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Common fixed point result for multivalued mappings with applications to systems of integral and functional equations

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Abstract. In this paper, we establish a common fixed point theorem for two multivalued mappings satisfying some dominated conditions on a complete metric space. This new rational type contractive inequality refines various results in the literature. The main theorem is illustrated with examples. As application, we found the existence of solutions of system nonlinear integral equations and functional equations.

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1. Introduction

Metric fixed point theory is widely recognized to have been originated in the work of S. Banach in 1922 [2], where he proved the famous contraction mapping principle. The conventional proofs of the Banach's fixed point theorem along with the proofs of several other contractive fixed point results like those in [3, 4, 11, 17, 22], do not require the respective contraction conditions to be satisfied between every pair of points. Taking into account the above fact, it was shown in many of the recent works that under some suitable conditions several of these results hold if a partial ordering $' \preceq'$ is introduced in the metric space, the contraction is restricted to hold for pairs of points related by partial ordering, there is an initial condition $x_0 \preceq Tx_0$ for some x_0 for the fixed point iteration. Some works in this direction are [14, 19, 20]. This has opened a new line of research which is the fixed point theory in partially ordered metric spaces.

An alternative way was suggested by Samet et al. [21] where the concept of admissible mappings was introduced. There is no need to introduce any partial ordering. A generalization of Banach's contraction mapping principle was proved by assuming a contraction condition for certain pairs of points and an initial condition to start with fixed point iteration. The α- admissible mappings have been used in several fixed point results like [1, 5, 6, 9].

We use rational terms in our inequality. The use of rational terms in contraction inequalities in the domain of metric fixed point theory was initiated by Dass et al. in their work [12] in which they extended the Banach's contraction principle [2] by using a contractive rational inequality. After that the rational inequalities have been used in fixed point and related problems in several works as for instances in [5, 6, 16].

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The extension of Banach's contraction mapping principal [2] into the domain of set valued analysis was obtain in 1969 in the work of Nadler [18] where the Hausdorff distance was used. There are several works which have utilized Hausdorff distance [5, 6, 9]. An alternative concept of distance which appeared in the subsequent literature of set valued fixed point theory is the δ- distance [7, 8, 13].

2. Mathematical background

The following are the concepts from setvalued analysis which we use in this paper. Let $(X,\ d)$ be a metric space. Then $N(X)$ denotes the collection of all nonempty subset of X; $CB(X)$ denotes the collection of all nonempty closed and bounded subset of X and $C(X)$ denotes the collection of all nonempty compact subset of X. Let $x \in X$ and $A,\ B \in CB(X)$. Then the functions D, δ and H are defined as follows.

$$D(x,\ B) = \inf\{d(x,\ y) : y \in B\},\quad D(A,\ B) = \inf\{d(a,\ b) : a \in A,\ b \in B\},$$

$$H(A,\ B) = \max\{\sup_{x\in A} D(x,\ B),\ \sup_{y\in B} D(y,\ A)\},\ \ \delta(A,\ B) = \sup\{d(a,\ b) : a \in A,\ b \in B\}.$$

If $A = \{a\}$, then we write $D(A,\ B) = D(a,\ B)$ and $\delta(A,\ B) = \delta(a,\ B)$. Also in addition, if $B = \{b\}$, then $D(A,\ B) = d(a,\ b)$ and $\delta(A,\ B) = d(a,\ b)$. For all $A,\ B,\ C \in CB(X)$, the definition of $\delta(A,\ B)$ yields that $\delta(A,\ B) = \delta(B,\ A)$, $\delta(A,\ B) \leq \delta(A,\ C)+\delta(C,\ B)$, $\delta(A,\ B) = 0$ iff $A = B = \{a\}$, $\delta(A,\ A) = \operatorname{diam} A$ [13]. H is known as the Hausdorff metric induced by d on $CB(X)$ [18]. Further, if $(X,\ d)$ is complete then $(CB(X),\ H)$ is also complete. The δ - distance is not a metric like the Hausdorff distance, but shares most of the properties of a metric.

Lemma 2.1 ([6])**.** Let $(X,\ d)$ be a metric space and $B \in C(X)$. Then for every $x \in X$ there exists $y \in B$ such that $d(x,\ y) = D(x,\ B)$.

Definition 2.1. Let X be a nonempty set and $T : X \longrightarrow X$. An element $x \in X$ is called a fixed point of the mapping T if $x = Tx$.

Definition 2.2. Let X be a nonempty set and $T : X \longrightarrow N(X)$ be a multivalued mapping. An element $x \in X$ is called a fixed point of the mapping T if $x \in Tx$.

Definition 2.3. Let X be a nonempty set and $S,\ T : X \longrightarrow X$. An element $x \in X$ is called a common fixed point of the mappings S and T if $x = Sx = Tx$.

Definition 2.4. Let X be a nonempty set and $S,\ T : X \longrightarrow N(X)$ be two multivalued mappings. An element $x \in X$ is called a common fixed point of the mappings S and T if $x \in Sx \bigcap Tx$.

Definition 2.5. Let X be a nonempty set, $T : X \longrightarrow N(X)$ and $\alpha : X \times X \longrightarrow [0,\ \infty)$. Then T is said to be α- dominated if $\alpha(x,\ u) \geq 1$, for all $x \in X$ and $u \in Tx$.

Definition 2.6. Let X be a nonempty set, $S,\ T : X \longrightarrow N(X)$ be multivalued mapping and $\alpha : X \times X \longrightarrow [0,\ \infty)$. We say that S and T are α-dominated by each other if for $x \in X$,

(i) $\alpha(u,\ v) \geq 1$, for all $u \in Tx$ and $v \in Su$ and (ii) $\alpha(a,\ b) \geq 1$, for all $a \in Sx$ and $b \in Ta$.

Definition 2.7. Let $(X, \preceq)$ be a partial ordered set and $S, T : X \longrightarrow N(X)$ be two multivalued mappings. Then S and T are said to be dominated by each other if for $x \in X$,

(i) $v \preceq u$, for all $u \in Tx$ and $v \in Su$; and (ii) $b \preceq a$, for all $a \in Sy$ and $b \in Ta$.

Definition 2.8. Let (X, d) be a metric space and $\alpha : X \times X \longrightarrow [0, \infty)$. Then X is said to have α- regular property if for every sequence $\{x_n\}$ in X with $x_n \longrightarrow x$ and $\alpha(x_n, x_{n+1}) \geq 1$ implies $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Definition 2.9. Let (X, d) be a metric space with a partial order $\preceq$. Then X is said to have regular property if for a monotonic increasing (respectively monotonic decreasing) sequence $\{x_n\}$ in X with $x_n \longrightarrow x$ implies $x \preceq x_n$ (respectively $x_n \preceq x$) for all $n \in \mathbb{N}$.

3. Main results

Theorem 3.1. *Let (X, d) is a complete metric space, $S, T : X \longrightarrow C(X)$ be two multivalued mappings and $\alpha : X \times X \longrightarrow [0, \infty)$. Suppose that (i) X has α- regular property; (ii) S, T are α- dominated by each other; (iii) there exist non-negative real numbers a, b, f, g with $a+b+f+g < 1$ such that for all $x, y \in X$ with $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$,*

$$H(Sx, Ty) \leq a\ d(x, y) + b\ \frac{D(x, Sx)D(y, Ty) + D(y, Sx)D(x, Ty)}{1 + \delta(x, Ty) + d(x, y) + \delta(y, Sx)}$$
$$+ f\ D(x, Sx) + g\ D(y, Ty).$$

Then S, T have a common fixed point in X.

Proof. Let $x_0 \in X$. As $Sx_0 \in C(X)$, by Lemma 2.1, there exists $x_1 \in Sx_0$ such that $d(x_0, x_1) = D(x_0, Sx_0)$. Again $Tx_1 \in C(X)$. By Lemma 2.1, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = D(x_1, Tx_1)$. Also by the assumption (ii), we have $\alpha(x_1, x_2) \geq 1$. Similarly by Lemma 2.1, as $Sx_2 \in C(X)$, there exists $x_3 \in Sx_2$ such that $d(x_3, x_2) = D(Sx_2, x_2)$ and by the assumption (ii), $\alpha(x_2, x_3) \geq 1$. Continuing this process we construct a sequence $\{x_n\}$ in X such that for all $n \in \mathbb{N}$,

$$x_{2n+1} \in Sx_{2n}, \quad \text{and} \quad x_{2n+2} \in Tx_{2n+1}, \text{ with } \alpha(x_n, x_{n+1}) \geq 1, \tag{3.1}$$

$$d(x_{2n+1}, x_{2n+2}) = D(x_{2n+1}, Tx_{2n+1}) \text{ and } d(x_{2n+3}, x_{2n+2}) = D(Sx_{2n+2}, x_{2n+2}). \tag{3.2}$$

Now as $\alpha(x_{2n}, x_{2n+1}) \geq 1$, by assumption (iii) and from (3.1) and (3.2), we have

$$\begin{aligned}
&d(x_{2n+1}, x_{2n+2}) = D(x_{2n+1}, Tx_{2n+1}) \leq H(Sx_{2n}, Tx_{2n+1})\\
&\leq a\ d(x_{2n}, x_{2n+1}) + b\ \frac{D(x_{2n}, Sx_{2n})D(x_{2n+1}, Tx_{2n+1}) + D(x_{2n+1}, Sx_{2n})D(x_{2n}, Tx_{2n+1})}{1 + \delta(x_{2n}, Tx_{2n+1}) + d(x_{2n}, x_{2n+1}) + \delta(x_{2n+1}, Sx_{2n})}\\
&\quad + f\ D(x_{2n}, Sx_{2n}) + g\ D(x_{2n+1}, Tx_{2n+1})\\
&\leq a\ d(x_{2n}, x_{2n+1}) + b\ \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})d(x_{2n}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+2}) + d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+1})}\\
&\quad + f\ d(x_{2n}, x_{2n+1}) + g\ d(x_{2n+1}, x_{2n+2})
\end{aligned}$$

$= a\ d(x_{2n},\ x_{2n+1}) + b\ \dfrac{d(x_{2n},\ x_{2n+1})d(x_{2n+1},\ x_{2n+2})}{1 + d(x_{2n}, x_{2n+2}) + d(x_{2n}, x_{2n+1})} + f\ d(x_{2n},\ x_{2n+1}) + g\ d(x_{2n+1},\ x_{2n+2})$

$\leq a\ d(x_{2n},\ x_{2n+1}) + b\ \dfrac{d(x_{2n},\ x_{2n+1})d(x_{2n+1},\ x_{2n+2})}{1 + d(x_{2n+1},\ x_{2n+2})} + f\ d(x_{2n},\ x_{2n+1}) + g\ d(x_{2n+1},\ x_{2n+2})$

$= a\ d(x_{2n},\ x_{2n+1}) + b\ d(x_{2n},\ x_{2n+1}) + f\ d(x_{2n},\ x_{2n+1}) + g\ d(x_{2n+1},\ x_{2n+2}).$

Therefore,

$$(1-g)d(x_{2n+1},\ x_{2n+2}) \leq (a+b+f)\ d(x_{2n},\ x_{2n+1}).$$

That is,

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{(a+b+f)}{(1-g)}\ d(x_{2n}, x_{2n+1}) = p\ d(x_{2n}, x_{2n+1}), \tag{3.3}$$

where $p = \frac{(a+b+f)}{(1-g)} < 1$. Similarly, as $\alpha(x_{2n+1},\ x_{2n+2}) \geq 1$, by assumption (iii) and using (3.1) and (3.2), we have

$d(x_{2n+3},\ x_{2n+2}) = D(Sx_{2n+2},\ x_{2n+2}) \leq H(Sx_{2n+2},\ Tx_{2n+1})$

$\leq a\ d(x_{2n+2},\ x_{2n+1}) + f\ D(x_{2n+2},\ Sx_{2n+2}) + g\ D(x_{2n+1},\ Tx_{2n+1}) +$

$+ b\ \dfrac{D(x_{2n+2},\ Sx_{2n+2})D(x_{2n+1},\ Tx_{2n+1}) + D(x_{2n+1},\ Sx_{2n+2})D(x_{2n+2},\ Tx_{2n+1})}{1 + \delta(x_{2n+2},\ Tx_{2n+1}) + d(x_{2n+2},\ x_{2n+1}) + \delta(x_{2n+1},\ Sx_{2n+2})}$

$\leq a\ d(x_{2n+2},\ x_{2n+1}) + b\ \dfrac{d(x_{2n+2},\ x_{2n+3})d(x_{2n+1},\ x_{2n+2}) + d(x_{2n+1},\ x_{2n+3})d(x_{2n+2},\ x_{2n+2})}{1 + d(x_{2n+2},\ x_{2n+2}) + d(x_{2n+2},\ x_{2n+1}) + d(x_{2n+1},\ x_{2n+3})}$

$+ f\ d(x_{2n+2},\ x_{2n+3}) + g\ d(x_{2n+1},\ x_{2n+2})$

$= a\ d(x_{2n+2},\ x_{2n+1}) + b\ \dfrac{d(x_{2n+2},\ x_{2n+3})d(x_{2n+1},\ x_{2n+2})}{1 + d(x_{2n+2},\ x_{2n+1}) + d(x_{2n+1},\ x_{2n+3})}$

$+ f\ d(x_{2n+2},\ x_{2n+3}) + g\ d(x_{2n+1},\ x_{2n+2})$

$\leq a\ d(x_{2n+2},\ x_{2n+1}) + b\ \dfrac{d(x_{2n+2},\ x_{2n+3})d(x_{2n+1},\ x_{2n+2})}{1 + d(x_{2n+2},\ x_{2n+3})} + f\ d(x_{2n+2},\ x_{2n+3}) + g\ d(x_{2n+1},\ x_{2n+2})$

$= a\ d(x_{2n+2},\ x_{2n+1}) + b\ d(x_{2n+1},\ x_{2n+2}) + f\ d(x_{2n+2},\ x_{2n+3}) + g\ d(x_{2n+1},\ x_{2n+2}).$

Therefore,

$$(1-f)d(x_{2n+3},\ x_{2n+2}) \leq (a+b+g)\ d(x_{2n+2},\ x_{2n+1}).$$

That is,

$$d(x_{2n+3}, x_{2n+2}) \leq \frac{(a+b+g)}{(1-f)}\ d(x_{2n+2}, x_{2n+1}) = q\ d(x_{2n+2}, x_{2n+1}), \tag{3.4}$$

where $q = \frac{(a+b+g)}{(1-f)} < 1$. Late $k = \text{Max}\ \{p,\ q\}$, then $k < 1$. So by (3.3) and (3.4), we have

$$d(x_{n+1},\ x_{n+2}) \leq k\ d(x_n,\ x_{n+1}). \tag{3.5}$$

By repeated application of (3.5), we have

$$d(x_{n+1},\ x_{n+2}) \leq k\ d(x_n,\ x_{n+1}) \leq k^2\ d(x_{n-1},\ x_n) \leq ... \leq k^{n+1}\ d(x_0,\ x_1). \tag{3.6}$$

For $m > n > 0$, we have from (3.6),

$$\begin{aligned} d(x_n,\ x_m) &\leq d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2}) + ... + d(x_{m-1},\ x_m) \\ &\leq [k^n + k^{n+1} + ... + k^{m-1}]d(x_0,\ x_1) \\ &\leq \frac{k^n}{(1-k)} d(x_0,\ x_1). \end{aligned} \tag{3.7}$$

Taking the limit as $m,\ n \to \infty$, in (3.7), we have

$$\lim_{m,\ n\to\infty} d(x_n,\ x_m) = 0.$$

So $\{x_n\}$ is a Cauchy sequence in X. As $(X,\ d)$ is complete, there exists $x \in X$, such that

$$\lim_{n\to\infty} x_n = x. \tag{3.8}$$

We prove that this x is a common fixed point of S and T.
As $\alpha(x_n,\ x_{n+1}) \geq 1$, by the condition (i), we have $\alpha(x_n,\ x) \geq 1$, for all n. Since $\alpha(x_{2n+1},\ x) \geq 1$, by equation (3.1) and the condition (iii), we have

$$\begin{aligned}
&D(Sx,\ x_{2n+2}) \leq H(Sx,\ Tx_{2n+1})\\
&\leq a\ d(x,\ x_{2n+1}) + b\ \frac{D(x,\ Sx)D(x_{2n+1},\ Tx_{2n+1}) + D(x_{2n+1},\ Sx)D(x\ Tx_{2n+1})}{1 + \delta(x,\ Tx_{2n+1}) + d(x,\ x_{2n+1}) + \delta(x_{2n+1},\ Sx)}\\
&\quad + f\ D(x,\ Sx) + g\ D(x_{2n+1},\ Tx_{2n+1})\\
&\leq a\ d(x,\ x_{2n+1}) + b\ \frac{D(x,\ Sx)d(x_{2n+1},\ x_{2n+2}) + D(x_{2n+1},\ Sx)d(x,\ x_{2n+2})}{1 + d(x,\ x_{2n+2}) + d(x,\ x_{2n+1}) + D(x_{2n+1},\ Sx)}\\
&\quad + f\ D(x,\ Sx) + g\ d(x_{2n+1},\ x_{2n+2})
\end{aligned}$$

taking the limit as $n \to \infty$ in the above inequality and applying (3.8) we have

$$D(Sx,\ x) \leq f\ D(x,\ Sx)$$

implies $D(Sx,\ x) = 0$. Since Sx is compact so it is closed. Hence $x \in \overline{Sx} = Sx$, where $\overline{Sx}$ is the closure of Sx. Therefore, x is a fixed point of S.

Again as $\alpha(x_{2n+2},\ x) \geq 1$, so by equation (3.1) and the condition (iii), we have

$$\begin{aligned}
&D(x_{2n+3},\ Tx) \leq H(Sx_{2n+2},\ Tx)\\
&\leq a\ d(x_{2n+2},\ x) + b\ \frac{D(x_{2n+2},\ Sx_{2n+2})D(x,\ Tx) + D(x,\ Sx_{2n+2})D(x_{2n+2},\ Tx)}{1 + \delta(x_{2n+2},\ Tx) + d(x_{2n+2},\ x) + \delta(x,\ Sx_{2n+2})}\\
&\qquad + f\ D(x_{2n+2},\ Sx_{2n+2}) + g\ D(x,\ Tx)\\
&\leq a\ d(x_{2n+2},\ x) + b\ \frac{d(x_{2n+2},\ x_{2n+3})D(x,\ Tx) + d(x,\ x_{2n+3})D(x_{2n+2},\ Tx)}{1 + D(x_{2n+2},\ Tx) + d(x_{2n+2},\ x) + d(x,\ x_{2n+3})}\\
&\qquad + f\ d(x_{2n+2},\ x_{2n+3}) + g\ D(x, T\ x).
\end{aligned}$$

Now taking limit as $n \to \infty$ and applying (3.8) , we have

$$D(x,\ Tx) \leq g\ D(x,\ Tx)$$

implies $D(x,\ Tx) = 0$. Since Tx is compact so it is closed. Hence $x \in \overline{Tx} = Tx$, where $\overline{Tx}$ is the closure of Tx. Hence x is a fixed point of T. Therefore, x is a common fixed point of S and T.

Example 3.1. Let $X = [0,\ 1]$ with the usual metric 'd' and $S,\ T : X \longrightarrow C(X)$ and $\alpha : X \times X \longrightarrow [0,\ \infty)$ be defined as follows:

$$Tx = [0,\ \frac{x}{16}], \qquad \text{forall} \quad x \in [0,\ 1]$$

and

$$Sx = [0,\ \frac{\sin x}{16}], \qquad \text{forall} \quad x \in [0,\ 1]$$

$$\alpha(x,\ y) = \begin{cases} 1, & \text{if } 0 \le x \le 1,\ 0 \le y \le \dfrac{1}{16}; \\ \dfrac{1}{e^{x+y}}, & \text{otherwise.} \end{cases}$$

Take $a = \dfrac{1}{2},\ b = \dfrac{1}{3},$ and $f = g = 0$, then all the conditions of Theorem 3.1 are satisfied and 0 is a common fixed point of S and T.

Example 3.2. Let $X = [0,\ 1]$ with the usual metric 'd'. Let $S,\ T : X \longrightarrow C(X)$ and $\alpha : X \times X \longrightarrow [0,\ \infty)$ be defined as follows:

$$Tx = [\frac{x^2}{4},\ 1], \qquad \text{forall } x \in [0,\ 1]$$

and

$$Sx = [\frac{x^2}{4(1+x)},\ 1], \qquad \text{forall } \ x \in [0,\ 1]$$

$$\alpha(x,\ y) = e^{x+y}, \qquad \text{forall } \ x,\ y \in [0,\ 1]$$

Take $a = \frac{1}{2}$, $b = f = g = \frac{1}{7}$, then all the conditions of Theorem 3.1 are satisfied and 0 and 1 are two common fixed points of S and T.

4. Consequences in Meir Keeler Khan type contraction

We have also obtain following Meir Keeler Khan type result for common fixed point.

Theorem 4.1. *Let $(X,\ d)$ is a complete metric space, $S,\ T : X \longrightarrow C(X)$ be two multivalued mappings and $\alpha : X \times X \longrightarrow [0,\ \infty)$ be a mapping. Suppose that (i) X has α - regular property; (ii) S, T are α- dominated by each other; (iii) there exist non-negative real numbers $a,\ b,\ f,\ g$ with $a+b+f+g < 1$ and for every $\epsilon > 0$ there exists $\delta(\epsilon)$ such that for all $x,\ y \in X$ with $\alpha(x,\ y) \ge 1$ or $\alpha(y,\ x) \ge 1$,*

$$\epsilon \le a\ d(x,y) + b\ \frac{D(x,Sx)D(y,Ty) + D(y,Sx)D(x,Ty)}{1 + \delta(x,\ Ty) + d(x,\ y) + \delta(y,\ Sx)} + f\ D(x,Sx) + g\ D(y,Ty) < \epsilon + \delta(\epsilon)$$

implies $H(Sx,\ Ty) < \epsilon$. Then $S,\ T$ have a common fixed point in X.

Proof. Result follows from the fact

$$H(Sx,\ Ty) \le \epsilon \le a\ d(x,\ y) + b\ \frac{D(x,\ Sx)D(y,\ Ty) + D(y,\ Sx)D(x,\ Ty)}{1 + \delta(x,\ Ty) + d(x,\ y) + \delta(y,\ Sx)} + f\ D(x,\ Sx) + g\ D(y,\ Ty).$$

5. Application on partial ordered metric space

We prove our next result in ordered metric space.

Theorem 5.1. *Let $(X,\ d)$ is a complete metric space with a partial order $\preceq$ and $S,\ T : X \longrightarrow C(X)$ be two multivalued mappings. Suppose that i) X has regular property, ii) S, T are dominated by each other, iii) there exist non-negative real numbers a, b, f, g with $a+b+f+g<1$ such that for all comparable $x,\ y \in X$*

$$H(Sx,\ Ty) \leq a.d(x,\ y) + b\ \frac{D(x,\ Sx)D(y,\ Ty) + D(y,\ Sx)D(x,\ Ty)}{1+\delta(x,\ Ty)+d(x,\ y)+\delta(y,\ Sx)}$$

$$+ f\ D(x,\ Sx) + g\ D(y,\ Ty).$$

Then S and T have a common fixed point in X.

Proof. Let us define a mapping $\alpha : X \times X \longrightarrow [0,\ \infty)$ by $\alpha(x,\ y) = \begin{cases} 1, & \text{if } y \preceq x \\ 0, & \text{otherwise} \end{cases}$.

We take the assumption (i) of the theorem, Suppose that $\{x_n\}$ is a monotonic increasing (or decreasing) and convergent sequence in X with $x_n \to x$. Then $x_n \preceq x$ (resp. $x \preceq x_n$) for all n. By the definition of α, $\alpha(x_n,\ x_{n+1}) = 1$, for all n, and $x_n \preceq x$ (resp. $x \preceq x_n$) for all n, implies $\alpha(x_n,\ x) = 1$. Hence the assumption (i) of the above theorem reduces to the assumption (i) of Theorem 3.1.

Now by assumption (ii) of the theorem, $S,\ T$ are dominated by each other. So $x \in X$, $u \in Tx \implies v \preceq u$, for all $v \in Su$ and for $a \in Sy \implies b \preceq a$, for all $b \in Ta$. From the definition of α it follows that for $x \in X$, $u \in Tx \implies \alpha(u,\ v) = 1$, $v \in Su$ and $a \in Sx \implies \alpha(a,\ b) = 1$, $b \in Ta$. Hence the assumption (ii) of the theorem reduces to the assumption (ii) of Theorem 3.1.

It observed that,the inequality in assumption (iii) holds for all comparable elements x and y, i.e $y \preceq x$ or $x \preceq y$ implies that the inequality holds for all $x,\ y$ for which $\alpha(x,\ y) = 1$ or $\alpha(y,\ x) = 1$. Hence the assumption (iii) of the theorem reduces to the assumption (iii) of Theorem 3.1.

Therefore all the condition of theorem 3.1 are satisfied. Then by an application of theorem 3.1, we conclude that S and T have a common fixed point.

6. Application to the solution of system of nonlinear second kind Volterra type integral equations

Every singleton set $\{x\}$ is a member of $C(X)$, that is, $\{x\} \in C(X)$, for every $x \in X$. Let $S,\ T : X \to X$ be two multivalued mappings in which case Sx and Tx is a singleton set for every $x \in X$. Hence the following result is a special case of Theorem 3.1 when S and T are single valued mappings and $\alpha(x, y) = 1$, for all $x,\ y \in X$.

Theorem 6.1. *Let $(X,\ d)$ is a complete metric space and $S,\ T : X \longrightarrow X$ be two mappings. Suppose that there exist nonnegative real numbers a, b, f, g with $a+b+f+g<1$ such that for all $x, y \in X$*

$$d(Sx,\ Ty) \leq ad(x,\ y) + b\ \frac{d(x,\ Sx)d(y,\ Ty) + d(y,\ Sx)d(x,\ Ty)}{1+d(x,\ Ty)+d(x,\ y)+d(y,\ Sx)}$$

$$+ f\ d(x,\ Sx) + g\ d(y,\ Ty).$$

Then S and T have a common fixed point in X.

In this section, we shall prove the existence of solution for system of integral equations by using Theorem 6.1. Fixed point theorems are widely investigated and have found applications in differential and integral equations (see [15, 19] and references therein). We consider, here a system of nonlinear second kind Volterra type integral equation as follows

$$\left.\begin{array}{l} x(t) = f(t) + \int_0^t K(t,\ s,\ x(s))\ ds \ \text{ and} \\ x(t) = f(t) + \int_0^t J(t,\ s,\ x(s))\ ds,\ \text{where } t \in [0,\ \lambda], \end{array}\right\} \tag{6.1}$$

with $\lambda > 0$ and the unknown function $x(t)$ takes real values. Let $X = C([0,\ \lambda])$, where $\lambda > 0$ be the set of all real valued continuous functions defined on $[0,\ \lambda]$. It is well known that $C([0,\ \lambda])$ endowed with the metric

$$\rho(x,\ y) = \max_{t\in[0,\ \lambda]} \mid x(t) - y(t) \mid . \tag{6.2}$$

is a complete metric space.

Define two mappings $S,\ T : X \to X$ by

$$\left.\begin{array}{l} Sx(t) = f(t) + \int_0^t K(t,\ s,\ x(s))\ ds \ \text{ and} \\ Tx(t) = f(t) + \int_0^t J(t,\ s,\ x(s))\ ds,\ \text{where } t \in [0,\ \lambda] \end{array}\right\} \tag{6.3}$$

with $\lambda > 0$.

We designate the following assumptions by A_1 and A_2

A_1 : $f \in C([0,\ \lambda]$ and $J,\ K : [0,\ \lambda] \times [0,\ \lambda] \times \mathbb{R} \to \mathbb{R}$ are continuous mappings;

A_2 : $\mid K(t,s,x(s)) - J(t,s,y(s)) \mid \leq q \mid x(s) - y(s) \mid$, for all $x, y \in X$and $s,t \in [0,\lambda]$; where $a = q\ \lambda \in (0,1)$.

Theorem 6.2. Let $X = C([0,\ \lambda])$, $(X,\ \rho)$, $S,\ T,\ h,\ K(t,s,x(s)),\ J(t,s,x(s))$ satisfy the assumptions A_1 and A_2. Then system of nonlinear second kind Volterra type integral equation (6.1) has a solution $w \in C([0,\ \lambda])$.

Proof. Consider the functions $S,\ T : X \to X$ defined by (6.3). From assumptions A_1 and A_2, for all $x,\ y \in C([0,\ \lambda])$, we have

$$\begin{aligned} &\mid Sx(t) - Ty(t) \mid = \mid \int_0^t (K(t,\ s,\ x(s)) - J(t,\ s,\ y(s)))ds \mid \\ &= \int_0^t \mid K(t,\ s,\ x(s)) - J(t,\ s,\ y(s)) \mid ds \leq \int_0^t q \mid x(s) - y(s) \mid ds \\ &= q \max_{s\in[0,\ \lambda]} \mid x(s) - y(s) \mid \int_0^t 1 \times ds = q\ t \max_{s\in[0,\ \lambda]} \mid x(s) - y(s) \mid \quad [\text{ by } A_2] \\ &\leq q\ \lambda \max_{s\in[0,\ \lambda]} \mid x(s) - y(s) \mid = a \max_{s\in[0,\ \lambda]} \mid x(s) - y(s) \mid \leq a\ \rho(x,y), \quad \text{for all } t \in [0,\lambda]. \end{aligned}$$

Taking $b = f = g = 0$, we have for all $x,\ y \in X$

$$d(Sx,\ Ty) \leq a\, d(x,y) + b\, \frac{d(x,Sx)d(y,Ty) + d(y,Sx)d(x,Ty)}{1 + d(x,Ty) + d(x,y) + d(y,Sx)} + f\ d(x,\ Sx) + g\ d(y,\ Ty).$$

Hence, all the conditions of Theorem 6.1 are satisfied. Therefore, Theorem 6.1 guarantees the existence of a common fixed point w of S and T in X. That is, w is a common solution of system of integral equations (6.1) in $C([0,\ \lambda])$.

7. Application to the solution of system of functional equations

The solutions of functional equations and system of functional equations using different fixed point theorems have been studied in [10] and reference there in. We will apply Theorem 6.1 to prove the existence of solution system of functional equations.

Through out this section, we assume U and V are Banach spaces, $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space.

We consider the following system of functional equations

$$\left.\begin{array}{l} x(\kappa) = \sup_{\mu \in D}\{p(\kappa,\ \mu) + g(\kappa,\ \mu,\ x(\eta(\kappa,\ \mu)))\},\ \kappa \in W \text{ and} \\ y(\kappa) = \sup_{\mu \in D}\{p(\kappa,\ \mu) + k(\kappa,\ \mu,\ y(\eta(\kappa,\ \mu)))\},\ \kappa \in W, \end{array}\right\} \tag{7.1}$$

where $\eta : W \times D \to W$, $p,\ q : W \times D \to \mathbb{R}$ and $g,\ k : W \times D \times \mathbb{R} \to \mathbb{R}$. Equations in (7.1) find their application in mathematical optimization, computer programming and in dynamic programming, which gives tools for solving boundary value problems arising in engineering and physical science.

Let $B(W)$ denote the set of all real valued bounded functions defined on a nonempty Banach spaces W, that is, $x \in B(W)$ if and only if $x : W \to \mathbb{R}$ is a bounded function. It is well known that $B(W)$ endowed with the metric

$$\varrho(x,\ y) = \sup_{\kappa \in W} |\ x(\kappa) - y(\kappa)\ | \quad \text{for all } \ x,\ y \in B(W) \tag{7.2}$$

is a complete metric space.

We designate the following assumptions by F_1 and F_2

F_1 : $p : W \times D \to \mathbb{R}$ and $g,\ k : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions.

F_2 : for $r_1,\ r_2 \in \mathbb{R}$ and $\kappa \in W,\ \mu \in D$ imply

$$0 \le g(\kappa,\ \mu,\ r_1) - k(\kappa,\ \mu,\ r_2) \le a\ [r_1 - r_2], \quad \text{where } \ a \in [0,\ 1).$$

Theorem 7.1. Let $(X,\ d) = (B_\varrho(W),\ \varrho)$, $g,\ k,\ p,\ \eta,\ \mu$ satisfy all the assumptions F_1 and F_2. Then system of functional equations (7.1) has a common bounded solution in W.

Proof. Define two mappings $S,\ T : B_\varrho(W) \to B_\varrho(W)$ by

$$\left.\begin{array}{l} S(\omega)(\kappa) = \sup_{\mu \in D}\{p(\kappa,\ \mu) + g(\kappa,\ \mu,\ \omega(\eta(\kappa,\ \mu)))\}, \quad \text{and} \\ T(\omega)(\kappa) = \sup_{\mu \in D}\{p(\kappa,\ \mu) + k(\kappa,\ \mu,\ \omega(\eta(\kappa,\ \mu)))\},\ \kappa \in W, \end{array}\right\} \tag{7.3}$$

Let λ be an arbitrary positive number, $\kappa \in W$ and $\omega_1,\ \omega_2 \in X$. Then there exist $\mu_1,\ \mu_2 \in D$ such that

$$S(\omega_1)(\kappa) < p(\kappa,\ \mu_1) + g(\kappa,\ \mu_1,\ \omega_1(\eta(\kappa,\ \mu_1))) + \lambda. \tag{7.4}$$

$$T(\omega_2)(\kappa) < p(\kappa,\ \mu_2) + k(\kappa,\ \mu_2,\ \omega_2(\eta(\kappa,\ \mu_2))) + \lambda. \tag{7.5}$$

Also from the definition of S and T, we have

$$S(\omega_1)(\kappa) \ge p(\kappa,\ \mu_2) + g(\kappa,\ \mu_2,\ \omega_1(\eta(\kappa,\ \mu_2))). \tag{7.6}$$

$$T(\omega_2)(\kappa) \ge p(\kappa,\ \mu_1) + k(\kappa,\ \mu_1,\ \omega_2(\eta(\kappa,\ \mu_1))). \tag{7.7}$$

Using (7.4) and (7.7), we have

$$\begin{aligned} S(\omega_1)(\kappa) - T(\omega_2)(\kappa) &< g(\kappa,\ \mu_1,\ \omega_1(\eta(\kappa,\ \mu_1))) - k(\kappa,\ \mu_1,\ \omega_2(\eta(\kappa,\ \mu_1))) + \lambda \\ &\leq \mid g(\kappa,\ \mu_1,\ \omega_1(\eta(\kappa,\ \mu_1))) - k(\kappa,\ \mu_1,\ \omega_2(\eta(\kappa,\ \mu_1))) \mid + \lambda \\ &\leq a\ \mid \omega_1(\eta(\kappa,\ \mu_1)) - \omega_2(\eta(\kappa,\ \mu_1)) \mid + \lambda.\ [\text{ using } F_2\ ] \end{aligned} \tag{7.8}$$

Analogously, using (7.5) and (7.6), we have

$$T(\omega_2)(\kappa) - S(\omega_1)(\kappa) < a\ \mid \omega_1(\eta(\kappa,\ \mu_2)) - \omega_2(\eta(\kappa,\ \mu_2)) \mid + \lambda. \tag{7.9}$$

Therefore, by (7.8) and (7.9), we have

$$\varrho(S(\omega_1),\ T(\omega_2)) \leq a\ \varrho(\omega_1,\ \omega_2) + \lambda.$$

Since $\lambda > 0$ be arbitrary, we obtain

$$\varrho(S(\omega_1),\ T(\omega_2)) \leq a\ \varrho(\omega_1,\ \omega_2).$$

Taking $b = f = g = 0$, for all $\omega_1,\ \omega_2 \in X$, the above relation can be written as

$$\varrho(S(\omega_1),\ T(\omega_2)) \leq a\ \varrho(\omega_1,\ \omega_2) + b\ \frac{\varrho(\omega_1, S(\omega_1))\varrho(\omega_2, T(\omega_2)) + \varrho(\omega_2, S(\omega_1))\varrho(\omega_1, T(\omega_2)}{1 + \varrho(\omega_1, T(\omega_2)) + \varrho(\omega_1, \omega_2) + \varrho(\omega_2, S(\omega_1))}$$
$$+ f\ \varrho(\omega_1,\ S(\omega_1)) + g\ \varrho(\omega_2,\ T(\omega_2)).$$

Hence, all the conditions of Theorem 6.1 are satisfied. Therefore, Theorem 6.1 guarantees the existence of a common fixed point x of S and T in X. That is, system of functional equations (7.1) has a common bounded solution in W.

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