lemma 1.1 of [47] (see also Lemma 2 of [36]), Lemma 2.1 from [42], Lemma 2.1 from [59], Lemma 2.1 from [54], Lemma 1.6 from [13], Lemma 2.1 from [1], Lemma 1 of [5], Lemma 2.1 of [45], and Lemma 2.1 of [40].

Remark 2.10. With $w(\tau) = (1-\tau)^{\alpha}$ and $\ell = 0$, we obtain a new result, for Riemann-Liouville integrals.

Remark 2.11. If $\ell = 1$, Lemma 1 of [2] and Lemma 1 of [50] can be obtained from our result, under the appropriate definition of $w(\tau) = w_1 + w_2$ (see also [51]).

In addition to the previous remarks, we want to stop at a very special case: the Katugampola integral. Let's put in the (6) $w(\tau) = \tau^{\rho\alpha}$ and let's use appropriate notations, then we have the following result:

Lemma 2.3.

$$\begin{aligned}
&-\left(\psi\left(\frac{\nu_1^{\rho}+\ell\nu_2^{\rho}}{\ell+1}\right)+\psi\left(\frac{\ell\nu_1^{\rho}+\nu_2^{\rho}}{\ell+1}\right)\right) \\
&\quad+\frac{\rho^{\alpha}(\ell+1)^{\alpha}\Gamma(\alpha+1)}{(\nu_2^{\rho}-\nu_1^{\rho})^{\alpha}}\left(\left({}^{\rho}\mathcal{I}^{\alpha}_{\nu_1^{\rho}+}\psi\right)\left(\frac{\ell\nu_1^{\rho}+\nu_2^{\rho}}{\ell+1}\right)-\left({}^{\rho}\mathcal{I}^{\alpha}_{\nu_2^{\rho}-}\psi\right)\left(\frac{\nu_1^{\rho}+\ell\nu_2^{\rho}}{\ell+1}\right)\right) \\
&=\frac{\nu_2^{\rho}-\nu_1^{\rho}}{\ell+1}\int_0^1 \tau^{\rho\alpha}\left[\psi'\left(\frac{\tau^{\rho}}{\ell+1}\nu_1^{\rho}+\left(1-\frac{\tau^{\rho}}{\ell+1}\right)\nu_2^{\rho}\right)\right. \\
&\quad\left.-\psi'\left(\frac{\tau^{\rho}}{\ell+1}\nu_2^{\rho}+\left(1-\frac{\tau^{\rho}}{\ell+1}\right)\nu_1^{\rho}\right)d\tau\right].
\end{aligned} \tag{7}$$

Remark 2.12. In the case $\ell = 0$, this Lemma contains as a particular case, with slight modification, Lemma 2.1 of [14]. If $\ell = 1$ the above result covers Lemma 2.1 of [58], Lemma 3.1 of [15] for k-fractional integrals ($\rho = 1$) and Lemma 3 of [41] and Lemma 2.1 of [48] for Riemann-Liouville fractional integrals (with $\rho = 1$).

Our first main result relative to the Katugampola integral is the following.

Theorem 2.4. *Let $\psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° such that $\psi' \in L_1\left[\nu_1^{\rho}, \frac{\nu_2^{\rho}}{m}\right]$. Under the assumptions of Lemma 2.2, if $|\psi'| \in N^s_{h,m}[\nu_1^{\rho}, \frac{\nu_2^{\rho}}{m}]$, then* *for all integers $\ell \geq 0$ we have the following inequality:*

$$\left|\mathbb{A} + \frac{\rho^{\alpha}(\ell+1)^{\alpha}\Gamma(\alpha+1)}{(\nu_2^{\rho}-\nu_1^{\rho})^{\alpha}}\left(\left({}^{\rho}\mathcal{I}^{\alpha}_{\nu_1^{\rho}+}\psi\right)\left(\frac{\ell\nu_1^{\rho}+\nu_2^{\rho}}{\ell+1}\right)-\left({}^{\rho}\mathcal{I}^{\alpha}_{\nu_2^{\rho}-}\psi\right)\left(\frac{\nu_1^{\rho}+\ell\nu_2^{\rho}}{\ell+1}\right)\right)\right| \tag{8}$$
$$\leq \frac{\nu_2^{\rho}-\nu_1^{\rho}}{\ell+1}\left\{\left(\left(|\psi'(\nu_1^{\rho})|+|\psi'(\nu_2^{\rho})|\right)\mathbb{B}+m\mathbb{C}\left[\left|\psi'\left(\frac{\nu_1^{\rho}}{m}\right)\right|+\left|\psi'\left(\frac{\nu_2^{\rho}}{m}\right)\right|\right]\right)\right\}$$

with

$$\begin{aligned}
\mathbb{A} &= -\left[\psi\left(\frac{\nu_1^{\rho}+\ell\nu_2}{\ell+1}\right)+\psi\left(\frac{\ell\nu_1+\nu_2^{\rho}}{\ell+1}\right)\right],\\
\mathbb{B} &= \int_0^1 \tau^{\rho\alpha}h^{s}\left(\frac{\tau^{\rho}}{\ell+1}\right)d\tau\\
\mathbb{C} &= \int_0^1 \tau^{\rho\alpha}\left(1-h\left(1-\frac{\tau}{\ell+1}\right)\right)^{s}d\tau.
\end{aligned}$$

Proof. From Lemma 2.2 we obtain

$$\left|\int_0^1 \tau^{\rho\alpha}\left[\psi'\left(\frac{\tau^{\rho}\nu_1^{\rho}}{\ell+1}+\left(1-\frac{\tau^{\rho}}{\ell+1}\right)\nu_2^{\rho}\right)-\psi'\left(\frac{\tau^{\rho}\nu_2^{\rho}}{\ell+1}+\left(1-\frac{\tau^{\rho}}{\ell+1}\right)\nu_1^{\rho}\right)\right]d\tau\right|$$
$$\leq \int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^{\rho}}{\ell+1}\nu_1^{\rho}+\left(1-\frac{\tau^{\rho}}{\ell+1}\right)\nu_2^{\rho}\right)\right|d\tau$$
$$+\int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^{\rho}}{\ell+1}\nu_2^{\rho}+\left(1-\frac{\tau^{\rho}}{\ell+1}\right)\nu_1^{\rho}\right)\right|d\tau.$$

Using the modified $(h, m.s)$-convexity of $|\psi'|$, we get

$$\int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^{\rho}}{\ell+1}\nu_1^{\rho}+\left(1-\frac{\tau^{\rho}}{\ell+1}\right)\nu_2^{\rho}\right)\right|d\tau \tag{9}$$
$$\leq \int_0^1 \tau^{\rho\alpha}\left[h^{s}\left(\frac{\tau^{\rho}}{\ell+1}\right)|\psi'\nu_1^{\rho}|+m\left(1-h\left(1-\frac{\tau^{\rho}}{\ell+1}\right)\right)^{s}\left|\psi'\left(\frac{\nu_2^{\rho}}{m}\right)\right|\right]d\tau$$
$$=|\psi'(\nu_1^{\rho})|\int_0^1 \tau^{\rho\alpha}h^{s}\left(\frac{\tau^{\rho}}{\ell+1}\right)d\tau+m\left|\psi'\left(\frac{\nu_2^{\rho}}{m}\right)\right|\int_0^1 \tau^{\rho\alpha}\left(1-h\left(1-\frac{\tau}{\ell+1}\right)\right)^{s}d\tau.$$

In the same way

$$\int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^{\rho}}{\ell+1}\nu_2^{\rho}+\left(1-\frac{\tau}{\ell+1}\right)\nu_1^{\rho}\right)\right|d\tau \tag{10}$$
$$\leq |\psi'(\nu_2^{\rho})|\int_0^1 \tau^{\rho\alpha}h^{s}\left(\frac{\tau}{\ell+1}\right)d\tau+m\left|\psi'\left(\frac{\nu_1^{\rho}}{m}\right)\right|\int_0^1 \tau^{\rho\alpha}\left(1-h\left(1-\frac{\tau}{\ell+1}\right)\right)^{s}d\tau.$$

From (9) and (10) we easily obtain (8). In this way the theorem is proved. □

Remark 2.13. From this result we can obtain, for $\ell = 0$, a modified version of the Theorem 2.3 of [28] for $s-$convex functions, the Theorem 2.3 of [10] for convex functions and Theorem 2.2 and Corollary 2.3 for $h-$convex functions of [14]. If we put $\ell = 1$ we obtain the Theorem 2. and Corollary 2.2 of [58] for $s-$convex functions.

Refinements of the previous results, can be obtained by imposing new additional conditions on $|\psi'|$.

Theorem 2.5. *Let $\psi : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $\psi' \in L_1\left[\nu_1^\rho, \frac{\nu_2^\rho}{m}\right]$. Under the assumptions of Lemma 2.2, if $|\psi'|^q \in N_{h,m}^s[\nu_1^\rho, \frac{\nu_2^\rho}{m}]$, $q \geq 1$, then for all integers $\ell \geq 0$ we have the following inequality:*

$$\begin{aligned} &\left|\mathbb{A} + \frac{\rho^\alpha(\ell+1)^\alpha\Gamma(\alpha+1)}{(\nu_2^\rho - \nu_1^\rho)^\alpha}\left(\left({}^\rho\mathcal{I}^\alpha_{\nu_1^\rho +}\psi\right)(\frac{\ell\nu_1^\rho + \nu_2^\rho}{\ell+1}) - \left({}^\rho\mathcal{I}^\alpha_{\nu_2^\rho -}\psi\right)(\frac{\nu_1^\rho + \ell\nu_2^\rho}{\ell+1})\right)\right| \\ &\quad \leq \frac{\nu_2^\rho - \nu_1^\rho}{\ell+1}\frac{1}{(p\alpha\rho+1)^{\frac{1}{p}}}\left\{(p_{12}C_{11} + mp_2C_{12})^{\frac{1}{q}} + (p_{12}C_{11} + mp_1C_{12})^{\frac{1}{q}}\right\} \end{aligned} \tag{11}$$

with $\mathbb{A}$ as before,

$$\begin{aligned} p_1 &= \left|\psi'\left(\frac{\nu_1^\rho}{m}\right)\right|^q, p_{12} = \left|\psi'\left(\frac{\nu_1^\rho + \nu_2^\rho}{2}\right)\right|^q, \\ p_2 &= \left|\psi'\left(\frac{\nu_2^\rho}{m}\right)\right|^q, C_{11} = \int_0^1 h^s\left(\frac{\tau^\rho}{\ell+1}\right)d\tau, \\ C_{12} &= \int_0^1 \left(1 - h\left(1 - \frac{\tau^\rho}{\ell+1}\right)\right)^s d\tau. \end{aligned}$$

Proof. As previous result, from Lemma 2.2 we obtain

$$\begin{aligned} &\left|\int_0^1 \tau^{\rho\alpha}\left[\psi'\left(\frac{\tau^\rho\nu_1^\rho}{\ell+1} + \left(1 - \frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right) - \psi'\left(\frac{\tau^\rho\nu_2^\rho}{\ell+1} + \left(1 - \frac{\tau^\rho}{\ell+1}\right)\nu_1^\rho\right)\right]d\tau\right| \\ &\leq \int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho + \left(1 - \frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right)\right|d\tau \\ &\quad + \int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho + \left(1 - \frac{\tau^\rho}{\ell+1}\right)\nu_1^\rho\right)\right|d\tau. \end{aligned}$$

From Hölder's inequality, we obtain

$$\begin{aligned} &\int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho + \left(1 - \frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right)\right|d\tau \\ &\leq \left(\int_0^1 \tau^{p\rho\alpha}d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho + \left(1 - \frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right)\right|^q d\tau\right)^{\frac{1}{q}} \\ &= \frac{1}{(p\alpha\rho+1)^{\frac{1}{p}}}\left(\int_0^1\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho + \left(1 - \frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right)\right|^q d\tau\right)^{\frac{1}{q}} \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho + \left(1 - \frac{\tau}{\ell+1}\right)\nu_1^\rho\right)\right|d\tau \\ &\leq \left(\int_0^1 \tau^{p\rho\alpha}d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho + \left(1 - \frac{\tau}{\ell+1}\right)\nu_1^\rho\right)\right|^q d\tau\right)^{\frac{1}{q}} \\ &= \frac{1}{(p\alpha\rho+1)^{\frac{1}{p}}}\left(\int_0^1\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho + \left(1 - \frac{\tau}{\ell+1}\right)\nu_1^\rho\right)\right|^q d\tau\right)^{\frac{1}{q}} \end{aligned} \tag{13}$$

for $\frac{1}{p}+\frac{1}{q}=1$. Using the (h,m,s)-convexity of the second type of $|\psi'|^q$, we obtain from (12) and (13):

$$\int_0^1\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho+\left(1-\frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right)\right|^q d\tau \tag{14}$$
$$\leq |\psi'(\nu_1^\rho)|^q\int_0^1 h^s\left(\frac{\tau^\rho}{\ell+1}\right)d\tau+m\left|\psi'\left(\frac{\nu_2^\rho}{m}\right)\right|^q\int_0^1\left(1-h\left(1-\frac{\tau^\rho}{\ell+1}\right)\right)^s d\tau,$$

$$\int_0^1\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho+\left(1-\frac{\tau}{\ell+1}\right)\nu_1^\rho\right)\right|^q d\tau \tag{15}$$
$$\leq |\psi'(\nu_2^\rho)|^q\int_0^1 h^s\left(\frac{\tau^\rho}{\ell+1}\right)d\tau+m\left|\psi'\left(\frac{\nu_1^\rho}{m}\right)\right|^q\int_0^1\left(1-h\left(1-\frac{\tau}{\ell+1}\right)\right)^s d\tau.$$

Substituting (14), (15) in (12) and (13), we obtain the required inequality. □

Remark 2.14. It can be verified, without much difficulty, that the following results can be derived from the previous result, the Theorem 2.5 of [14] for h-convex functions with $\ell=0$, the Theorem 2.3 for s-convex functions of [58] $\ell=1$, the Theorem 3.2 for convex functions, $\ell=1$, from [15], the Theorem 2.2 for (a,m)-convex functions of [48] and the Theorem 6 for convex functions of [41] with $\ell=1$ and $\rho=1$.

Theorem 2.6. *Let $\psi: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° such that $\psi'\in L_1\left[\nu_1^\rho,\frac{\nu_2^\rho}{m}\right]$. Under the assumptions of Lemma 2.2, if $|\psi'|^q\in N_{h,m}^s[\nu_1^\rho,\frac{\nu_2^\rho}{m}]$, $q>1$, then for all integers $\ell\geq 0$ we have the following inequality:*

$$\left|\mathbb{A}+\frac{\rho^\alpha(\ell+1)^\alpha\Gamma(\alpha+1)}{(\nu_2^\rho-\nu_1^\rho)^\alpha}\left(\left({}^\rho\mathcal{I}^\alpha_{\nu_1^\rho+}\psi\right)\left(\frac{\ell\nu_1^\rho+\nu_2^\rho}{\ell+1}\right)-\left({}^\rho\mathcal{I}^\alpha_{\nu_2^\rho-}\psi\right)\left(\frac{\nu_1^\rho+\ell\nu_2^\rho}{\ell+1}\right)\right)\right| \tag{16}$$
$$\leq\frac{\nu_2^\rho-\nu_1^\rho}{\ell+1}\Delta\left\{(p_{12}D_{11}+mp_2D_{12})^{\frac{1}{q}}+(p_{12}D_{21}+mp_2D_{22})^{\frac{1}{q}}\right\}$$

with $\mathbb{A}$, p_1, p_{12} and p_2 as before,

$$\begin{aligned}D_{11}&=\int_0^1\tau^{\rho\alpha}h^s\left(\frac{\tau^\rho}{\ell+1}\right)d\tau,\ \Delta=\left(\frac{1}{\alpha\rho+1}\right)^{1-\frac{1}{q}}\\ D_{12}&=\int_0^1\tau^{\rho\alpha}\left(1-h\left(1-\frac{\tau^\rho}{\ell+1}\right)\right)^s d\tau,\\ D_{21}&=\int_0^1\tau^{\rho\alpha}h^s\left(1-\frac{\tau^\rho}{\ell+1}\right)d\tau \text{ and } D_{22}=\int_0^1\tau^{\rho\alpha}\left(1-h\left(\frac{\tau^\rho}{\ell+1}\right)\right)^s d\tau.\end{aligned}$$

Proof. As before, from the Lemma 2.2 we have:

$$\left|\int_0^1\tau^{\rho\alpha}\left[\psi'\left(\frac{\tau^\rho\nu_1^\rho}{\ell+1}+\left(1-\frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right)-\psi'\left(\frac{\tau^\rho\nu_2^\rho}{\ell+1}+\left(1-\frac{\tau^\rho}{\ell+1}\right)\nu_1^\rho\right)\right]d\tau\right|$$
$$\leq\int_0^1\tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho+\left(1-\frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right)\right|d\tau$$
$$+\int_0^1\tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho+\left(1-\frac{\tau^\rho}{\ell+1}\right)\nu_1^\rho\right)\right|d\tau.$$

and using well known power mean inequality, we have

$$\int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho+\left(1-\frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right)\right|d\tau \tag{17}$$
$$\leq \left(\frac{1}{\alpha\rho+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho+\left(1-\frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right)\right|^q d\tau\right)^{\frac{1}{q}}$$

and

$$\int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho+\left(1-\frac{\tau}{\ell+1}\right)\nu_1^\rho\right)\right|d\tau \tag{18}$$
$$\leq \left(\frac{1}{\alpha\rho+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho+\left(1-\frac{\tau}{\ell+1}\right)\nu_1^\rho\right)\right|^q d\tau\right)^{\frac{1}{q}}.$$

Using the modified (h,m,s)-convexity of $|\psi'|^q$, we get

$$\int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_1^\rho+\left(1-\frac{\tau^\rho}{\ell+1}\right)\nu_2^\rho\right)\right|^q d\tau \tag{19}$$
$$\leq \int_0^1 \tau^{\rho\alpha}\left[h^s\left(\frac{\tau^\rho}{\ell+1}\right)|\psi'(\nu_1^\rho)|^q+m\left(1-h\left(1-\frac{\tau^\rho}{\ell+1}\right)\right)^s\left|\psi'\left(\frac{\nu_2^\rho}{m}\right)\right|^q\right]d\tau$$
$$=|\psi'(\nu_1^\rho)|^q\int_0^1 \tau^{\rho\alpha}h^s\left(\frac{\tau^\rho}{\ell+1}\right)d\tau+m\left|\psi'\left(\frac{\nu_2^\rho}{m}\right)\right|^q\int_0^1 \tau^{\rho\alpha}\left(1-h\left(1-\frac{\tau^\rho}{\ell+1}\right)\right)^s d\tau.$$

Similarly

$$\int_0^1 \tau^{\rho\alpha}\left|\psi'\left(\frac{\tau^\rho}{\ell+1}\nu_2^\rho+\left(1-\frac{\tau}{\ell+1}\right)\nu_1^\rho\right)\right|^q d\tau \tag{20}$$
$$\leq |\psi'(\nu_2^\rho)|^q\int_0^1 \tau^{\rho\alpha}h^s\left(1-\frac{\tau^\rho}{\ell+1}\right)d\tau+m\left|\psi'\left(\frac{\nu_1^\rho}{m}\right)\right|^q\int_0^1 \tau^{\rho\alpha}\left(1-h\left(\frac{\tau^\rho}{\ell+1}\right)\right)^s d\tau.$$

If we put (19) and (20), in (17) and in (18), it allows us to obtain the inequality (16). In this way the proof is completed. □

Remark 2.15. The Theorem 2.9 of [28] can be obtained from Theorem 2.5 putting $\ell = 0$ and considering $s-$convex functions. Additionally, the following results: Theorem 3.1 of [15] for convex functions and $\ell = 1$, Theorem 2.8 of [14] for h-convex functions and $\ell = 0$, Theorem 5 of [4], Theorem 9 of [36], Theorem 7 [51], Theorem 2.3 of [48] for (a,m)-convex functions and $\ell = 1$, and the Theorem 5 of [41] for convex functions, $\ell = 1$ and $\rho = 1$, can be obtained as particular cases from the previous.

Remark 2.16. For $\ell = 1$ and $\rho = 1$ this result complements Theorem 6 of [44] for convex functions con $\ell = 0$.

3. Conclusions

In this paper, using the generalized integral operator Katugampola, we obtained some integral inequalities that generalize a number of results available in the literature. It should be emphasized that the results obtained are valid for various classes of convex functions, for example, for $h-$convex functions, $m-$convex and $s-$convex functions in the second sense, defined on a closed interval of non–negative real numbers. These results can be extended to the case of $(h, m, s)-$convex modified functions of the first type.

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