Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 49(2), 2022, Pages 16–28, DOI: 10.52846/ami.v50i1.1600
ISSN: 1223-6934

$k-$fractional Ostrowski type inequalities via $(s,r)-$convex

Ali Hassan and Asif R. Khan

Abstract. We are introducing very first time a generalized class named it the class of $(s,r)-$convex in mixed kind, this class includes $s-$convex in 1^{st} and 2^{nd} kind, $P-$convex, quasi convex and the class of ordinary convex. Also, we would like to state the generalization of the classical Ostrowski inequality via $k-$fractional integrals, which is obtained for functions whose first derivative in absolute values is $(s,r)-$convex in mixed kind. Moreover we establish some Ostrowski type inequalities via $k-$fractional integrals and their particular cases for the class of functions whose absolute values at certain powers of derivatives are $(s,r)-$convex in mixed kind by using different techniques including Hölder's inequality and power mean inequality. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means would also be given.

2010 Mathematics Subject Classification. 26A33, 26A51, 26D15, 26D99, 47A30, 33B10.
Key words and phrases. Ostrowski inequality, convex, power mean inequality, Hölder's inequality.

1. Introduction

The main aim of this study is to reveal new generalized-Ostrowski-type inequalities via $(s,r)-$convex using $k-$fractional operator which generalizes Riemann-Liouville integral operator.

Definition 1.1. [2] The $\eta : I \subset (0,\infty) \to \mathbb{R}$ is convex (concave), if

$$\eta\left(\rho_t\rho_x + (1-\rho_t)\rho_y\right) \leq (\geq)\rho_t\eta(\rho_x) + (1-\rho_t)\eta(\rho_y),$$

$\forall \rho_x, \rho_y \in I, \rho_t \in [0,1]$.

Definition 1.2. [12] Let $s \in (0,1]$, the $\eta : I \subset (0,\infty) \to [0,\infty)$ is the $s-$convex (concave) in 1^{st} kind, if

$$\eta\left(\rho_t\rho_x + (1-\rho_t)\rho_y\right) \leq (\geq)\rho_t^s\eta(\rho_x) + (1-\rho_t^s)\eta(\rho_y),$$

$\forall \rho_x, \rho_y \in I, \rho_t \in [0,1]$.

Definition 1.3. [8] The $\eta : I \subset (0,\infty) \to [0,\infty)$ is quasi convex (concave), if

$$\eta\left(\rho_t\rho_x + (1-\rho_t)\rho_y\right) \leq (\geq)\max\{\eta(\rho_x), \eta(\rho_y)\}$$

$\forall \rho_x, \rho_y \in I, \rho_t \in [0,1]$.

Received January 5, 2022. April 22, 2022.

Definition 1.4. [12] Let $s \in (0,1]$, the $\eta : I \subset (0,\infty) \to [0,\infty)$ is the $s-$convex (concave) in 2^{nd} kind, if

$$\eta\left(\rho_t\rho_x + (1-\rho_t)\rho_y\right) \le (\ge) \rho_t^s \eta(\rho_x) + (1-\rho_t)^s \eta(\rho_y),$$

$\forall \rho_x, \rho_y \in I, \rho_t \in [0,1]$.

Definition 1.5. [2] The $\eta : I \subset (0,\infty) \to [0,\infty)$ is a $P-$convex (concave), if $\eta(\rho_x) \ge 0$ and $\forall \rho_x, \rho_y \in I$ and $\rho_t \in [0,1]$,

$$\eta\left(\rho_t\rho_x + (1-\rho_t)\rho_y\right) \le (\ge) \eta(\rho_x) + \eta(\rho_y).$$

Next we present the clasical ostrowski inequality.

Theorem 1.1. [10] *Let $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be differentiable function on (ρ_a, ρ_b), $|\varphi'(\rho_t)| \le M$, $\forall \rho_t \in (\rho_a, \rho_b)$. Then*

$$\left|\varphi(\rho_x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b} \varphi(\rho_t)d\rho_t\right| \le M(\rho_b-\rho_a)\left[\frac{1}{4} + \left(\frac{\rho_x - \frac{\rho_a+\rho_b}{2}}{\rho_b-\rho_a}\right)^2\right], \tag{1}$$

$\forall \rho_x \in (\rho_a, \rho_b)$.

Also, one can see the numerous variants and applications in [4]-[6].

Definition 1.6. [11] The Riemann-Liouville integrals $I^{\zeta}_{\rho_a^+}\varphi$ and $I^{\zeta}_{\rho_b^-}\varphi$ of $\varphi \in L_1([\rho_a, \rho_b])$ having order $\zeta > 0$ with $\rho_a \ge 0, \rho_a < \rho_b$ are defined by

$$I^{\zeta}_{\rho_a^+}\varphi(\rho_x) = \frac{1}{\Gamma(\zeta)}\int_{\rho_a}^{\rho_x} \frac{\varphi(\rho_t)}{(\rho_x-\rho_t)^{1-\zeta}}d\rho_t,\ \rho_x > \rho_a$$

and

$$I^{\zeta}_{\rho_b^-}\varphi(\rho_x) = \frac{1}{\Gamma(\zeta)}\int_{\rho_x}^{\rho_b} \frac{\varphi(\rho_t)}{(\rho_t-\rho_x)^{1-\zeta}}d\rho_t,\ \rho_x < \rho_b,$$

respectively. Here $\Gamma(\zeta) = \int_0^\infty e^{-u}u^{\zeta-1}du$ is the Gamma function and $I^{0}_{\rho_a^+}\varphi(\rho_x) = I^{0}_{\rho_b^-}\varphi(\rho_x) = \varphi(\rho_x)$. We also make use of Euler's beta function, which is for $x, y > 0$ defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Definition 1.7. [15] The $k-$fractional integrals ${}^kJ^{\zeta}_{\rho_a^+}\varphi$ and ${}^kJ^{\zeta}_{\rho_b^-}\varphi$ of $\varphi \in L_1([\rho_a, \rho_b])$ having order $\zeta > 0$ with $0 \le \rho_a < \rho_b, k > 0$ are defined by

$${}^kJ^{\zeta}_{\rho_a^+}\varphi(\rho_x) = \frac{1}{k\Gamma_k(\zeta)}\int_{\rho_a}^{\rho_x} \frac{\varphi(\rho_t)}{(\rho_x-\rho_t)^{1-\frac{\zeta}{k}}}d\rho_t,\ \rho_x > \rho_a$$

and

$${}^kJ^{\zeta}_{\rho_b^-}\varphi(\rho_x) = \frac{1}{k\Gamma_k(\zeta)}\int_{\rho_x}^{\rho_b} \frac{\varphi(\rho_t)}{(\rho_t-\rho_x)^{1-\frac{\zeta}{k}}}d\rho_t,\ \rho_x < \rho_b,$$

respectively. Here $\Gamma_k(\zeta) = \int_0^\infty e^{-\frac{u^k}{k}}u^{\zeta-1}du$ is the generalized Gamma function and ${}^1J^{0}_{\rho_a^+}\varphi(\rho_x) = {}^1J^{0}_{\rho_b^-}\varphi(\rho_x) = \varphi(\rho_x)$.

Throught this paper, we denote,

$$\begin{aligned}Y_{\varphi}(\zeta,k,\rho_a,\rho_x,\rho_b) &= \left[\frac{(\rho_x-\rho_a)^{\frac{\zeta}{k}}+(\rho_b-\rho_x)^{\frac{\zeta}{k}}}{(\rho_b-\rho_a)}\right]\varphi(\rho_x)\\ &\quad -\frac{k\Gamma_k(\zeta+1)}{\rho_b-\rho_a}\left[{}^kJ^{\zeta}_{\rho_x^-}\varphi(\rho_a)+{}^kJ^{\zeta}_{\rho_x^+}\varphi(\rho_b)\right],\end{aligned}$$

$$\begin{aligned}Z_{\varphi}(\zeta,\rho_x,\rho_a,\rho_b) = &\left[\frac{(\rho_x-\rho_a)^{\zeta}+(\rho_b-\rho_x)^{\zeta}}{\rho_b-\rho_a}\right]\varphi(\rho_x)\\ &-\frac{\Gamma(\zeta+1)}{\rho_b-\rho_a}\left[I^{\zeta}_{\rho_x^-}\varphi(\rho_a)+I^{\zeta}_{\rho_x^+}\varphi(\rho_b)\right].\end{aligned}$$

$${}^{\zeta}\kappa^{\rho_b}_{\rho_a}(\rho_x)=\left[\frac{(\rho_x-\rho_a)^{\zeta+1}+(\rho_b-\rho_x)^{\zeta+1}}{\rho_b-\rho_a}\right].$$

In order to prove our main results we need the following Lemma.

Lemma 1.2. [15] *Let $\varphi : I \subset (0,\infty) \to \mathbb{R}$ be an absolutely continuous, and $\rho_a,\rho_b \in I, \rho_a<\rho_b$. If $\varphi' \in L_1[\rho_a,\rho_b], \zeta,k>0$ then*

$$\begin{aligned}Y_{\varphi}(\zeta,k,\rho_a,\rho_x,\rho_b) &= \frac{(\rho_x-\rho_a)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}\int_0^1 \rho_t^{\frac{\zeta}{k}}\varphi'\left(\rho_t\rho_x+(1-\rho_t)\rho_a\right)d\rho_t\\ &\quad -\frac{(\rho_b-\rho_x)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}\int_0^1 \rho_t^{\frac{\zeta}{k}}\varphi'\left(\rho_t\rho_x+(1-\rho_t)\rho_b\right)d\rho_t.\end{aligned}$$

Theorem 1.3. [11] *Let $\varphi : I \to \mathbb{R}$ be differentiable mapping on I^0, with $\rho_a,\rho_b \in I, \rho_a<\rho_b$ $\varphi' \in L_1[\rho_a,\rho_b]$ and for $\zeta,k>1$, Montgomery identity for $k-$fractional integrals holds:*

$$\begin{aligned}\varphi(\rho_x) &= \frac{k\Gamma_k(\zeta)}{\rho_b-\rho_a}(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}\ {}^kJ^{\zeta}_{\rho_a}\varphi(\rho_b)\\ &\quad -{}^kJ^{\zeta-1}_{\rho_a}(P_1(\rho_x,\rho_b)\varphi(\rho_b))+{}^kJ^{\zeta}_{\rho_a}(P_1(\rho_x,\rho_b)\varphi'(\rho_b)),\end{aligned}\tag{2}$$

where $P_1(x,t)$ is the fractional Peano Kernel defined by:

$$P_1(x,t)=\begin{cases}\dfrac{\rho_t-\rho_a}{\rho_b-\rho_a}\dfrac{k\Gamma_k(\zeta)}{(\rho_b-\rho_x)^{\frac{\zeta}{k}-1}}, & \text{if } t\in[\rho_a,x],\\[2ex] \dfrac{\rho_t-\rho_b}{\rho_b-\rho_a}\dfrac{k\Gamma_k(\zeta)}{(\rho_b-\rho_x)^{\frac{\zeta}{k}-1}}, & \text{if } \rho_t\in(\rho_x,\rho_b].\end{cases}$$

Let $[\rho_a,\rho_b]\subseteq(0,+\infty)$, we may define special means as follows:

(a) The arithmetic mean

$$A=A(\rho_a,\rho_b):=\frac{\rho_a+\rho_b}{2};$$

(b) The geometric mean

$$G=G(\rho_a,\rho_b):=\sqrt{\rho_a\rho_b};$$

(c) The harmonic mean

$$H = H(\rho_a, \rho_b) := \frac{2}{\frac{1}{\rho_a} + \frac{1}{\rho_b}};$$

(d) The logarithmic mean

$$L = L(\rho_a, \rho_b) := \begin{cases} \rho_a & \text{if } \rho_a = \rho_b \\ \dfrac{\rho_b - \rho_a}{\ln \rho_b - \ln \rho_a}, & \text{if } \rho_a \neq \rho_b \end{cases};$$

(e) The identric mean

$$I = I(\rho_a, \rho_b) := \begin{cases} \rho_a & \text{if } \rho_a = \rho_b \\ \dfrac{1}{e}\left(\dfrac{{\rho_b}^{\rho_b}}{{\rho_a}^{\rho_a}}\right)^{\frac{1}{\rho_b - \rho_a}}, & \text{if } \rho_a \neq \rho_b. \end{cases};$$

(f) The $p-$logarithmic mean

$$L_p = L_p(\rho_a, \rho_b) := \begin{cases} \rho_a & \text{if } \rho_a = \rho_b \\ \left[\dfrac{\rho_b^{p+1} - {\rho_a}^{p+1}}{(p+1)(\rho_b - \rho_a)}\right]^{\frac{1}{p}}, & \text{if } \rho_a \neq \rho_b. \end{cases};$$

where $p \in \mathbb{R} \setminus \{0, -1\}$.

2. $k-$fractional Ostrowski type inequalties via $(s, r)-$convex

In this section, we are introducing very first time the concept of $(s, r)-$convex in mixed kind, this class contains many classes of convex from literature of convex analysis.

Definition 2.1. Let $(s, r) \in (0, 1]^2$, the $\eta : I \subset (0, \infty) \to [0, \infty)$ is the $(s, r)-$convex (concave) in mixed kind, if

$$\eta\left(\rho_t \rho_x + (1 - \rho_t)\rho_y\right) \leq (\geq) \rho_t^{rs} \eta(\rho_x) + (1 - \rho_t^r)^s \eta(\rho_y), \tag{3}$$

$\forall \rho_x, \rho_y \in I, \rho_t \in [0, 1]$.

Remark 2.1. In Definition 2.1, we can see the following:
(1) If $s = 1$ and $r \in [0, 1]$ in (3), we get $r-$convex in 1^{st} kind.
(2) If $r \to 0$, and $s = 1$, in (3), we get quasi convex.
(3) If $r = 1$ and $s \in [0, 1]$ in (3), we get $s-$convex in 2^{nd} kind.
(4) If $s \to 0$, and $r = 1$ in (3), we get $P-$convex.
(5) If $s = r = 1$ in (3), gives us ordinary convex.

Now, we will generalize the Ostrowski type inequalties via $(s, r)-$convex (concave) by using $k-$fractional integral operator.

Theorem 2.1. *Let $\varphi : [\rho_a, \rho_b] \subset (0, \infty) \to \mathbb{R}$ be an absolutely continuous, and $\varphi' \in L_1[\rho_a, \rho_b]$. If $|\varphi'|$ is $(s, r)-$convex, $|\varphi'(\rho_x)| \leq M, \forall \rho_x \in [\rho_a, \rho_b]$, and $\zeta, k > 0$*

then

$$|Y_{\varphi}(\zeta,k,\rho_a,\rho_x,\rho_b)| \leq M\left\{\begin{array}{l}\int_0^1 \rho_t^{\frac{\zeta}{k}}\rho_t^{rs}d\rho_t \\ +\int_0^1 \rho_t^{\frac{\zeta}{k}}(1-\rho_t^r)^s d\rho_t\end{array}\right\} \times\left[\frac{(\rho_x-\rho_a)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}+\frac{(\rho_b-\rho_x)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}\right]. \tag{4}$$

Proof. By using the Lemma 1.2,

$$\begin{aligned}|Y_{\varphi}(\zeta,k,\rho_a,\rho_x,\rho_b)| \leq & \frac{(\rho_x-\rho_a)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}\int_0^1 \rho_t^{\frac{\zeta}{k}}\left|\varphi'\left(\rho_t\rho_x+(1-\rho_t)\rho_a\right)\right|d\rho_t \\ & +\frac{(\rho_b-\rho_x)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}\int_0^1 \rho_t^{\frac{\zeta}{k}}\left|\varphi'\left(\rho_t\rho_x+(1-\rho_t)\rho_b\right)\right|d\rho_t \end{aligned}\tag{5}$$

Since $|\varphi'|$ is $(s,r)-$convex and by using $|\varphi'(\rho_x)| \leq M$, we get

$$\begin{aligned}|Y_{\varphi}(\zeta,k,\rho_a,\rho_x,\rho_b)| \leq & \frac{(\rho_x-\rho_a)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}\int_0^1 \rho_t^{\frac{\zeta}{k}}\left[\rho_t^{rs}|\varphi'(\rho_x)|+(1-\rho_t^r)^s|\varphi'(\rho_a)|\right]d\rho_t \\ & +\frac{(\rho_b-\rho_x)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}\int_0^1 \rho_t^{\frac{\zeta}{k}}\left[\rho_t^{rs}|\varphi'(\rho_x)|+(1-\rho_t^r)^s|\varphi'(\rho_b)|\right]d\rho_t, \end{aligned}\tag{6}$$

$$\begin{aligned}|Y_{\varphi}(\zeta,k,\rho_a,\rho_x,\rho_b)| \leq & \frac{(\rho_x-\rho_a)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}\left\{\begin{array}{l}|\varphi'(\rho_x)|\int_0^1 \rho_t^{\frac{\zeta}{k}}\rho_t^{rs}d\rho_t \\ +|\varphi'(\rho_a)|\int_0^1 \rho_t^{\frac{\zeta}{k}}(1-\rho_t^r)^s d\rho_t\end{array}\right\} \\ & +\frac{(\rho_b-\rho_x)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}\left\{\begin{array}{l}|\varphi'(\rho_x)|\int_0^1 \rho_t^{\frac{\zeta}{k}}\rho_t^{rs}d\rho_t \\ +|\varphi'(\rho_b)|\int_0^1 \rho_t^{\frac{\zeta}{k}}(1-\rho_t^r)^s d\rho_t,\end{array}\right\}, \end{aligned}\tag{7}$$

$$\begin{aligned}&|Y_{\varphi}(\zeta,k,\rho_a,\rho_x,\rho_b)| \\ &\leq M\left\{\begin{array}{l}\int_0^1 \rho_t^{\frac{\zeta}{k}}\rho_t^{rs}d\rho_t \\ +\int_0^1 \rho_t^{\frac{\zeta}{k}}(1-\rho_t^r)^s d\rho_t\end{array}\right\}\times\left[\frac{(\rho_x-\rho_a)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}+\frac{(\rho_b-\rho_x)^{\frac{\zeta}{k}+1}}{\rho_b-\rho_a}\right]. \end{aligned}\tag{8}$$

□

Remark 2.2. In Theorem 2.1, one can also capture the inequalties for $s-$convex in 1^{st} and 2^{nd} kind, $P-$convex and convex via $k-$fractional integrals by using Remark 2.1.

Corollary 2.2. *In Theorem 2.1, one can see for $k=1$ the following.*
(1) The Ostrowski inequality for $(s,r)-$convex in mixed kind via fractional integrals:

$$|Z_{\varphi}(\zeta,\rho_x,\rho_a,\rho_b)| \leq M\left(\frac{1}{\zeta+rs+1}+\frac{B\left(\frac{\zeta+1}{r},s+1\right)}{r}\right)\ {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

(2) If $s = 1$ and $r \in (0,1]$ in inequality (4), then the Ostrowski inequality for $r-$convex in 1^{st} kind via fractional integrals:

$$|Z_{\varphi}(\zeta, \rho_x, \rho_a, \rho_b)| \leq M \left(\frac{1}{\zeta + r + 1} + \frac{B\left(\frac{\zeta+1}{r}, 2\right)}{r} \right) \ {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

(3) If $r = 1$ and $s \in (0,1]$ in inequality (4), then the Ostrowski inequality for $s-$convex in 2^{nd} kind via fractional integrals:

$$|Z_{\varphi}(\zeta, \rho_x, \rho_a, \rho_b)| \leq M \left(\frac{1}{\zeta + s + 1} + B(\zeta + 1, s + 1) \right) \ {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

(4) If $\zeta = r = 1$ and $s \in (0,1]$ in inequality (4), then the inequality (2.1) *of Theorem* 2 *in* [1].

(5) If $r = 1$ and $s \in (0,1]$ in inequality (4), then the inequality (2.6) *of Theorem* 7 *in* [12].

(6) If $s \to 0$ and $r = 1$, in inequality (4), then the Ostrowski inequality for $P-$convex via fractional integrals:

$$|Z_{\varphi}(\zeta, \rho_x, \rho_a, \rho_b)| \leq M \left(\frac{1}{\zeta + 1} + B(\zeta + 1, 1) \right) \ {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

(7) If $r = s = 1$, in inequality (4), then the Ostrowski inequality for convex via fractional integrals:

$$|Z_{\varphi}(\zeta, \rho_x, \rho_a, \rho_b)| \leq M \left(\frac{1}{\zeta + 2} + B(\zeta + 1, 2) \right) \ {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

(8) If $\zeta = r = s = 1$, in inequality (4), then the Ostrowski inequality (1.1) for convex.

Theorem 2.3. *Let $\varphi : [\rho_a, \rho_b] \subset (0, \infty) \to \mathbb{R}$ be an absolutely continuous, and $\varphi' \in L[\rho_a, \rho_b]$. If $|\varphi'|^q$ is $(s,r)-$convex for $q > 1$ and $|\varphi'(\rho_x)| \leq M, \forall \rho_x \in [\rho_a, \rho_b]$, and $\zeta, k > 0$ then*

$$\begin{aligned} &|Y_{\varphi}(\zeta, k, \rho_a, \rho_x, \rho_b)| \\ &\leq \frac{M}{L^{\frac{1}{q}-1}} \left[\frac{(\rho_x - \rho_a)^{\frac{\zeta}{k}+1}}{\rho_b - \rho_a} + \frac{(\rho_b - \rho_x)^{\frac{\zeta}{k}+1}}{(\rho_b - \rho_a)} \right] \times \left\{ \begin{array}{l} \int_0^1 \rho_t^{\frac{\zeta}{k}} \rho_t^{rs} d\rho_t \\ + \int_0^1 \rho_t^{\frac{\zeta}{k}} (1 - \rho_t^r)^s d\rho_t \end{array} \right\}^{\frac{1}{q}} \end{aligned} \tag{9}$$

where

$$L = \int_0^1 \rho_t^{\frac{\zeta}{k}} d\rho_t.$$

Proof. By using the Lemma 1.2, and Power mean inequality [13],

$$\begin{aligned} &|Y_{\varphi}(\zeta, k, \rho_a, \rho_x, \rho_b)| \\ &\leq \frac{(\rho_x - \rho_a)^{\frac{\zeta}{k}+1}}{\rho_b - \rho_a} \left(\int_0^1 \rho_t^{\frac{\zeta}{k}} d\rho_t \right)^{1-\frac{1}{q}} \times \left(\int_0^1 \rho_t^{\frac{\zeta}{k}} |\varphi'(\rho_t \rho_x + (1 - \rho_t)\rho_a)|^q d\rho_t \right)^{\frac{1}{q}} \\ &\quad + \frac{(\rho_b - \rho_x)^{\frac{\zeta}{k}+1}}{\rho_b - \rho_a} \left(\int_0^1 \rho_t^{\frac{\zeta}{k}} d\rho_t \right)^{1-\frac{1}{q}} \times \left(\int_0^1 \rho_t^{\frac{\zeta}{k}} |\varphi'(\rho_t \rho_x + (1 - \rho_t)\rho_b)|^q d\rho_t \right)^{\frac{1}{q}}. \end{aligned} \tag{10}$$

Since $|\varphi'|^q$ is $(s,r)-$convex and $|\varphi'(\rho_x)| \leq M$,

$$\begin{aligned} |Y_\varphi(\zeta,k,\rho_a,\rho_x,\rho_b)| &\leq \frac{M(\rho_x-\rho_a)^{\frac{\zeta}{k}+1}}{L^{\frac{1}{q}-1}(\rho_b-\rho_a)}\left\{\begin{array}{l} \int_0^1 \rho_t^{\frac{\zeta}{k}}\rho_t^{rs}d\rho_t \\ +\int_0^1 \rho_t^{\frac{\zeta}{k}}(1-\rho_t^r)^s d\rho_t \end{array}\right\}^{\frac{1}{q}} \\ &+\frac{M(\rho_b-\rho_x)^{\frac{\zeta}{k}+1}}{L^{\frac{1}{q}-1}(\rho_b-\rho_a)}\left\{\begin{array}{l} \int_0^1 \rho_t^{\frac{\zeta}{k}}\rho_t^{rs}d\rho_t \\ +\int_0^1 \rho_t^{\frac{\zeta}{k}}(1-\rho_t^r)^s d\rho_t \end{array}\right\}^{\frac{1}{q}}. \end{aligned}$$

□

Remark 2.3. In Theorem 2.3, one can also capture the inequalties for $s-$convex in 1^{st} and 2^{nd} kind, $P-$convex and convex via $k-$fractional integrals by using Remark 2.1.

Corollary 2.4. *In Theorem 2.3, one can see for $k=1$ the following.*

(1) The Ostrowski inequality for $(s,r)-$convex in mixed kind via fractional integrals:

$$|Z_\varphi(\zeta,\rho_x,\rho_a,\rho_b)| \leq \frac{M}{(\zeta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\zeta+rs+1}+\frac{B\left(\frac{\zeta+1}{r},s+1\right)}{r}\right)^{\frac{1}{q}} {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

(2) If $s=1$ and $r\in(0,1]$ in inequality (9), then the Ostrowski inequality for $r-$convex in 1^{st} kind via fractional integrals:

$$|Z_\varphi(\zeta,\rho_x,\rho_a,\rho_b)| \leq \frac{M}{(\zeta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\zeta+s+1}+\frac{B\left(\frac{\zeta+1}{s},2\right)}{s}\right)^{\frac{1}{q}} {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

(3) If $r=1$ and $s\in(0,1]$ in inequality (9), then the Ostrowski inequality for $s-$convex in 2^{nd} kind via fractional integrals:

$$|Z_\varphi(\zeta,\rho_x,\rho_a,\rho_b)| \leq \frac{M}{(\zeta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\zeta+s+1}+B(\zeta+1,s+1)\right)^{\frac{1}{q}} {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

(4) If $\zeta=r=1$, and $s\in(0,1]$ in inequality (9), then the inequality (2.3) *of Theorem* 4 *in* [1].

(5) If $r=1$ and $s\in(0,1]$ in inequality (9), then the inequality (2.8) *of Theorem* 9 *in* [12].

(6) If $r=1$ and $s\to 0$ in inequality (9), then the Ostrowski inequality for $P-$convex via fractional integrals:

$$|Z_\varphi(\zeta,\rho_x,\rho_a,\rho_b)| \leq \frac{M}{(\zeta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\zeta+1}+B(\zeta+1,1)\right)^{\frac{1}{q}} {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

(7) If $r=s=1$, in inequality (9), then the Ostrowski inequality for convex via fractional integrals:

$$|Z_\varphi(\zeta,\rho_x,\rho_a,\rho_b)| \leq \frac{M}{(\zeta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\zeta+2}+B(\zeta+1,2)\right)^{\frac{1}{q}} {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

Theorem 2.5. *Let $\varphi : [\rho_a, \rho_b] \subset (0, \infty) \to \mathbb{R}$ be an absolutely continuous, $\varphi' \in L[\rho_a, \rho_b]$. If $|\varphi'|^q$ is $(s, r)-$convex, $|\varphi'(\rho_x)| \le M, \forall \rho_x \in [\rho_a, \rho_b]$, $\zeta, k > 0$, and $p, z > 1$ with $\frac{1}{z} + \frac{1}{q} = 1$, then*

$$\begin{aligned}|Y_\varphi(\zeta, k, \rho_a, \rho_x, \rho_b)| &\le \frac{MK^{\frac{1}{z}}}{\rho_b - \rho_a}\left(\frac{1}{rs+1} + \frac{1}{r}B\left(\frac{1}{r}, s+1\right)\right)^{\frac{1}{q}} \\ &\quad \times \left[(\rho_x - \rho_a)^{\frac{\zeta}{k}+1} + (\rho_b - \rho_x)^{\frac{\zeta}{k}+1}\right]. \qquad (11)\end{aligned}$$

where

$$K = \int_0^1 \rho_t^{\frac{\zeta z}{k}} d\rho_t.$$

Proof. By using the Lemma 1.2, and Hölder's inequality [14],

$$\begin{aligned}&|Y_\varphi(\zeta, k, \rho_a, \rho_x, \rho_b)| \\ &\le \frac{(\rho_x - \rho_a)^{\frac{\zeta}{k}+1}}{\rho_b - \rho_a}\left(\int_0^1 \rho_t^{\frac{\zeta z}{k}} d\rho_t\right)^{\frac{1}{z}} \times \left(\int_0^1 |\varphi'(\rho_t\rho_x + (1-\rho_t)\rho_a)|^q d\rho_t\right)^{\frac{1}{q}} \\ &\quad + \frac{(\rho_b - \rho_x)^{\frac{\zeta}{k}+1}}{\rho_b - \rho_a}\left(\int_0^1 \rho_t^{\frac{\zeta z}{k}} d\rho_t\right)^{\frac{1}{z}} \times \left(\int_0^1 |\varphi'(\rho_t\rho_x + (1-\rho_t)\rho_b)|^q d\rho_t\right)^{\frac{1}{q}}. \qquad (12)\end{aligned}$$

Since $|\varphi'|^q$ is $(s, r)-$convex and $|\varphi'(\rho_x)| \le M$,

$$\begin{aligned}|Y_\varphi(\zeta, k, \rho_a, \rho_x, \rho_b)| &\le \frac{K^{\frac{1}{z}}(\rho_x - \rho_a)^{\frac{\zeta}{k}+1}}{\rho_b - \rho_a}\left(\frac{M^q}{rs+1} + \frac{M^q}{r}B\left(\frac{1}{r}, s+1\right)\right)^{\frac{1}{q}} \\ &\quad + \frac{K^{\frac{1}{z}}(\rho_b - \rho_x)^{\frac{\zeta}{k}+1}}{\rho_b - \rho_a}\left(\frac{M^q}{rs+1} + \frac{M^q}{r}B\left(\frac{1}{r}, s+1\right)\right)^{\frac{1}{q}}.\end{aligned}$$

□

Remark 2.4. In Theorem 2.5, one can also capture the inequalties for $s-$convex in 1^{st} and 2^{nd} kind, $P-$convex and convex via $k-$fractional integrals by using Remark 2.1.

Corollary 2.6. *In Theorem 2.5, one can see for $k = 1$ the following.*

(1) The Ostrowski inequality for $(s, r)-$convex in mixed kind via fractional integrals:

$$|Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \le \frac{M}{(\zeta z + 1)^{\frac{1}{z}}}\left(\frac{1}{rs+1} + \frac{B\left(\frac{1}{r}, s+1\right)}{r}\right)^{\frac{1}{q}} {}^{\zeta}K_{\rho_a}^{\rho_b}(\rho_x).$$

(2) If $s = 1$ and $r \in (0, 1]$ in inequality (11), then the Ostrowski inequality for $r-$convex in 1^{st} kind via fractional integrals:

$$|Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \le \frac{M}{(\zeta z + 1)^{\frac{1}{z}}}\left(\frac{1}{s+1} + \frac{B\left(\frac{1}{s}, 2\right)}{s}\right)^{\frac{1}{q}} {}^{\zeta}K_{\rho_a}^{\rho_b}(\rho_x).$$

(3) If $r = 1$ and $s \in (0, 1]$ in inequality (11), then the Ostrowski inequality for $s-$convex in 2^{nd} kind via fractional integrals:

$$|Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \le \frac{M}{(\zeta z + 1)^{\frac{1}{z}}}\left(\frac{1}{s+1} + B(1, s+1)\right)^{\frac{1}{q}} {}^{\zeta}K_{\rho_a}^{\rho_b}(\rho_x).$$

(4) *If $\zeta = r = 1$ and $s \in (0,1]$ in inequality (11), then the inequality (2.2) of Theorem 3 in [1].*
(5) *If $r = 1$ and $s \in (0,1]$ in inequality (11), then the inequality (2.7) of Theorem 8 in [12].*
(6) *If $r = 1$, and $s \to 0$ in inequality (11), then the Ostrowski inequality for $P-$convex via fractional integrals:*

$$|Z_{\varphi}(\zeta, \rho_x, \rho_a, \rho_b)| \leq \frac{(2)^{\frac{1}{q}}\, M}{(\zeta z + 1)^{\frac{1}{z}}} \ {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

(7) *If $r = s = 1$, in inequality (11), then the Ostrowski inequality for convex via fractional integrals:*

$$|Z_{\varphi}(\zeta, \rho_x, \rho_a, \rho_b)| \leq \frac{M}{(\zeta z + 1)^{\frac{1}{z}}} \ {}^{\zeta}\kappa_{\rho_a}^{\rho_b}(\rho_x).$$

Theorem 2.7. *Let $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be differentiable on (ρ_a, ρ_b), $\varphi' : [\rho_a, \rho_b] \to \mathbb{R}$ be integrable on $[\rho_a, \rho_b]$ and $\eta : I \subset \mathbb{R} \to \mathbb{R}$, be a $(s,r)-$convex function in mixed sense, then we have the inequalities*

$$\begin{aligned}
&\eta\left[\varphi(\rho_x) - \frac{k\Gamma_k(\zeta)}{\rho_b - \rho_a}(\rho_b - \rho_x)^{1-\frac{\zeta}{k}} \ {}^{k}J_{\rho_a}^{\zeta}\varphi(\rho_b) + \ {}^{k}J_{\rho_a}^{\zeta-1}(P_1(\rho_x, \rho_b)\varphi(\rho_b))\right] \\
&\leq \frac{(\rho_b - \rho_x)^{1-\frac{\zeta}{k}}}{(\rho_b - \rho_a)^{rs}}\left[(\rho_x - \rho_a)^{rs-1}\int_{\rho_a}^{\rho_x}\eta\left[\frac{(\rho_t - \rho_a)\varphi'(\rho_t)}{(\rho_b - \rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right. \\
&\left. + \frac{((\rho_b - \rho_a)^r - (\rho_x - \rho_a)^r)^s}{\rho_b - \rho_x}\int_{\rho_x}^{\rho_b}\eta\left[\frac{(\rho_t - \rho_b)\varphi'(\rho_t)}{(\rho_b - \rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right], \qquad (13)
\end{aligned}$$

$\forall \rho_x \in [\rho_a, \rho_b]$.

Proof. Utilizing the Theorem 1.3, we get

$$\begin{aligned}
&\varphi(\rho_x) - \frac{k\Gamma_k(\zeta)}{\rho_b - \rho_a}(\rho_b - \rho_x)^{1-\frac{\zeta}{k}} \ {}^{k}J_{\rho_a}^{\zeta}\varphi(\rho_b) + \ {}^{k}J_{\rho_a}^{\zeta-1}(P_1(\rho_x, \rho_b)\varphi(\rho_b)) \\
&= \ {}^{k}J_{\rho_a}^{\zeta}(P_1(\rho_x, \rho_b)\varphi'(\rho_b)) = \frac{1}{k\Gamma_k(\zeta)}\int_{\rho_a}^{\rho_b}P_1(x,t)\frac{\varphi'(\rho_t)}{(\rho_b - \rho_t)^{1-\frac{\zeta}{k}}}d\rho_t \\
&= \left(\frac{x - \rho_a}{\rho_b - \rho_a}\right)\left[\frac{(\rho_b - \rho_x)^{1-\frac{\zeta}{k}}}{x - \rho_a}\int_{\rho_a}^{\rho_x}\frac{\{\rho_t - \rho_a\}\varphi'(\rho_t)}{(\rho_b - \rho_t)^{1-\frac{\zeta}{k}}}d\rho_t\right] \\
&\quad + \left(1 - \left(\frac{x - \rho_a}{\rho_b - \rho_a}\right)\right)\left[\frac{(\rho_b - \rho_x)^{1-\frac{\zeta}{k}}}{\rho_b - \rho_x}\int_{\rho_x}^{\rho_b}\frac{\{\rho_t - \rho_b\}\varphi'(\rho_t)}{(\rho_b - \rho_t)^{1-\frac{\zeta}{k}}}d\rho_t\right],
\end{aligned}$$

$\forall \rho_x \in [\rho_a, \rho_b]$. Next by using the $(s,r)-$convex function in mixed sense of $\eta : I \subset [0, \infty) \to \mathbb{R}$, we get

$$\eta\left[\varphi(\rho_x)-\frac{k\Gamma_k(\zeta)}{\rho_b-\rho_a}(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}\ {}^kJ^{\zeta}_{\rho_a}\varphi(\rho_b)+\ {}^kJ^{\zeta-1}_{\rho_a}(P_1(\rho_x,\rho_b)\varphi(\rho_b))\right]$$
$$\leq\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{rs}\eta\left[\frac{(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}}{x-\rho_a}\int_{\rho_a}^{\rho_x}\frac{\{\rho_t-\rho_a\}\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}d\rho_t\right]$$
$$+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{r}\right)^{s}\eta\left[\frac{(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}}{\rho_b-\rho_x}\int_{\rho_x}^{\rho_b}\frac{\{\rho_t-\rho_b\}\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}d\rho_t\right],$$

$\forall\rho_x\in[\rho_a,\rho_b]$. Applying Jensen's integral inequality [5], we get the Inequality (13). $\square$

Corollary 2.8. *In Theorem 2.7, one can see the following.*

(1) If $s=1$ and $r\in(0,1]$ in (13), then Ostrowski inequality for $r-$convex functions in 1^{st} kind:

$$\eta\left[\varphi(\rho_x)-\frac{k\Gamma_k(\zeta)}{\rho_b-\rho_a}(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}\ {}^kJ^{\zeta}_{\rho_a}\varphi(\rho_b)+\ {}^kJ^{\zeta-1}_{\rho_a}(P_1(\rho_x,\rho_b)\varphi(\rho_b))\right]$$
$$\leq\frac{(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}}{(\rho_b-\rho_a)^{r}}\left[(\rho_x-\rho_a)^{r-1}\int_{\rho_a}^{\rho_x}\eta\left[\frac{(\rho_t-\rho_a)\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right.$$
$$\left.+\frac{(\rho_b-\rho_a)^{r}-(\rho_x-\rho_a)^{r}}{(\rho_b-\rho_x)}\int_{\rho_x}^{\rho_b}\eta\left[\frac{(\rho_t-\rho_b)\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right].$$

(2) If $s=1$ and $r\to0$ in (13), we get quasi$-$convex function.

$$\eta\left[\varphi(\rho_x)-\frac{k\Gamma_k(\zeta)}{\rho_b-\rho_a}(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}\ {}^kJ^{\zeta}_{\rho_a}\varphi(\rho_b)+\ {}^kJ^{\zeta-1}_{\rho_a}(P_1(\rho_x,\rho_b)\varphi(\rho_b))\right]$$
$$\leq\frac{(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}}{(\rho_x-\rho_a)}\left[\int_{\rho_a}^{\rho_x}\eta\left[\frac{(\rho_t-\rho_a)\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right].$$

(3) If $r=1$ and $s\in[0,1)$ (13), then fractional Ostrowski type inequality for $s-$convex functions in 2^{nd} kind:

$$\eta\left[\varphi(\rho_x)-\frac{k\Gamma_k(\zeta)}{\rho_b-\rho_a}(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}\ {}^kJ^{\zeta}_{\rho_a}\varphi(\rho_b)+\ {}^kJ^{\zeta-1}_{\rho_a}(P_1(\rho_x,\rho_b)\varphi(\rho_b))\right]$$
$$\leq\frac{(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}}{(\rho_b-\rho_a)^{s}}\left[(\rho_x-\rho_a)^{s-1}\int_{\rho_a}^{\rho_x}\eta\left[\frac{(\rho_t-\rho_a)\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right.$$
$$\left.+(\rho_b-\rho_x)^{s-1}\int_{\rho_x}^{\rho_b}\eta\left[\frac{(\rho_t-\rho_b)\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right].$$

(4) *If $r=1$ and $s\to 0$ in (13), then fractional Ostrowski type inequality for $P-$convex functions:*

$$\eta\left[\varphi(\rho_x)-\frac{k\Gamma_k(\zeta)}{\rho_b-\rho_a}(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}\ {}^kJ^{\zeta}_{\rho_a}\varphi(\rho_b)+\ {}^kJ^{\zeta-1}_{\rho_a}(P_1(\rho_x,\rho_b)\varphi(\rho_b))\right]$$
$$\leq(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{\rho_x}\eta\left[\frac{(\rho_t-\rho_a)\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right.$$
$$\left.+\frac{1}{\rho_b-\rho_x}\int_{\rho_x}^{\rho_b}\eta\left[\frac{(\rho_t-\rho_b)\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right].$$

(5) *If $s=r=1$ in (13), then fractional Ostrowski type inequality for convex functions:*

$$\eta\left[\varphi(\rho_x)-\frac{k\Gamma_k(\zeta)}{\rho_b-\rho_a}(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}\ {}^kJ^{\zeta}_{\rho_a}\varphi(\rho_b)+\ {}^kJ^{\zeta-1}_{\rho_a}(P_1(\rho_x,\rho_b)\varphi(\rho_b))\right]$$
$$\leq\frac{(\rho_b-\rho_x)^{1-\frac{\zeta}{k}}}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\rho_x}\eta\left[\frac{(\rho_t-\rho_a)\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t+\int_{\rho_x}^{\rho_b}\eta\left[\frac{(\rho_t-\rho_b)\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right].$$

3. Applications to Sspecial means

If we replace φ by $-\varphi$ and $x=\frac{\rho_a+\rho_b}{2}$ in Theorem 2.7, we get

Theorem 3.1. *Let $\varphi:[\rho_a,\rho_b]\to\mathbb{R}$ be differentiable on (ρ_a,ρ_b), $\varphi':[\rho_a,\rho_b]\to\mathbb{R}$ be integrable on $[\rho_a,\rho_b]$ and $\eta:I\subset\mathbb{R}\to\mathbb{R}$, be a $(s,r)-$convex function in mixed sense, then*

$$\eta\left[\frac{k\Gamma_k(\zeta)\left(\frac{\rho_b-\rho_a}{2}\right)^{1-\frac{\zeta}{k}}}{\rho_b-\rho_a}\ {}^kJ^{\zeta}_{\rho_a}\varphi(\rho_b)-\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-\ {}^kJ^{\zeta-1}_{\rho_a}\left(P_1\left(\frac{\rho_a+\rho_b}{2},b\right)\varphi(\rho_b)\right)\right]$$
$$\leq\frac{2^{\zeta-1}}{(\rho_b-\rho_a)^{\zeta}}\left[\frac{1}{2^{sr-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_a}\eta\left[\frac{(\rho_t-\rho_a)\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right.$$
$$\left.+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\rho_b}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{(\rho_t-\rho_b)\varphi'(\rho_t)}{(\rho_b-\rho_t)^{1-\frac{\zeta}{k}}}\right]d\rho_t\right]. \tag{14}$$

Remark 3.1. In Theorem 3.1, if we put $\zeta=k=1$ in (14). we get

$$\eta\left(\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(\rho_t)d\rho_t-\varphi\left(\frac{\rho_a+\rho_b}{2}\right)\right)$$
$$\leq\frac{1}{\rho_b-\rho_a}\left[\frac{1}{2^{sr-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta[(\rho_a-\rho_t)\varphi'(\rho_t)]d\rho_t\right.$$
$$\left.+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta[(\rho_b-\rho_t)\varphi'(\rho_t)]d\rho_t\right].$$

Remark 3.2. Assume that $\eta:I\subset[0,\infty)\to\mathbb{R}$ be an $(s,r)-$convex function in mixed kind:

(1) If $\zeta=k=1, \varphi(\rho_t)=\frac{1}{\rho_t}$ in inequality (14) where $\rho_t\in[\rho_a,\rho_b]\subset(0,\infty)$, then we have

$$\begin{aligned}&(\rho_b-\rho_a)\eta\left[\frac{A(\rho_a,\rho_b)-L(\rho_a,\rho_b)}{A(\rho_a,\rho_b)L(\rho_a,\rho_b)}\right]\\&\leq\frac{1}{2^{sr-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{\rho_t-\rho_a}{\rho_t^2}\right]d\rho_t+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[\frac{\rho_t-\rho_b}{\rho_t^2}\right]d\rho_t.\end{aligned}$$

(2) If $\zeta=k=1, \varphi(\rho_t)=-\ln\rho_t$ in inequality (14), where $\rho_t\in[\rho_a,\rho_b]\subset(0,\infty)$, then we have

$$\begin{aligned}&(\rho_b-\rho_a)\eta\left[\ln\left(\frac{A(\rho_a,\rho_b)}{I(\rho_a,\rho_b)}\right)\right]\\&\leq\frac{1}{2^{sr-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{\rho_t-\rho_a}{\rho_t}\right]d\rho_t+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[\frac{\rho_t-\rho_b}{\rho_t}\right]d\rho_t.\end{aligned}$$

(3) If $\zeta=k=1, \varphi(\rho_t)=\rho_t^p, p\in\mathbb{R}\setminus\{0,-1\}$ in inequality (14), where $\rho_t\in[\rho_a,\rho_b]\subset(0,\infty)$, then we have

$$\begin{aligned}&(\rho_b-\rho_a)\eta\left[L_p^p(\rho_a,\rho_b)-A^p(\rho_a,\rho_b)\right]\\&\leq\frac{1}{2^{sr-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{p\left(\rho_a-\rho_t\right)}{\rho_t^{1-p}}\right]d\rho_t+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[\frac{p\left(\rho_b-\rho_t\right)}{\rho_t^{1-p}}\right]d\rho_t.\end{aligned}$$

Remark 3.3. In Theorem 2.3, one can see for $\zeta=k=1$ the following.

(1) Let $\rho_x=\frac{\rho_a+\rho_b}{2}, 0<\rho_a<\rho_b$, $q\geq1$ and $\varphi:\mathbb{R}\to\mathbb{R}^+$, $\varphi(\rho_t)=\rho_t^n$ in (9). Then

$$\left|A^n\left(\rho_a,\rho_b\right)-L_n^n\left(\rho_a,\rho_b\right)\right|\leq\frac{M\left(\rho_b-\rho_a\right)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)^{\frac{1}{q}}.$$

(2) Let $\rho_x=\frac{\rho_a+\rho_b}{2}, 0<\rho_a<\rho_b$, $q\geq1$ and $\varphi:(0,1]\to\mathbb{R}$, $\varphi(\rho_t)=-\ln\rho_t$ in (9). Then

$$\left|\ln\left(\frac{A\left(\rho_a,\rho_b\right)}{I\left(\rho_a,\rho_b\right)}\right)\right|\leq\frac{M\left(\rho_b-\rho_a\right)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)^{\frac{1}{q}}.$$

Remark 3.4. In Theorem 2.5, one can see for $\zeta=k=1$ the following.

(1) Let $\rho_x=\frac{\rho_a+\rho_b}{2}, 0<\rho_a<\rho_b$, $p^{-1}+q^{-1}=1$ and $\varphi:\mathbb{R}\to\mathbb{R}^+$, $\varphi(\rho_t)=\rho_t^n$ in (11). Then

$$\left|A^n\left(\rho_a,\rho_b\right)-L_n^n\left(\rho_a,\rho_b\right)\right|\leq\frac{M\left(\rho_b-\rho_a\right)}{2\left(z+1\right)^{\frac{1}{z}}}\left(\frac{1}{rs+1}+\frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}.$$

(2) Let $\rho_x=\frac{\rho_a+\rho_b}{2}, 0<\rho_a<\rho_b$, $p^{-1}+q^{-1}=1$ and $\varphi:(0,1]\to\mathbb{R}$, $\varphi(\rho_t)=-\ln\rho_t$ in (11). Then

$$\left|\ln\left(\frac{A\left(\rho_a,\rho_b\right)}{I\left(\rho_a,\rho_b\right)}\right)\right|\leq\frac{M\left(\rho_b-\rho_a\right)}{2\left(z+1\right)^{\frac{1}{z}}}\left(\frac{1}{rs+1}+\frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}.$$

4. Conclusion

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $(s,r)-$convex in mixed kind, this class of functions contains many important classes including class of $s-$convex in 1^{st} and 2^{nd} kind [3], $P-$convex, quasi convex and the class of convex. In this study, theorems are put forward to obtain new upper bounds by $k-$fractional operator for Ostrowski type inequalities. We have stated our first main result in section 2, the generalization of Ostrowski inequality [10] via $k-$fractional integral and others results obtained by using different techniques including Hölder's inequality [14] and power mean inequality [13]. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means presented in the end.

References

[1] M. Alomari, M. Darus, S.S. Dragomir, and P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, *Appl. Math. Lett.* **23** (2010), 1071–1076.
[2] E.F. Beckenbach, convex, *Bull. Amer. Math. Soc.* **54** (1948), 439–460.
[3] M.J.V. Cortez and J.E. Hernández, Ostrowski and Jensen-type inequalities via (s, m)-convex in the second sense, *Bol. Soc. Mat. Mex.* **26** (2020), no. 3, 287–302.
[4] S.S. Dragomir, A Companion of Ostrowski's Inequality for Functions of Bounded Variation and Applications, *Int. J. Nonlinear Anal. Appl.* **5** (2014), no. 4, 89–97.
[5] S.S. Dragomir, A Functional Generalization of Ostrowski Inequality via Montgomery identity, *Acta Math. Univ. Comenianae* **LXXXIV** (2015), 63–78.
[6] S.S. Dragomir, P. Cerone, and J. Roumeliotis, A new Generalization of Ostrowski Integral Inequality for Mappings whose Derivatives are Bounded and Applications in Numerical Integration and for Special Means, *Appl. Math. Lett.* **13** (2000), no. 2, 19–25.
[7] S. S.Dragomir, J. Pečarić, and L. Persson, Some inequalities of Hadamard type, *Soochow J. Math.* **21** (1995), 335–341.
[8] E.K. Godunova and, V.I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions, *Numerical Mathematics and Mathematical Physics* **166** (1985), 138–142.
[9] S. Mubeen and G.M. Habibullah, K-Fractional integrals and application, *J. Contemp. Math. Sci.* **7** (2012), no. 3, 89–94.
[10] A.M. Ostrowski, Uber die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert, *Comment. Math. Helv.* **10** (1938), no. 1, 226–227.
[11] S.G. Samko, A.A. Kilbas, and O.I. Marichev, Fractional Integrals and Derivatives, *Theory and Applications Gordon and Breach New York* **1** (1993).
[12] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are $s-$convex in the second sense via fractional integrals, *Comput Math Appl* **63** (2012), 1147–1154.
[13] Z.G. Xiao and A.H. Zhang, Mixed power mean inequalities, *Research Communication on Inequalities* **8** (2002), 15–17.
[14] X. Yang, A note on Hölder inequality, *Appl. Math. Comput.* **134** (2003), no. 2, 319–322.
[15] S. Mubeen and G.M. Habibullah, K-Fractional integrals and application, *J. Contemp. Math. Sci.* **7** (2012), no. 1, 89–94.

(Ali Hassan) Department of Mathematics, Shah Abdul Latif University Khairpur-66020, Pakistan.
E-mail address: alihassan.iiui.math@gmail.com

(Asif Raza Khan) Department of Mathematics, University of Karachi, University Road, Karachi-75270, Pakistan.
E-mail address: asifrk@uok.edu.pk