$k-$fractional Ostrowski type inequalities via $(s, r)-$convex

ALI HASSAN AND ASIF R. KHAN

Abstract. We are introducing very first time a generalized class named it the class of $(s, r)-$convex in mixed kind, this class includes $s-$convex in 1st and 2nd kind, $P-$convex, quasi convex and the class of ordinary convex. Also, we would like to state the generalization of the classical Ostrowski inequality via $k-$fractional integrals, which is obtained for functions whose first derivative in absolute values is $(s, r)-$convex in mixed kind. Moreover we establish some Ostrowski type inequalities via $k-$fractional integrals and their particular cases for the class of functions whose absolute values at certain powers of derivatives are $(s, r)-$convex in mixed kind by using different techniques including Hölder’s inequality and power mean inequality. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means would also be given.

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1. Introduction

The main aim of this study is to reveal new generalized-Ostrowski-type inequalities via $(s, r)-$convex using $k-$fractional operator which generalizes Riemann-Liouville integral operator.

Definition 1.1. [2] The $\eta : I \subset (0, \infty) \to \mathbb{R}$ is convex (concave), if

$$\eta(\rho t \rho x + (1 - \rho t)\rho y) \leq (\geq)\rho t \eta(\rho x) + (1 - \rho t)\eta(\rho y),$$

$\forall \rho x, \rho y \in I, \rho t \in [0, 1].$

Definition 1.2. [12] Let $s \in (0, 1]$, the $\eta : I \subset (0, \infty) \to [0, \infty)$ is the $s-$convex (concave) in 1st kind, if

$$\eta(\rho t \rho x + (1 - \rho t)\rho y) \leq (\geq)\rho t^s \eta(\rho x) + (1 - \rho t)^s\eta(\rho y),$$

$\forall \rho x, \rho y \in I, \rho t \in [0, 1].$

Definition 1.3. [8] The $\eta : I \subset (0, \infty) \to [0, \infty)$ is quasi convex (concave), if

$$\eta(\rho t \rho x + (1 - \rho t)\rho y) \leq (\geq)\max\{\eta(\rho x), \eta(\rho y)\}$$

$\forall \rho x, \rho y \in I, \rho t \in [0, 1].$

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Definition 1.4. [12] Let $s \in (0,1]$, the $\eta : I \subset (0,\infty) \to [0,\infty)$ is the $s$–convex (concave) in 2nd kind, if
\[
\eta(\rho_t \rho_x + (1-\rho_t)\rho_y) \leq (\geq) \rho_t^s \eta(\rho_x) + (1-\rho_t)^s \eta(\rho_y),
\]
forall $\rho_x, \rho_y \in I, \rho_t \in [0,1]$.

Definition 1.5. [2] The $\eta : I \subset (0,\infty) \to [0,\infty)$ is a $P$–convex (concave), if $\eta(\rho_x) \geq 0$ and $\forall \rho_x, \rho_y \in I$ and $\rho_t \in [0,1]$,
\[
\eta(\rho_t \rho_x + (1-\rho_t)\rho_y) \leq (\geq) \eta(\rho_x) + \eta(\rho_y).
\]

Next we present the classical Ostrowski inequality.

Theorem 1.1. [10] Let $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be differentiable function on $(\rho_a, \rho_b)$, $|\varphi'(\rho_t)| \leq M$, $\forall \rho_t \in (\rho_a, \rho_b)$. Then
\[
|\varphi(\rho_x) - \frac{1}{\rho_b - \rho_a} \int_{\rho_a}^{\rho_b} \varphi(\rho_t) d\rho_t| \leq M(\rho_b - \rho_a) \left[ \frac{1}{4} + \left( \frac{\rho_x - \rho_a + \rho_b}{\rho_b - \rho_a} \right)^2 \right],
\]
forall $\rho_x \in (\rho_a, \rho_b)$.

Also, one can see the numerous variants and applications in [4]-[6].

Definition 1.6. [11] The Riemann-Liouville integrals $I^\zeta_{\rho_a} \varphi$ and $I^\zeta_{\rho_b} \varphi$ of $\varphi \in L_1([\rho_a, \rho_b])$ having order $\zeta > 0$ with $\rho_a \geq 0, \rho_a < \rho_b$ are defined by
\[
I^\zeta_{\rho_a} \varphi(\rho_x) = \frac{1}{\Gamma(\zeta)} \int_{\rho_a}^{\rho_x} \frac{\varphi(\rho_t)}{(\rho_x - \rho_t)^{1-\zeta}} d\rho_t, \rho_x > \rho_a
\]
and
\[
I^\zeta_{\rho_b} \varphi(\rho_x) = \frac{1}{\Gamma(\zeta)} \int_{\rho_x}^{\rho_b} \frac{\varphi(\rho_t)}{(\rho_t - \rho_x)^{1-\zeta}} d\rho_t, \rho_x < \rho_b,
\]
respectively. Here $\Gamma(\zeta) = \int_{0}^{\infty} e^{-u}u^{\zeta-1} du$ is the Gamma function and $I^0_{\rho_a} \varphi(\rho_x) = I^0_{\rho_b} \varphi(\rho_x) = \varphi(\rho_x)$. We also make use of Euler’s beta function, which is for $x, y > 0$ defined as
\[
B(x, y) = \int_{0}^{1} t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\]

Definition 1.7. [15] The $k$–fractional integrals $^k\!I_{\rho_a}^\zeta \varphi$ and $^k\!I_{\rho_b}^\zeta \varphi$ of $\varphi \in L_1([\rho_a, \rho_b])$ having order $\zeta > 0$ with $0 \leq \rho_a < \rho_b, k > 0$ are defined by
\[
^k\!I_{\rho_a}^\zeta \varphi(\rho_x) = \frac{1}{k\Gamma_k(\zeta)} \int_{\rho_a}^{\rho_x} \frac{\varphi(\rho_t)}{(\rho_x - \rho_t)^{1-\zeta}} d\rho_t, \rho_x > \rho_a
\]
and
\[
^k\!I_{\rho_b}^\zeta \varphi(\rho_x) = \frac{1}{k\Gamma_k(\zeta)} \int_{\rho_x}^{\rho_b} \frac{\varphi(\rho_t)}{(\rho_t - \rho_x)^{1-\zeta}} d\rho_t, \rho_x < \rho_b,
\]
respectively. Here $\Gamma_k(\zeta) = \int_{0}^{\infty} e^{-\frac{k}{\zeta}u^{\zeta-1}u^{\zeta-1}} du$ is the generalized Gamma function and $^1\!J_{\rho_a}^0 \varphi(\rho_x) = ^1\!J_{\rho_b}^0 \varphi(\rho_x) = \varphi(\rho_x)$.  

GENERALIZATION OF OSTROWSKI INEQUALITIES VIA $k$–FRACTIONAL
Lemma 1.2. [15] Let \( \varphi : I \subset (0, \infty) \rightarrow \mathbb{R} \) be an absolutely continuous, and \( \rho_a, \rho_b \in I, \rho_a < \rho_b \). If \( \varphi' \in L_1[\rho_a, \rho_b], \zeta, k > 0 \) then

\[
Y_{\varphi}(\zeta, k, \rho_a, \rho_x, \rho_b) = \left[ \frac{(\rho_x - \rho_a)^{\frac{\zeta}{k}} + (\rho_b - \rho_x)^{\frac{\zeta}{k}}}{(\rho_b - \rho_a)} \right] \varphi(\rho_x) - \frac{k \Gamma_k(\zeta + 1)}{\rho_b - \rho_a} \left[ k J_{\rho_a}^\zeta \varphi(\rho_a) + k J_{\rho_x}^\zeta \varphi(\rho_b) \right],
\]

\[
Z_{\varphi}(\zeta, \rho_x, \rho_a, \rho_b) = \left[ \frac{(\rho_x - \rho_a)^{\zeta} + (\rho_b - \rho_x)^{\zeta}}{\rho_b - \rho_a} \right] \varphi(\rho_x) - \frac{\Gamma(\zeta + 1)}{\rho_b - \rho_a} \left[ I_{\rho_a}^\zeta \varphi(\rho_a) + I_{\rho_x}^\zeta \varphi(\rho_b) \right].
\]

\[
\zeta_{k \rho_b}(\rho_x) = \left[ \frac{(\rho_x - \rho_a)^{\zeta + 1} + (\rho_b - \rho_x)^{\zeta + 1}}{\rho_b - \rho_a} \right].
\]

In order to prove our main results we need the following Lemma.

Theorem 1.3. [11] Let \( \varphi : I \rightarrow \mathbb{R} \) be differentiable mapping on \( I^0 \), with \( \rho_a, \rho_b \in I, \rho_a < \rho_b \). If \( \varphi' \in L_1[\rho_a, \rho_b] \) and for \( \zeta, k > 1 \), Montgomery identity for \( k \)-fractional integrals holds:

\[
\varphi(\rho_x) = k \frac{\Gamma_k(\zeta)}{\rho_b - \rho_a} (\rho_b - \rho_x)^{1-\frac{\zeta}{k}} + k J_{\rho_a}^\zeta \varphi(\rho_b) - k J_{\rho_x}^{\zeta-1}(P_1(\rho_x, \rho_a) \varphi(\rho_a)) + k J_{\rho_a}^{\zeta-1}(P_1(\rho_x, \rho_b) \varphi(\rho_b)), \quad (2)
\]

where \( P_1(x, t) \) is the fractional Peano Kernel defined by:

\[
P_1(x, t) = \begin{cases} 
\frac{\rho_t - \rho_a}{\rho_b - \rho_a} \frac{k \Gamma_k(\zeta)}{\rho_b - \rho_a (\rho_b - \rho_x)^{\frac{\zeta}{k}-1}}, & \text{if } t \in [\rho_a, x], \\
\frac{\rho_t - \rho_b}{\rho_b - \rho_a} \frac{k \Gamma_k(\zeta)}{\rho_b - \rho_a (\rho_b - \rho_x)^{\frac{\zeta}{k}-1}}, & \text{if } t \in (\rho_x, \rho_b].
\end{cases}
\]

Let \( [\rho_a, \rho_b] \subseteq (0, +\infty) \), we may define special means as follows:

(a) The arithmetic mean

\[
A = A(\rho_a, \rho_b) := \frac{\rho_a + \rho_b}{2};
\]

(b) The geometric mean

\[
G = G(\rho_a, \rho_b) := \sqrt{\rho_a \rho_b};
\]
(c) The harmonic mean

\[H = H(\rho_a, \rho_b) := \frac{2}{\frac{1}{\rho_a} + \frac{1}{\rho_b}};\]

(d) The logarithmic mean

\[L = L(\rho_a, \rho_b) := \begin{cases} \frac{\rho_a \rho_b - \rho_a}{\ln \rho_b - \ln \rho_a}, & \text{if } \rho_a \neq \rho_b; \\ \frac{1}{\rho_b^{\rho_b} - \rho_a^{\rho_a}}, & \text{if } \rho_a = \rho_b. \end{cases}\]

(e) The identric mean

\[I = I(\rho_a, \rho_b) := \begin{cases} \frac{\rho_a}{1 + \left(\frac{\rho_b^{\rho_b} - \rho_a^{\rho_a}}{\rho_b^{\rho_b} - \rho_a^{\rho_a}}\right)^{\rho_a}}, & \text{if } \rho_a = \rho_b; \\ \frac{e^{\rho_b^{\rho_b} - \rho_a^{\rho_a}} - 1}{(p + 1)(\rho_b^{\rho_b} - \rho_a^{\rho_a})}, & \text{if } \rho_a \neq \rho_b. \end{cases}\]

(f) The \(p\)-logarithmic mean

\[L_p = L_p(\rho_a, \rho_b) := \begin{cases} \rho_a & \text{if } \rho_a = \rho_b; \\ \left[\frac{\rho_a^{p+1} - \rho_a^{p+1}}{(p + 1)(\rho_b^{\rho_b} - \rho_a^{\rho_a})}\right]^{\frac{1}{p}}, & \text{if } \rho_a \neq \rho_b. \end{cases}\]

where \(p \in \mathbb{R} \setminus \{0, -1\}\).

2. \(k\)-fractional Ostrowski type inequalities via \((s, r)\)-convex

In this section, we are introducing very first time the concept of \((s, r)\)-convex in mixed kind, this class contains many classes of convex from literature of convex analysis.

**Definition 2.1.** Let \((s, r) \in (0, 1)^2\), the \(\eta : I \subset (0, \infty) \rightarrow [0, \infty)\) is the \((s, r)\)-convex (concave) in mixed kind, if

\[\eta(t \rho_x + (1 - t) \rho_y) \leq \begin{cases} \geq \rho_t^s \eta(t \rho_x) + (1 - \rho_t^s) \eta(t \rho_y), & \text{if } \rho_a = \rho_b; \\ \leq \rho_t^s \eta(t \rho_x) + (1 - \rho_t^s) \eta(t \rho_y), & \text{if } \rho_a \neq \rho_b. \end{cases}\]

\(\forall \rho_x, \rho_y \in I, \rho_t \in [0, 1]\).

**Remark 2.1.** In Definition 2.1, we can see the following:

(1) If \(s = 1\) and \(r \in [0, 1]\) in (3), we get \(r\)-convex in 1st kind.
(2) If \(r \rightarrow 0\), and \(s = 1\), in (3), we get quasi convex.
(3) If \(r = 1\) and \(s \in [0, 1]\) in (3), we get \(s\)-convex in 2nd kind.
(4) If \(s \rightarrow 0\), and \(r = 1\) in (3), we get \(P\)-convex.
(5) If \(s = r = 1\) in (3), gives us ordinary convex.

Now, we will generalize the Ostrowski type inequalities via \((s, r)\)-convex (concave) by using \(k\)-fractional integral operator.

**Theorem 2.1.** Let \(\varphi : [\rho_a, \rho_b] \subset (0, \infty) \rightarrow \mathbb{R}\) be an absolutely continuous, and \(\varphi' \in L_1[\rho_a, \rho_b]\). If \(|\varphi'|\) is \((s, r)\)-convex, \(|\varphi'(\rho_x)| \leq M, \forall \rho_x \in [\rho_a, \rho_b]\), and \(\zeta, k > 0\)
Corollary 2.2. In Theorem 2.1, one can also capture the inequalities for $s$—convex in $1^{st}$ and $2^{nd}$ kind, $P$—convex and convex via $k$—fractional integrals by using Remark 2.1.

**Corollary 2.2.** In Theorem 2.1, one can see for $k = 1$ the following.

(1) The Ostrowski inequality for $(s, r)$—convex in mixed kind via fractional integrals:

$$|Y_{\varphi}(\zeta, \rho_x, \rho_a, \rho_b)| \leq M \left\{ \int_0^1 \rho_t^\frac{\zeta}{1} \rho_t^{rs} d\rho_t + \int_0^1 \rho_t^\frac{\zeta}{1} (1 - \rho_t^s)^s d\rho_t \right\} \times \left[ \frac{(\rho_x - \rho_a)^{\frac{\zeta}{1}+1}}{\rho_b - \rho_a} + \frac{(\rho_b - \rho_x)^{\frac{\zeta}{1}+1}}{\rho_b - \rho_a} \right].$$
(2) If \( s = 1 \) and \( r \in (0,1) \) in inequality (4), then the Ostrowski inequality for \( r \)-convex in 1st kind via fractional integrals:

\[
|Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq M \left( \frac{1}{\zeta + r + 1} + B \left( \frac{\zeta + 1}{r} \right) \right) \zeta K_{\rho_a}^{\rho_b}(\rho_x).
\]

(3) If \( r = 1 \) and \( s \in (0,1) \) in inequality (4), then the Ostrowski inequality for \( s \)-convex in 2nd kind via fractional integrals:

\[
|Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq M \left( \frac{1}{\zeta + s + 1} + B (\zeta + 1, s + 1) \right) \zeta K_{\rho_a}^{\rho_b}(\rho_x).
\]

(4) If \( \zeta = r = 1 \) and \( s \in (0,1) \) in inequality (4), then the inequality (2.1) of Theorem 2 in [1].

(5) If \( r = 1 \) and \( s \in (0,1) \) in inequality (4), then the inequality (2.6) of Theorem 7 in [12].

(6) If \( s \to 0 \) and \( r = 1 \), in inequality (4), then the Ostrowski inequality for \( P \)-convex via fractional integrals:

\[
|Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq M \left( \frac{1}{\zeta + 1} + B (\zeta + 1, 1) \right) \zeta K_{\rho_a}^{\rho_b}(\rho_x).
\]

(7) If \( r = s = 1 \), in inequality (4), then the Ostrowski inequality for convex via fractional integrals:

\[
|Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq M \left( \frac{1}{\zeta + 2} + B (\zeta + 1, 2) \right) \zeta K_{\rho_a}^{\rho_b}(\rho_x).
\]

(8) If \( \zeta = r = s = 1 \), in inequality (4), then the Ostrowski inequality (1.1) for convex.

**Theorem 2.3.** Let \( \varphi : [\rho_a, \rho_b] \subset (0, \infty) \to \mathbb{R} \) be an absolutely continuous, and \( \varphi' \in L[\rho_a, \rho_b] \). If \( |\varphi'|^q \) is \((s,r)-\)convex for \( q > 1 \) and \( |\varphi'(\rho_x)| \leq M, \forall \rho_x \in [\rho_a, \rho_b] \), and \( \zeta, k > 0 \) then

\[
|Y_\varphi(\zeta, k, \rho_a, \rho_x, \rho_b)| \\
\leq \frac{M}{L^{\frac{1}{q}-1}} \left[ \frac{(\rho_x - \rho_a)^{\frac{1}{q}} + 1}{\rho_b - \rho_a} + \frac{(\rho_b - \rho_x)^{\frac{1}{q}} + 1}{(\rho_b - \rho_a)} \right] \times \left\{ \int_0^1 \rho_t^{\frac{\zeta}{k}} \rho_t^{\frac{r}{s}} d\rho_t + \int_0^1 \rho_t^{\frac{\zeta}{k}} (1 - \rho_t)^{\frac{r}{s}} d\rho_t \right\}^{\frac{1}{q}}
\]

where

\[
L = \int_0^1 \rho_t^{\frac{\zeta}{k}} d\rho_t.
\]

**Proof.** By using the Lemma 1.2, and Power mean inequality [13],

\[
|Y_\varphi(\zeta, k, \rho_a, \rho_x, \rho_b)| \\
\leq \frac{(\rho_x - \rho_a)^{\frac{1}{q}} + 1}{\rho_b - \rho_a} \left( \int_0^1 \rho_t^{\frac{\zeta}{k}} d\rho_t \right)^{1 - \frac{1}{q}} \times \left( \int_0^1 \rho_t^{\frac{\zeta}{k}} |\varphi' (\rho_t \rho_x + (1 - \rho_t) \rho_a)|^q d\rho_t \right)^{\frac{1}{q}}
\]

\[
+ \frac{(\rho_b - \rho_x)^{\frac{1}{q}} + 1}{\rho_b - \rho_a} \left( \int_0^1 \rho_t^{\frac{\zeta}{k}} d\rho_t \right)^{1 - \frac{1}{q}} \times \left( \int_0^1 \rho_t^{\frac{\zeta}{k}} |\varphi' (\rho_t \rho_x + (1 - \rho_t) \rho_b)|^q d\rho_t \right)^{\frac{1}{q}}.
\]
Since $|\varphi'|^q$ is $(s,r)$–convex and $|\varphi'(|\rho_x|) \leq M$,

$$
|Y_\varphi(\zeta, k, \rho_a, \rho_x, \rho_b)| \leq \frac{M(\rho_x - \rho_a)^{\frac{1}{q} + 1}}{L_{\frac{1}{q} - 1}(\rho_b - \rho_a)} \left\{ \int_0^1 \rho_t^\zeta \rho_t^{rs} d\rho_t + \int_0^1 \rho_t^\zeta (1 - \rho_t)^s d\rho_t \right\} \frac{1}{q} + \frac{M(\rho_b - \rho_x)^{\frac{1}{q} + 1}}{L_{\frac{1}{q} - 1}(\rho_b - \rho_a)} \left\{ \int_0^1 \rho_t^\zeta \rho_t^{rs} d\rho_t + \int_0^1 \rho_t^\zeta (1 - \rho_t)^s d\rho_t \right\} \frac{1}{q}.
$$

\[\square\]

**Remark 2.3.** In Theorem 2.3, one can also capture the inequalities for $s$–convex in $1^{st}$ and $2^{nd}$ kind, $P$–convex and convex via $k$–fractional integrals by using Remark 2.1.

**Corollary 2.4.** In Theorem 2.3, one can see for $k = 1$ the following.

1. The Ostrowski inequality for $(s,r)$–convex in mixed kind via fractional integrals:

   $$
   |Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq \frac{M}{(\zeta + 1)^{1 - \frac{1}{q}}} \left( \frac{1}{\zeta + rs + 1} + \frac{B(\zeta + 1, r, s + 1)}{r} \right)^\frac{1}{q} \zeta_{K_{\rho_a}^p}(\rho_x).
   $$

2. If $s = 1$ and $r \in (0,1]$ in inequality (9), then the Ostrowski inequality for $r$–convex in $1^{st}$ kind via fractional integrals:

   $$
   |Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq \frac{M}{(\zeta + 1)^{1 - \frac{1}{q}}} \left( \frac{1}{\zeta + s + 1} + \frac{B(\zeta + 1, s + 1)}{s} \right)^\frac{1}{q} \zeta_{K_{\rho_a}^p}(\rho_x).
   $$

3. If $r = 1$ and $s \in (0,1]$ in inequality (9), then the Ostrowski inequality for $s$–convex in $2^{nd}$ kind via fractional integrals:

   $$
   |Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq \frac{M}{(\zeta + 1)^{1 - \frac{1}{q}}} \left( \frac{1}{\zeta + s + 1} + B(\zeta + 1, s + 1) \right)^\frac{1}{q} \zeta_{K_{\rho_a}^p}(\rho_x).
   $$

4. If $\zeta = r = 1, \text{ and } s \in (0,1]$ in inequality (9), then the inequality (2.3) of Theorem 4 in [1].

5. If $r = 1$ and $s \in (0,1]$ in inequality (9), then the inequality (2.8) of Theorem 9 in [12].

6. If $r = 1$ and $s \to 0$ in inequality (9), then the Ostrowski inequality for $P$–convex via fractional integrals:

   $$
   |Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq \frac{M}{(\zeta + 1)^{1 - \frac{1}{q}}} \left( \frac{1}{\zeta + 1} + B(\zeta + 1, 1) \right)^\frac{1}{q} \zeta_{K_{\rho_a}^p}(\rho_x).
   $$

7. If $r = s = 1, \text{ in inequality (9), then the Ostrowski inequality for convex via fractional integrals:}$

   $$
   |Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq \frac{M}{(\zeta + 1)^{1 - \frac{1}{q}}} \left( \frac{1}{\zeta + 2} + B(\zeta + 1, 2) \right)^\frac{1}{q} \zeta_{K_{\rho_a}^p}(\rho_x).
   $$
Theorem 2.5. Let \( \varphi : [\rho_a, \rho_b] \subset (0, \infty) \rightarrow \mathbb{R} \) be an absolutely continuous, \( \varphi' \in L[\rho_a, \rho_b] \). If \( |\varphi'|^q \) is \((s, r)\)-convex, \( |\varphi'(\rho_x)| \leq M, \forall \rho_x \in [\rho_a, \rho_b], \) \( \zeta, k > 0 \), and \( p, z > 1 \) with \( \frac{1}{z} + \frac{1}{q} = 1 \), then

\[
|Y_\varphi(\zeta, k, \rho_a, \rho_x, \rho_b)| \leq \frac{MK^{\frac{1}{z}}}{\rho_b - \rho_a} \left( \frac{1}{rs + 1} + \frac{1}{r} B \left( \frac{1}{r}, s + 1 \right) \right) \frac{1}{q} \\
\times \left[ (\rho_x - \rho_a)^{\frac{k}{q} + 1} + (\rho_b - \rho_x)^{\frac{k}{q} + 1} \right].
\]  
where

\[
K = \int_0^1 \rho_t^{\frac{k}{q}} \, d\rho_t.
\]

Proof. By using the Lemma 1.2, and Hölder’s inequality [14],

\[
|Y_\varphi(\zeta, k, \rho_a, \rho_x, \rho_b)| \leq \frac{(\rho_x - \rho_a)^{\frac{k}{q} + 1}}{\rho_b - \rho_a} \left( \int_0^1 \rho_t^{\frac{k}{q}} \, d\rho_t \right) \frac{1}{q} \left( \int_0^1 |\varphi'(\rho_t \rho_x + (1 - \rho_t)\rho_a)|^q \, d\rho_t \right)^{\frac{1}{q}}
\]

\[
+ \frac{(\rho_b - \rho_x)^{\frac{k}{q} + 1}}{\rho_b - \rho_a} \left( \int_0^1 \rho_t^{\frac{k}{q}} \, d\rho_t \right) \frac{1}{q} \left( \int_0^1 |\varphi'(\rho_t \rho_x + (1 - \rho_t)\rho_b)|^q \, d\rho_t \right)^{\frac{1}{q}}.
\]  

Since \( |\varphi'|^q \) is \((s, r)\)-convex and \( |\varphi'(\rho_x)| \leq M, \)

\[
|Y_\varphi(\zeta, k, \rho_a, \rho_x, \rho_b)| \leq \frac{K^{\frac{1}{z}}(\rho_x - \rho_a)^{\frac{k}{q} + 1}}{\rho_b - \rho_a} \left( \frac{M^q}{rs + 1} + \frac{M^q}{r} B \left( \frac{1}{r}, s + 1 \right) \right) \frac{1}{q}
\]

\[
+ \frac{K^{\frac{1}{z}}(\rho_b - \rho_x)^{\frac{k}{q} + 1}}{\rho_b - \rho_a} \left( \frac{M^q}{rs + 1} + \frac{M^q}{r} B \left( \frac{1}{r}, s + 1 \right) \right) \frac{1}{q}.
\]

Remark 2.4. In Theorem 2.5, one can also capture the inequalities for \((s, r)\)-convex in 1\textsuperscript{st} and 2\textsuperscript{nd} kind, \( P \)-convex and convex via \( k \)-fractional integrals by using Remark 2.1.

Corollary 2.6. In Theorem 2.5, one can see for \( k = 1 \) the following.

(1) The Ostrowski inequality for \((s, r)\)-convex in mixed kind via fractional integrals:

\[
|Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq \frac{M}{(\zeta + 1)^{\frac{1}{z}}} \left( \frac{1}{rs + 1} + \frac{B \left( \frac{1}{r}, s + 1 \right)}{r} \right) \zeta K^{\frac{1}{q}}(\rho_a, \rho_x).
\]

(2) If \( s = 1 \) and \( r \in (0, 1] \) in inequality (11), then the Ostrowski inequality for \( r \)-convex in 1\textsuperscript{st} kind via fractional integrals:

\[
|Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq \frac{M}{(\zeta + 1)^{\frac{1}{z}}} \left( \frac{1}{s + 1} + \frac{B \left( \frac{1}{s}, 2 \right)}{s} \right) \zeta K^{\frac{1}{q}}(\rho_a, \rho_x).
\]

(3) If \( r = 1 \) and \( s \in (0, 1) \) in inequality (11), then the Ostrowski inequality for \( s \)-convex in 2\textsuperscript{nd} kind via fractional integrals:

\[
|Z_\varphi(\zeta, \rho_x, \rho_a, \rho_b)| \leq \frac{M}{(\zeta + 1)^{\frac{1}{z}}} \left( \frac{1}{s + 1} + B(1, s + 1) \right) \zeta K^{\frac{1}{q}}(\rho_a, \rho_x).
\]
(4) If \( \zeta = r = 1 \) and \( s \in (0, 1] \) in inequality (11), then the inequality (2.2) of Theorem 3 in [1].

(5) If \( r = 1 \) and \( s \in (0, 1] \) in inequality (11), then the inequality (2.7) of Theorem 8 in [12].

(6) If \( r = 1 \), and \( s \to 0 \) in inequality (11), then the Ostrowski inequality for \( P \)-convex via fractional integrals:

\[
|Z_\varphi(\zeta, \rho, \rho_a, \rho_b)| \leq \frac{(2)^{\frac{1}{s}} M}{(\zeta + 1)^{\frac{1}{s}}} \zeta \kappa_{\rho_a}^s(\rho_x).
\]

(7) If \( r = s = 1 \), in inequality (11), then the Ostrowski inequality for convex via fractional integrals:

\[
|Z_\varphi(\zeta, \rho, \rho_a, \rho_b)| \leq \frac{M}{(\zeta + 1)^{\frac{1}{s}}} \zeta \kappa_{\rho_a}^s(\rho_x).
\]

**Theorem 2.7.** Let \( \varphi : [\rho_a, \rho_b] \to \mathbb{R} \) be differentiable on \( (\rho_a, \rho_b) \), \( \varphi' : [\rho_a, \rho_b] \to \mathbb{R} \) be integrable on \( [\rho_a, \rho_b] \) and \( \eta : I \subset \mathbb{R} \to \mathbb{R} \), be a \((s, r)\)-convex function in mixed sense, then we have the inequalities

\[
\eta \left[ \varphi(\rho_x) - \frac{k \Gamma_k(\zeta)}{\rho_b - \rho_a} (\rho_b - \rho_x)^{1 - \frac{s}{r}} k J_\rho_a^\zeta \varphi(\rho_b) + k J_\rho_a^\zeta - 1(P_1(\rho_x, \rho_b) \varphi(\rho_b)) \right]
\]

\[
\leq \frac{(\rho_b - \rho_x)^{1 - \frac{s}{r}}}{(\rho_b - \rho_a)^r s} \left[ (\rho_x - \rho_a)^{r s - 1} \int_{\rho_a}^{\rho_x} \eta \left[ \frac{(\rho_t - \rho_a) \varphi'(\rho_t)}{(\rho_b - \rho_t)^{1 - \frac{r}{s}}} \right] d\rho_t \right] + \frac{(\rho_b - \rho_a)^r - (\rho_x - \rho_a)^r}{\rho_b - \rho_x} \int_{\rho_x}^{\rho_a} \eta \left[ \frac{(\rho_t - \rho_b) \varphi'(\rho_t)}{(\rho_b - \rho_t)^{1 - \frac{r}{s}}} \right] d\rho_t, \tag{13}
\]

\( \forall \rho_x \in [\rho_a, \rho_b] \).

**Proof.** Utilizing the Theorem 1.3, we get

\[
\varphi(\rho_x) - \frac{k \Gamma_k(\zeta)}{\rho_b - \rho_a} (\rho_b - \rho_x)^{1 - \frac{s}{r}} k J_\rho_a^\zeta \varphi(\rho_b) + k J_\rho_a^\zeta - 1(P_1(\rho_x, \rho_b) \varphi(\rho_b))
\]

\[
= k J_\rho_a^\zeta(P_1(\rho_x, \rho_b) \varphi'(\rho_b)) = \frac{1}{k \Gamma_k(\zeta)} \int_{\rho_a}^{\rho_b} P_1(x, t) \frac{\varphi'(\rho_t)}{(\rho_b - \rho_t)^{1 - \frac{r}{s}}} d\rho_t
\]

\[
= \left( \frac{x - \rho_a}{\rho_b - \rho_a} \right) \left[ (\rho_b - \rho_x)^{1 - \frac{s}{r}} \int_{\rho_a}^{\rho_x} \frac{\{\rho_t - \rho_a\} \varphi'(\rho_t)}{(\rho_b - \rho_t)^{1 - \frac{r}{s}}} d\rho_t \right]
\]

\[
+ \left( 1 - \left( \frac{x - \rho_a}{\rho_b - \rho_a} \right) \right) \left[ (\rho_b - \rho_x)^{1 - \frac{s}{r}} \int_{\rho_x}^{\rho_b} \frac{\{\rho_t - \rho_b\} \varphi'(\rho_t)}{(\rho_b - \rho_t)^{1 - \frac{r}{s}}} d\rho_t \right],
\]

\( \forall \rho_x \in [\rho_a, \rho_b] \). Next by using the \((s, r)\)-convex function in mixed sense of \( \eta : I \subset [0, \infty) \to \mathbb{R} \), we get
Corollary 2.8. In Theorem 2.7, one can see the following.

1. If \( s = 1 \) and \( r \in (0, 1] \) in (13), then Ostrowski inequality for \( r \)-convex functions in 1st kind:

\[
\eta \left[ \varphi(\rho_x) - \frac{k\Gamma_k(\zeta)}{\rho_b - \rho_a} (\rho_b - \rho_x)^{1 - \frac{\zeta}{k}} k_{\rho_a}^k \varphi(\rho_b) + \frac{k\zeta^{-1}}{\rho_a} (P_1(\rho_x, \rho_b) \varphi(\rho_b)) \right] \\
\leq \frac{\rho_b - \rho_x}{(\rho_b - \rho_a)} \int_{\rho_a}^{\rho_b} \eta \left[ \frac{(\rho_t - \rho_a) \varphi'(\rho_t)}{(\rho_b - \rho_t)^{1 - \frac{\zeta}{k}}} \right] d\rho_t \\
+ \frac{(\rho_b - \rho_x)^r - (\rho_x - \rho_a)^r}{(\rho_b - \rho_x)} \int_{\rho_a}^{\rho_b} \eta \left[ \frac{(\rho_t - \rho_b) \varphi'(\rho_t)}{(\rho_b - \rho_t)^{1 - \frac{\zeta}{k}}} \right] d\rho_t.
\]

2. If \( s = 1 \) and \( r \to 0 \) in (13), we get quasi-convex function.

\[
\eta \left[ \varphi(\rho_x) - \frac{k\Gamma_k(\zeta)}{\rho_b - \rho_a} (\rho_b - \rho_x)^{1 - \frac{\zeta}{k}} k_{\rho_a}^k \varphi(\rho_b) + \frac{k\zeta^{-1}}{\rho_a} (P_1(\rho_x, \rho_b) \varphi(\rho_b)) \right] \\
\leq \frac{(\rho_b - \rho_x)^{1 - \frac{\zeta}{k}}}{(\rho_x - \rho_a)} \int_{\rho_a}^{\rho_b} \eta \left[ \frac{(\rho_t - \rho_a) \varphi'(\rho_t)}{(\rho_b - \rho_t)^{1 - \frac{\zeta}{k}}} \right] d\rho_t.
\]

3. If \( r = 1 \) and \( s \in [0, 1] \) (13), then fractional Ostrowski type inequality for \( s \)-convex functions in 2nd kind:

\[
\eta \left[ \varphi(\rho_x) - \frac{k\Gamma_k(\zeta)}{\rho_b - \rho_a} (\rho_b - \rho_x)^{1 - \frac{\zeta}{k}} k_{\rho_a}^k \varphi(\rho_b) + \frac{k\zeta^{-1}}{\rho_a} (P_1(\rho_x, \rho_b) \varphi(\rho_b)) \right] \\
\leq \frac{(\rho_b - \rho_x)^{1 - \frac{\zeta}{k}}}{(\rho_b - \rho_a)} \int_{\rho_a}^{\rho_b} \eta \left[ \frac{(\rho_t - \rho_a) \varphi'(\rho_t)}{(\rho_b - \rho_t)^{1 - \frac{\zeta}{k}}} \right] d\rho_t \\
+ (\rho_b - \rho_x)^s \int_{\rho_x}^{\rho_b} \eta \left[ \frac{(\rho_t - \rho_b) \varphi'(\rho_t)}{(\rho_b - \rho_t)^{1 - \frac{\zeta}{k}}} \right] d\rho_t.
\]
(4) If \( r = 1 \) and \( s \to 0 \) in (13), then fractional Ostrowski type inequality for \( P \)-convex functions:

\[
\eta \left[ \varphi(\rho_x) - \frac{k \Gamma_k(\zeta)}{\rho_b - \rho_a} (\rho_b - \rho_x)^{1 - \frac{k}{\xi}} k \int_{\rho_a}^{\zeta} (P_1(\rho_x, \rho_b)) \varphi(\rho_b) \right] \\
\leq (\rho_b - \rho_x)^{1 - \frac{k}{\xi}} \left[ \frac{1}{x - \rho_a} \int_{\rho_a}^{\rho_x} \eta \left[ \left( \frac{\rho_t - \rho_a}{\rho_b - \rho_t} \right)^{1 - \frac{k}{\xi}} \right] d\rho_t \right] \\
+ \frac{1}{\rho_b - \rho_x} \int_{\rho_x}^{\rho_b} \eta \left[ \left( \frac{\rho_t - \rho_b}{\rho_b - \rho_t} \right)^{1 - \frac{k}{\xi}} \right] d\rho_t .
\]

(5) If \( s = r = 1 \) in (13), then fractional Ostrowski type inequality for convex functions:

\[
\eta \left[ \varphi(\rho_x) - \frac{k \Gamma_k(\zeta)}{\rho_b - \rho_a} (\rho_b - \rho_x)^{1 - \frac{k}{\xi}} k \int_{\rho_a}^{\zeta} (P_1(\rho_x, \rho_b)) \varphi(\rho_b) \right] \\
\leq \frac{(\rho_b - \rho_x)^{1 - \frac{k}{\xi}}}{\rho_b - \rho_a} \int_{\rho_a}^{\rho_x} \eta \left[ \left( \frac{\rho_t - \rho_a}{\rho_b - \rho_t} \right)^{1 - \frac{k}{\xi}} \right] d\rho_t + \int_{\rho_x}^{\rho_b} \eta \left[ \left( \frac{\rho_t - \rho_b}{\rho_b - \rho_t} \right)^{1 - \frac{k}{\xi}} \right] d\rho_t .
\]

3. Applications to Special Means

If we replace \( \varphi \) by \( -\varphi \) and \( x = \frac{\rho_a + \rho_b}{2} \) in Theorem 2.7, we get

**Theorem 3.1.** Let \( \varphi : [\rho_a, \rho_b] \to \mathbb{R} \) be differentiable on \( (\rho_a, \rho_b) \), \( \varphi' : [\rho_a, \rho_b] \to \mathbb{R} \) be integrable on \([\rho_a, \rho_b]\) and \( \eta : I \subset \mathbb{R} \to \mathbb{R} \), be a \((s, r)\)-convex function in mixed sense, then

\[
\eta \left[ \frac{k \Gamma_k(\zeta)}{\rho_b - \rho_a} (\rho_b - \rho_x)^{1 - \frac{k}{\xi}} k \int_{\rho_a}^{\zeta} (P_1(\rho_x, \rho_b)) \varphi(\rho_b) \right] \\
\leq \frac{2^{k-1}}{(\rho_b - \rho_a)^{k-1}} \int_{\rho_a}^{\rho_x} \eta \left[ \left( \frac{\rho_t - \rho_a}{\rho_b - \rho_t} \right)^{1 - \frac{k}{\xi}} \right] d\rho_t \\
+ \frac{(2^r - 1)^s}{2^{rs-1}} \int_{\rho_x}^{\rho_b} \eta \left[ \left( \frac{\rho_t - \rho_b}{\rho_b - \rho_t} \right)^{1 - \frac{k}{\xi}} \right] d\rho_t .
\]

**Remark 3.1.** In Theorem 3.1, if we put \( \zeta = k = 1 \) in (14), we get

\[
\eta \left( \frac{1}{\rho_b - \rho_a} \int_{\rho_a}^{\rho_b} \varphi(\rho_t) d\rho_t - \varphi \left( \frac{\rho_a + \rho_b}{2} \right) \right) \\
\leq \frac{1}{\rho_b - \rho_a} \int_{\rho_a}^{\rho_b} \eta \left[ (\rho_a - \rho_t) \varphi'(\rho_t) \right] d\rho_t \\
+ \frac{(2^r - 1)^s}{2^{rs-1}} \int_{\rho_a}^{\rho_b} \eta \left[ (\rho_b - \rho_t) \varphi'(\rho_t) \right] d\rho_t .
\]

**Remark 3.2.** Assume that \( \eta : I \subset [0, \infty) \to \mathbb{R} \) be an \((s, r)\)-convex function in mixed kind:
Remark 3.4. In Theorem 2

(1) If $\zeta = k = 1$, $\varphi(\rho_t) = \frac{1}{\rho_t}$ in inequality (14) where $\rho_t \in [\rho_a, \rho_b] \subset (0, \infty)$, then we have

$$
(\rho_b - \rho_a)\eta \left[ \frac{A(\rho_a, \rho_b) - L(\rho_a, \rho_b)}{A(\rho_a, \rho_b)L(\rho_a, \rho_b)} \right] 
\leq \frac{1}{2sr-1} \int_{\rho_a}^{\rho_{a+\rho_b}} \eta \left[ \frac{\rho_t - \rho_a}{\rho_t^2} \right] d\rho_t + \frac{(2^r - 1)^s}{2^{rs-1}} \int_{\rho_{a+\rho_b}}^{\rho_b} \eta \left[ \frac{\rho_t - \rho_b}{\rho_t^2} \right] d\rho_t.
$$

(2) If $\zeta = k = 1$, $\varphi(\rho_t) = -\ln \rho_t$ in inequality (14), where $\rho_t \in [\rho_a, \rho_b] \subset (0, \infty)$, then we have

$$
(\rho_b - \rho_a)\eta \left[ \ln \left( \frac{A(\rho_a, \rho_b)}{I(\rho_a, \rho_b)} \right) \right] 
\leq \frac{1}{2sr-1} \int_{\rho_a}^{\rho_{a+\rho_b}} \eta \left[ \frac{\rho_t - \rho_a}{\rho_t^1} \right] d\rho_t + \frac{(2^r - 1)^s}{2^{rs-1}} \int_{\rho_{a+\rho_b}}^{\rho_b} \eta \left[ \frac{\rho_t - \rho_b}{\rho_t^1} \right] d\rho_t.
$$

(3) If $\zeta = k = 1$, $\varphi(\rho_t) = \rho_t^p, p \in \mathbb{R} \setminus \{0, -1\}$ in inequality (14), where $\rho_t \in [\rho_a, \rho_b] \subset (0, \infty)$, then we have

$$
(\rho_b - \rho_a)\eta \left[ L_p^p(\rho_a, \rho_b) - A^p(\rho_a, \rho_b) \right] 
\leq \frac{1}{2sr-1} \int_{\rho_a}^{\rho_{a+\rho_b}} \eta \left[ \frac{p(\rho_a - \rho_t)}{\rho_t^{1-p}} \right] d\rho_t + \frac{(2^r - 1)^s}{2^{rs-1}} \int_{\rho_{a+\rho_b}}^{\rho_b} \eta \left[ \frac{p(\rho_b - \rho_t)}{\rho_t^{1-p}} \right] d\rho_t.
$$

Remark 3.3. In Theorem 2.3, one can see for $\zeta = k = 1$ the following.

(1) Let $\rho_x = \frac{\rho_a + \rho_b}{2}, 0 < \rho_a < \rho_b, q \geq 1$ and $\varphi : \mathbb{R} \to \mathbb{R}^+, \varphi(\rho_t) = \rho_t^p$ in (9). Then

$$
|A^n(\rho_a, \rho_b) - L^n(\rho_a, \rho_b)| \leq \frac{M(\rho_b - \rho_a)}{(2)^{2-\frac{1}{q}} + \frac{B(\frac{2}{r}, s + 1)}{r}}.
$$

(2) Let $\rho_x = \frac{\rho_a + \rho_b}{2}, 0 < \rho_a < \rho_b, q \geq 1$ and $\varphi : (0, 1] \to \mathbb{R}, \varphi(\rho_t) = -\ln \rho_t$ in (9). Then

$$
\left| \ln \left( \frac{A(\rho_a, \rho_b)}{I(\rho_a, \rho_b)} \right) \right| \leq \frac{M(\rho_b - \rho_a)}{(2)^{2-\frac{1}{q}} + \frac{B(\frac{2}{r}, s + 1)}{r}}.
$$

Remark 3.4. In Theorem 2.5, one can see for $\zeta = k = 1$ the following.

(1) Let $\rho_x = \frac{\rho_a + \rho_b}{2}, 0 < \rho_a < \rho_b, p^{-1} + q^{-1} = 1$ and $\varphi : \mathbb{R} \to \mathbb{R}^+, \varphi(\rho_t) = \rho_t^p$ in (11). Then

$$
|A^n(\rho_a, \rho_b) - L^n(\rho_a, \rho_b)| \leq \frac{M(\rho_b - \rho_a)}{2(z + 1)^{\frac{1}{q}}} \left( \frac{1}{r s + 1} + \frac{B(\frac{1}{r}, s + 1)}{r} \right)^{\frac{1}{q}}.
$$

(2) Let $\rho_x = \frac{\rho_a + \rho_b}{2}, 0 < \rho_a < \rho_b, p^{-1} + q^{-1} = 1$ and $\varphi : (0, 1] \to \mathbb{R}, \varphi(\rho_t) = -\ln \rho_t$ in (11). Then

$$
\left| \ln \left( \frac{A(\rho_a, \rho_b)}{I(\rho_a, \rho_b)} \right) \right| \leq \frac{M(\rho_b - \rho_a)}{2(z + 1)^{\frac{1}{q}}} \left( \frac{1}{r s + 1} + \frac{B(\frac{1}{r}, s + 1)}{r} \right)^{\frac{1}{q}}.
$$
4. Conclusion

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of \((s, r)\)-convex in mixed kind, this class of functions contains many important classes including class of \(s\)-convex in 1st and 2nd kind [3], \(P\)-convex, quasi convex and the class of convex. In this study, theorems are put forward to obtain new upper bounds by \(k\)-fractional operator for Ostrowski type inequalities. We have stated our first main result in section 2, the generalization of Ostrowski inequality [10] via \(k\)-fractional integral and others results obtained by using different techniques including Hölder’s inequality [14] and power mean inequality [13]. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means presented in the end.

References