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Location of zeros of a Lacunary type polynomial

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Abstract. For a given polynomial $p(z)$ of degree n with real or complex coefficients, our basic aim is to determine the smallest region in which all the zeros of $p(z)$ lie. In the present paper, we have obtained a result by using Lacunary type polynomial which gives the region of zeros neither circular nor annular except in some particular cases. Our result plays an important role to reduce the region of polynomial zeros.

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1. Introduction and statement of results

Concerning the region for the location of zeros of a polynomial

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

of degree n with real or complex coefficients, the results obtained by Lagrange [9] (see [1, Theorem 1.1, p. 19]) and Cauchy [2] (see also [10, Ch. VIII, Sect. 27, Theorem 27.1, p. 122]) are well known. In fact, applying Cauchy result on the polynomial $q(z) = z^n p\left(\frac{1}{z}\right)$, one can easily obtain a circular region with centre origin in which no zeros of $p(z)$ lie when $a_n \neq 0$, and consequently, we can restate the Cauchy result as follows.

Theorem 1.1. *All the zeros of $p(z)$ with $a_n \neq 0$ lie in the ring shaped region*

$$\frac{1}{r_0'} \leq |z| \leq r_0,$$

where r_0, r_0' are the unique positive roots of the equations

$$h(t) \equiv |a_0| t^n - |a_1| t^{n-1} - |a_2| t^{n-2} - \cdots - |a_{n-1}| t - |a_n| = 0$$

and

$$h'(t) \equiv |a_n| t^n - |a_{n-1}| t^{n-1} - |a_{n-2}| t^{n-2} - \cdots - |a_1| t - |a_0| = 0$$

respectively.

In the literature, there are some results (see [3, 4, 5, 6, 7, 8, 11, 13]) dealing with the refinement and improvement of the Cauchy result that have been published to determine the circular region for estimating the location of zeros of a polynomial.

In this paper, we have obtained a result for the location of polynomial zeros by using Lacunary type of polynomial [10, Ch. VIII, Sect. 34, pp. 156] of the form

$$P(z) = a_0 z^n + a_\lambda z^{n-\lambda} + \cdots + a_{n-1} z + a_n,\ 1 \leq \lambda \leq n$$

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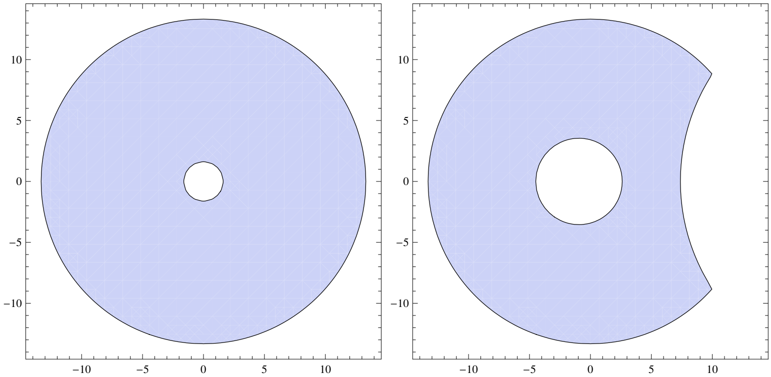

FIGURE 1. The left side region obtain by Theorem 1.1 whereas the right side region obtain by applying Theorem 1.2 on it.

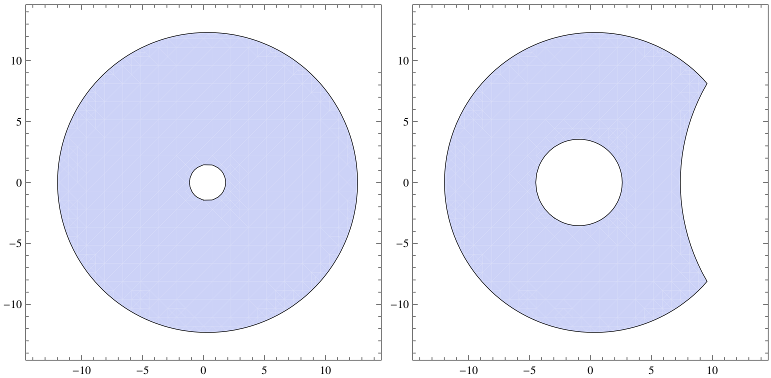

FIGURE 2. The left side region obtain by [3, Theorem 3] whereas the right side region obtain by applying Theorem 1.2 on it.

Using our result we can produce a region in which the zeros of a polynomial are simple. Also, we present some application of our result to produce more approximate region which is explained by few examples mentioned in the section 3. More precisely, we prove

Theorem 1.2. *Let*

$$P(z) = a_0 z^n + a_\lambda z^{n-\lambda} + \cdots + a_{n-1}z + a_n, \;\; 1 \le \lambda \le n, \;\; |a_n| \neq 0, |a_\lambda| \neq 0$$

be a polynomial of degree n with real or complex coefficients. Then all the zeros of $P(z)$ lie in the following region:

(i) $\left\{z : |z| \ge \frac{1}{t_0}\right\}$, *if* $\zeta_0 = 0$,

(ii) $\left\{z : \zeta_0 z + \overline{\zeta_0}\overline{z} \ge 1\right\}$, *if* $t_0 = |\zeta_0|\,(\neq 0)$,

(iii) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| \ge \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$, *if* $t_0 > |\zeta_0|\,(\neq 0)$,

(iv) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| \le \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$, *if* $t_0 < |\zeta_0|\,(\neq 0)$,
where t_0 *is an unique positive root of the equation*

$$f(t) \equiv |a_n|\,t^n - |b_2|\,t^{n-2} - |b_3|\,t^{n-3} - \cdots - |b_{n-1}|\,t - |b_n| = 0,$$

$$\begin{aligned} b_p &= \binom{n}{p} a_n \zeta_0^p + \binom{n-1}{p-1} a_{n-1} \zeta_0^{p-1} + \cdots + \binom{n-j}{p-j} a_{n-j} \zeta_0^{p-j} \\ &\quad + \cdots + \binom{n-p+1}{1} a_{n-p+1} \zeta_0 + \binom{n-p}{0} a_{n-p}, \; p = 2, \ldots, n; \end{aligned}$$

$$a_k = 0, \; k = 1, 2, \ldots, \lambda - 1; \; \zeta_0 = -\frac{a_{n-1}}{n a_n} \; \textit{with} \; \binom{0}{0} = 1.$$

Moreover, if $|b_n| \neq 0$*, then zeros of* $P(z)$ *lie in the following region:*

(i) $\left\{z : |z| = \frac{1}{t_0}\right\}$, *if* $t_0 t_0' = 1, |\zeta_0| = 0$,

(ii) $\left\{z : \zeta_0 z + \overline{\zeta_0}\,\overline{z} = 1\right\}$, *if* $t_0 t_0' = 1, t_0 = |\zeta_0|\,(\neq 0)$,

(iii) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| = \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$, *if* $t_0 t_0' = 1, t_0 \neq |\zeta_0|\,(\neq 0)$,

(iv) $\left\{z : \zeta_0 z + \overline{\zeta_0}\,\overline{z} \le 1, \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| \ge \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$,
if $t_0 t_0' > 1, \frac{1}{t_0'} = |\zeta_0|\,(\neq 0)$,

(v) $\left\{z : \zeta_0 z + \overline{\zeta_0}\,\overline{z} \ge 1, \left|z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right| \ge \left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right|\right\}$,
if $t_0 t_0' > 1, t_0 = |\zeta_0|\,(\neq 0)$,

(vi) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right| \le \left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right|, \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| \ge \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$,
if $t_0 t_0' > 1, \frac{1}{t_0'} > |\zeta_0|\,(\neq 0)$,

(vii) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right| \ge \left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right|, \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| \ge \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$,
if $t_0 t_0' > 1, \frac{1}{t_0'} < |\zeta_0|\,(\neq 0) < t_0$,

(viii) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| \le \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|, \left|z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right| \ge \left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right|\right\}$,
if $t_0 t_0' > 1, t_0 < |\zeta_0|\,(\neq 0)$,

(ix) $\left\{z : \frac{1}{t_0} \le |z| \le t_0'\right\}$, *if* $t_0 t_0' > 1, |\zeta_0| = 0$,
where t_0' *is an unique positive root of the equation*

$$f'(t) \equiv |b_n|\,t^n - |b_{n-1}|\,t^{n-1} - |b_{n-2}|\,t^{n-2} - \cdots - |b_2|\,t^2 - |a_n| = 0.$$

Using Theorem 1.2, we can easily obtain a region in which the zeros of $P(z)$ are simple.

Theorem 1.3. *Let*

$$P(z) = a_0 z^n + a_\lambda z^{n-\lambda} + \cdots + a_{n-1} z + a_n, \; 1 \le \lambda \le n, \; |a_{n-1}| \neq 0$$

be a polynomial of degree n with real or complex coefficients. Then the zeros of $P(z)$ are simple in the following region:

(i) $\left\{z:|z|<\frac{1}{t_0}\right\}$, *if* $\zeta_0=0$,

(ii) $\left\{z:\zeta_0 z+\overline{\zeta_0}\overline{z}<1\right\}$, *if* $t_0=|\zeta_0|\,(\neq 0)$,

(iii) $\left\{z:\left|z+\frac{\overline{\zeta_0}}{(t_0^2-|\zeta_0|^2)}\right|<\left|\frac{t_0}{(t_0^2-|\zeta_0|^2)}\right|\right\}$, *if* $t_0>|\zeta_0|\,(\neq 0)$,

(iv) $\left\{z:\left|z+\frac{\overline{\zeta_0}}{(t_0^2-|\zeta_0|^2)}\right|>\left|\frac{t_0}{(t_0^2-|\zeta_0|^2)}\right|\right\}$, *if* $t_0<|\zeta_0|\,(\neq 0)$,

where t_0 *is an unique positive root of the equation*

$$f(t)\equiv|a_{n-1}|\,t^{n-1}-|b_2|\,t^{n-3}-|b_3|\,t^{n-4}-\cdots-|b_{n-2}|\,t-|b_{n-1}|=0,$$

$$\begin{aligned} b_p \;=\;& \binom{n-1}{p}a_{n-1}\zeta_0^p+2\binom{n-2}{p-1}a_{n-2}\zeta_0^{p-1}+\cdots+(j+1)\binom{n-j-1}{p-j}a_{n-j-1}\zeta_0^{p-j} \\ &+\cdots+p\binom{n-p}{1}a_{n-p}\zeta_0+(p+1)\binom{n-p-1}{0}a_{n-p-1},\; p=2,\ldots,n-1; \end{aligned}$$

$$a_k=0,\; k=1,2,\ldots,\lambda-1;\; \zeta_0=-\frac{2a_{n-2}}{(n-1)\,a_{n-1}} \text{ with } \binom{0}{0}=1.$$

Moreover, if $|b_{n-1}|\neq 0$, *then zeros of* $P(z)$ *are simple in the region:*

(i) $\left\{z:|z|>\frac{1}{t_0}\right\}\cup\left\{z:|z|<\frac{1}{t_0}\right\}$, *if* $t_0t_0'=1, |\zeta_0|=0$,

(ii) $\left\{z:\zeta_0 z+\overline{\zeta_0}\overline{z}>1\right\}\cup\left\{z:\zeta_0 z+\overline{\zeta_0}\overline{z}<1\right\}$, *if* $t_0t_0'=1, t_0=|\zeta_0|\,(\neq 0)$,

(iii) $\left\{z:\left|z+\frac{\overline{\zeta_0}}{(t_0^2-|\zeta_0|^2)}\right|>\left|\frac{t_0}{(t_0^2-|\zeta_0|^2)}\right|\right\}\cup\left\{z:\left|z+\frac{\overline{\zeta_0}}{(t_0^2-|\zeta_0|^2)}\right|<\left|\frac{t_0}{(t_0^2-|\zeta_0|^2)}\right|\right\}$,
if $t_0t_0'=1, t_0\neq|\zeta_0|\,(\neq 0)$,

(iv) $\left\{z:\zeta_0 z+\overline{\zeta_0}\overline{z}>1\right\}\cup\left\{z:\left|z+\frac{\overline{\zeta_0}}{(t_0^2-|\zeta_0|^2)}\right|<\left|\frac{t_0}{(t_0^2-|\zeta_0|^2)}\right|\right\}$,
if $t_0t_0'>1, \frac{1}{t_0'}=|\zeta_0|\,(\neq 0)$,

(v) $\left\{z:\zeta_0 z+\overline{\zeta_0}\overline{z}<1\right\}\cup\left\{z:\left|z+\frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2}-|\zeta_0|^2\right)}\right|<\left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2}-|\zeta_0|^2\right)}\right|\right\}$,
if $t_0t_0'>1, t_0=|\zeta_0|\,(\neq 0)$,

(vi) $\left\{z:\left|z+\frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2}-|\zeta_0|^2\right)}\right|>\left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2}-|\zeta_0|^2\right)}\right|\right\}\cup\left\{z:\left|z+\frac{\overline{\zeta_0}}{(t_0^2-|\zeta_0|^2)}\right|<\left|\frac{t_0}{(t_0^2-|\zeta_0|^2)}\right|\right\}$,
if $t_0t_0'>1, \frac{1}{t_0'}>|\zeta_0|\,(\neq 0)$,

(vii) $\left\{z:\left|z+\frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2}-|\zeta_0|^2\right)}\right|<\left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2}-|\zeta_0|^2\right)}\right|\right\}\cup\left\{z:\left|z+\frac{\overline{\zeta_0}}{(t_0^2-|\zeta_0|^2)}\right|<\left|\frac{t_0}{(t_0^2-|\zeta_0|^2)}\right|\right\}$,
if $t_0t_0'>1, \frac{1}{t_0'}<|\zeta_0|\,(\neq 0)<t_0$,

(viii) $\left\{z:\left|z+\frac{\overline{\zeta_0}}{(t_0^2-|\zeta_0|^2)}\right|>\left|\frac{t_0}{(t_0^2-|\zeta_0|^2)}\right|\right\}\cup\left\{z:\left|z+\frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2}-|\zeta_0|^2\right)}\right|<\left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2}-|\zeta_0|^2\right)}\right|\right\}$,
if $t_0t_0'>1, t_0<|\zeta_0|\,(\neq 0)$,

(ix) $\left\{z:|z|<\frac{1}{t_0}\right\}\cup\left\{z:|z|>t_0'\right\}$, *if* $t_0t_0'>1, |\zeta_0|=0$, *where t_0' is an unique positive root of the equation*

$$f'(t) \equiv |b_{n-1}|\, t^{n-1} - |b_{n-2}|\, t^{n-2} - |b_{n-3}|\, t^{n-3} - \cdots - |b_2|\, t^2 - |a_{n-1}| = 0.$$

For example, we consider a class of polynomials define by

$$\Omega = \left\{ P_n(z) = z^{n+1} - (n+1)\, z + n : n \in N \right\},$$

where N is a set of natural numbers. With the help of Theorem 1.3, we can say that the zeros of any polynomial of Ω are simple in the region

$$\{z : |z| < 1\} \cup \{z : |z| > 1\},$$

which can be seen by observing that $t_0 = 1$, $t_0' = 1$ and $\zeta_0 = 0$ respectively. We also observe that $z = 1$ is only a multiple zero of multiplicity 2 for each polynomial of Ω.

Here we note that for $\lambda = 1$, $P(z)$ reduces to $p(z)$. So our results are also applicable for the polynomial $p(z)$.

2. Proof of Theorems

Proof of Theorem 1.2. First of all, we consider a transformation

$$\zeta = L(z), \; L(z) = \frac{1}{z}$$

from z-plane to ζ-plane. Using this transformation, $P(z)$ becomes

$$P\left(\frac{1}{\zeta}\right) = \frac{T(\zeta)}{\zeta^n}, \; T(\zeta) = a_n \zeta^n + a_{n-1}\zeta^{n-1} + \cdots + a_\lambda \zeta^\lambda + a_0$$

in the ζ- plane.

Here all the coefficients of $P(z)$ are finite with $|a_n| \neq 0$ and zeros of $P(z)$ are reciprocal of zeros of $T(\zeta)$ which give $0, \infty$, neither a zero of $P(z)$ nor $T(\zeta)$. Now we construct an Entire Linear Transformation from ζ-plane to η-plane define by

$$\eta = L'(\zeta), \; L'(\zeta) = \zeta - \zeta_0,$$

where the complex number ζ_0 is to be determined for which the polynomial $T(\zeta)$ in the ζ-plane becomes $R(\eta)$ under the transformation $\eta = L'(\zeta)$ in the η-plane with the property that the coefficient of η^{n-1} in $R(\eta)$ is absent. Clearly

$$R(\eta) = a_n \eta^n + b_1 \eta^{n-1} + b_2 \eta^{n-2} + \cdots + b_{n-1}\eta + b_n,$$

where

$$\begin{aligned} b_p \;=\; & \binom{n}{p} a_n \zeta_0^p + \binom{n-1}{p-1} a_{n-1}\zeta_0^{p-1} + \cdots + \binom{n-j}{p-j} a_{n-j}\zeta_0^{p-j} \\ & + \cdots + \binom{n-p+1}{1} a_{n-p+1}\zeta_0 + \binom{n-p}{0} a_{n-p}, \; p = 1, 2, \ldots, n; \end{aligned}$$

$$a_k = 0, k = 1, 2, \ldots, (\lambda - 1) \text{ with } \binom{0}{0} = 1.$$

As $b_1 = 0$ i.e., $n a_n \zeta_0 + a_{n-1} = 0$ which gives $\zeta_0 = -\frac{a_{n-1}}{n a_n}$, and consequently we have

$$R(\eta) = a_n \eta^n + b_2 \eta^{n-2} + \cdots + b_{n-1}\eta + b_n$$

Again for some $|\eta| > 0$,

$$|R(\eta)| \geq |a_n| |\eta|^n - |b_2| |\eta|^{n-2} - \cdots - |b_{n-1}| |\eta| - |b_n| .$$

Now, we introduce a function

$$f(t) = |a_n| t^n - |b_2| t^{n-2} - \cdots - |b_{n-1}| t - |b_n| .$$

By Descartes' rule of sign, $f(t) = 0$ has an unique positive root, say t_0, which shows that

$$f(t) > 0 \text{ if } t > t_0.$$

Consequently, all the zeros of $R(\eta)$ lie in the circular region

$$|\eta| \leq t_0.$$

As the inverse transformation $\zeta = L'^{-1}(\eta) = \eta + \zeta_0$ of $\eta = L'(\zeta) = \zeta - \zeta_0$ is an entire linear and it preserves the shape, so we get all the zeros of $T(\zeta)$ lie in the circular region

$$D : |\zeta - \zeta_0| \leq t_0$$

with the boundary $\Gamma_D : |\zeta - \zeta_0| = t_0$ in the ζ-plane.
Now, the circle Γ_D can be written as

$$\zeta\overline{\zeta} - \overline{\zeta_0}\zeta - \zeta_0\overline{\zeta} + |\zeta_0|^2 = t_0^2.$$

For finding the image of the circle Γ_D under $z = L^{-1}(\zeta) = \frac{1}{\zeta}$, we replace ζ by $\frac{1}{z}$ in the above equation and consequently we get

$$\frac{1}{z}\frac{1}{\overline{z}} - \overline{\zeta_0}\frac{1}{z} - \zeta_0\frac{1}{\overline{z}} = t_0^2 - |\zeta_0|^2 .$$

For $t_0 \neq |\zeta_0|$,

$$z\overline{z} + \frac{\zeta_0}{\left(t_0^2 - |\zeta_0|^2\right)} z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \overline{z} = \frac{1}{\left(t_0^2 - |\zeta_0|^2\right)}.$$

Adding both sides by $\frac{\zeta_0\overline{\zeta_0}}{\left(t_0^2-|\zeta_0|^2\right)^2}$, we have

$$z\overline{z} + \frac{\zeta_0}{\left(t_0^2-|\zeta_0|^2\right)} z + \frac{\overline{\zeta_0}}{\left(t_0^2-|\zeta_0|^2\right)} \overline{z} + \frac{\zeta_0\overline{\zeta_0}}{\left(t_0^2-|\zeta_0|^2\right)^2} = \frac{1}{\left(t_0^2-|\zeta_0|^2\right)} + \frac{\zeta_0\overline{\zeta_0}}{\left(t_0^2-|\zeta_0|^2\right)^2}$$

$$\text{or, } z\left(\overline{z} + \frac{\zeta_0}{\left(t_0^2-|\zeta_0|^2\right)}\right) + \frac{\overline{\zeta_0}}{\left(t_0^2-|\zeta_0|^2\right)}\left(\overline{z} + \frac{\zeta_0}{\left(t_0^2-|\zeta_0|^2\right)}\right) = \frac{1}{\left(t_0^2-|\zeta_0|^2\right)} + \frac{|\zeta_0|^2}{\left(t_0^2-|\zeta_0|^2\right)^2}$$

$$\text{or, } \left(z + \frac{\overline{\zeta_0}}{\left(t_0^2-|\zeta_0|^2\right)}\right)\left(\overline{z} + \frac{\zeta_0}{\left(t_0^2-|\zeta_0|^2\right)}\right) = \frac{t_0^2}{\left(t_0^2-|\zeta_0|^2\right)^2}$$

$$\text{or, } \left(z + \frac{\overline{\zeta_0}}{\left(t_0^2-|\zeta_0|^2\right)}\right)\overline{\left(z + \frac{\overline{\zeta_0}}{\left(t_0^2-|\zeta_0|^2\right)}\right)} = \frac{t_0^2}{\left(t_0^2-|\zeta_0|^2\right)^2}$$

$$\text{or, } \left|z + \frac{\overline{\zeta_0}}{\left(t_0^2-|\zeta_0|^2\right)}\right|^2 = \left|\frac{t_0}{\left(t_0^2-|\zeta_0|^2\right)}\right|^2$$

$$\text{or, } \left|z + \frac{\overline{\zeta_0}}{\left(t_0^2-|\zeta_0|^2\right)}\right| = \left|\frac{t_0}{\left(t_0^2-|\zeta_0|^2\right)}\right|.$$

In case of $t_0 = |\zeta_0|$,

$$\zeta_0 z + \overline{\zeta_0}\overline{z} = 1.$$

So, the image of Γ_D under $z = L^{-1}(\zeta) = \frac{1}{\zeta}$ is $L^{-1}(\Gamma_D) = \Lambda$, where

$$\Lambda = \begin{cases} \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| = \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right| & \text{a circle if } t_0 \neq |\zeta_0|, \\ \zeta_0 z + \overline{\zeta_0}\overline{z} = 1 & \text{a straight line if } t_0 = |\zeta_0|. \end{cases}$$

Clearly the zeros of $P(z)$ lie in $L^{-1}(D)$ in the z-plane where $L^{-1}(D)$ is the image of D under $z = L^{-1}(\zeta)$. Now we determine all the possibilities of $L^{-1}(D)$ which is depend on ζ_0 and t_0 as follows.

(i) For $\zeta_0 = 0$, the image of Γ_D under $z = L^{-1}(\zeta)$ is

$$\Lambda : |z| = \frac{1}{t_0} \ \left(\text{as } t_0 > 0, \text{ so } t_0 \neq |\zeta_0| \right).$$

As $0 \in Int(D)$ and the image of 0 goes to ∞ under $z = L^{-1}(\zeta)$, it imply

$$L^{-1}(D) = \left\{ z : |z| \geq \frac{1}{t_0} \right\}.$$

(ii) For $t_0 = |\zeta_0| \, (\neq 0)$, in this case,

$$\Lambda : \zeta_0 z + \overline{\zeta_0}\overline{z} = 1.$$

Clearly $\zeta_0 \in Int(D)$ and its image under $z = L^{-1}(\zeta)$ in the z-plane is

$$z_0 = \frac{1}{\zeta_0} = \frac{\overline{\zeta_0}}{|\zeta_0|^2}.$$

Also,

$$\zeta_0 z_0 + \overline{\zeta_0}\overline{z_0} = \zeta_0 \left(\frac{\overline{\zeta_0}}{|\zeta_0|^2} \right) + \overline{\zeta_0} \overline{\left(\frac{\overline{\zeta_0}}{|\zeta_0|^2} \right)} = \frac{\zeta_0 \overline{\zeta_0}}{|\zeta_0|^2} + \frac{\overline{\zeta_0}\,\overline{(\overline{\zeta_0})}}{|\zeta_0|^2} = 2 > 1,$$

which shows that

$$L^{-1}(D) = \left\{ z : \zeta_0 z + \overline{\zeta_0}\overline{z} \geq 1 \right\}.$$

(iii) For the case of $t_0 > |\zeta_0| \, (\neq 0)$, we have

$$\Lambda : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| = \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right|.$$

Since $0 \in Int(D)$ and the image of 0 goes to ∞ under $z = L^{-1}(\zeta)$, it give

$$L^{-1}(D) = \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \geq \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \right\}.$$

(iv) For $t_0 < |\zeta_0| \, (\neq 0)$, we have

$$\Lambda : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| = \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right|.$$

Clearly $0 \in Ext(D)$ and the image of 0 goes to ∞ under $z = L^{-1}(\zeta)$, we get

$$L^{-1}(D) = \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \leq \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \right\}.$$

In case of $b_n \neq 0$, we construct a polynomial in the η-plane define by

$$S(\eta) = \eta^n R\left(\frac{1}{\eta}\right)$$
$$= b_n\eta^n + b_{n-1}\eta^{n-1} + b_{n-2}\eta^{n-2} + \cdots + b_2\eta^2 + a_n$$

On $|\eta| > 0$,

$$|S(\eta)| \geq |b_n|\,|\eta|^n - |b_{n-1}|\,|\eta|^{n-1} - |b_{n-2}|\,|\eta|^{n-2} - \cdots - |b_2|\,|\eta|^2 - |a_n|\,.$$

Consider the equation

$$f'(t) \equiv |b_n|\,t^n - |b_{n-1}|\,t^{n-1} - |b_{n-2}|\,t^{n-2} - \cdots - |b_2|\,t^2 - |a_n| = 0.$$

Clearly $f'(t) = 0$ has exactly one positive root, say t'_0, and so

$$f'(t) > 0 \text{ when } t > t'_0,$$

which gives the zeros of $S(\eta)$ lie in the region

$$|\eta| \leq t'_0,$$

and therefore, all the zeros of $R(\eta)$ must be contained in the annular region

$$\frac{1}{t'_0} \leq |\eta| \leq t_0$$

in the η-plane.

Now, using the shape preserving property of an Entire Linear Transformation $\zeta = L'^{-1}(\eta) = \eta + \zeta_0$, we can easily say that the zeros of $T(\zeta)$ should lie in the region

$$\Omega : \frac{1}{t'_0} \leq |\zeta - \zeta_0| \leq t_0$$

with boundaries

$$\Gamma_D : |\zeta - \zeta_0| = t_0 \text{ and } \Gamma_\Omega : |\zeta - \zeta_0| = \frac{1}{t'_0}$$

in the ζ- plane.

Clearly, the images of Γ_D and Γ_Ω under $z = L^{-1}(\zeta) = \frac{1}{\zeta}$ are given by $L^{-1}(\Gamma_D) = \Lambda$ and $L^{-1}(\Gamma_\Omega) = \Delta$ respectively, where

$$\Lambda = \begin{cases} \left|z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)}\right| = \left|\frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)}\right| & \text{a circle if } t_0 \neq |\zeta_0|\,, \\ \zeta_0 z + \overline{\zeta_0}\,\overline{z} = 1 & \text{a straight line if } t_0 = |\zeta_0|\,. \end{cases}$$

and

$$\Delta = \begin{cases} \left|z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right| = \left|\frac{\frac{1}{t'_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right| & \text{a circle if } \frac{1}{t'_0} \neq |\zeta_0|\,, \\ \zeta_0 z + \overline{\zeta_0}\,\overline{z} = 1 & \text{a straight line if } \frac{1}{t'_0} = |\zeta_0|\,. \end{cases}$$

Also, the image of Ω under $z = L^{-1}(\zeta)$ is denoted by $L^{-1}(\Omega)$ which contains all the zeros of $P(z)$ in the z-plane.

Now we discuss about all the possibilities of $L^{-1}(\Omega)$ that depend on the values of t_0, t'_0, ζ_0 and the region

$$\Omega : \frac{1}{t'_0} \leq |\zeta - \zeta_0| \leq t_0$$

which are as follows.
(*i*) For $t_0t_0' = 1$ and $|\zeta_0| = 0$, in this case $\Gamma_D = \Gamma_\Omega : |\zeta| = t_0$ (as $t_0 > 0,\ t_0 \neq |\zeta_0|$), and $\Omega : |\zeta| = t_0$. So all the zeros of $P(z)$ lie in

$$L^{-1}(\Omega) = \left\{ z : |z| = \frac{1}{t_0} \right\}.$$

(*ii*) For $t_0t_0' = 1$ and $t_0 = |\zeta_0| (\neq 0)$, we get $\Gamma_D = \Gamma_\Omega = \Omega : |\zeta - \zeta_0| = t_0$ and the image of the boundary Γ_D under $z = L^{-1}(\zeta)$ becomes $L^{-1}(\Gamma_D) = \Lambda$, where

$$\Lambda : \zeta_0 z + \overline{\zeta_0}\,\overline{z} = 1$$

and therefore,

$$L^{-1}(\Omega) = \left\{ z : \zeta_0 z + \overline{\zeta_0}\,\overline{z} = 1 \right\}.$$

(*iii*) For $t_0t_0' = 1$ and $t_0 \neq |\zeta_0| (\neq 0)$, in this case $\Gamma_D = \Gamma_\Omega = \Omega : |\zeta - \zeta_0| = t_0$ and $L^{-1}(\Gamma_D) = \Lambda$, where

$$\Lambda : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| = \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right|$$

and so

$$L^{-1}(\Omega) = \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| = \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \right\}.$$

(*iv*) For $t_0t_0' > 1$ and $\frac{1}{t_0'} = |\zeta_0| (\neq 0)$, the images of

$$\Gamma_D : |\zeta - \zeta_0| = t_0 \text{ and } \Gamma_\Omega : |\zeta - \zeta_0| = \frac{1}{t_0'}$$

under the transformation $z = L^{-1}(\zeta)$ are

$$L^{-1}(\Gamma_D) = \Lambda : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| = \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right|$$

(as $t_0 > \frac{1}{t_0'} = |\zeta_0|$ i.e., $t_0 \neq |\zeta_0|$), a circle and

$$L^{-1}(\Gamma_\Omega) = \Delta : \zeta_0 z + \overline{\zeta_0}\,\overline{z} = 1 \ \left(\text{as } \frac{1}{t_0'} = |\zeta_0| \right),$$

a straight line in the z-plane.
Clearly $0 \in Int(\Gamma_D)$ and the image of 0 under $z = L^{-1}(\zeta)$ goes to ∞, which give

$$L^{-1}\left(\overline{Int(\Gamma_D)}\right) = \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \geq \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \right\}.$$

Again the image of $\zeta_0 \in Int(\Gamma_\Omega)$ under $z = L^{-1}(\zeta)$ in the z-plane becomes

$$z_0 = \frac{1}{\zeta_0} = \frac{\overline{\zeta_0}}{|\zeta_0|^2}$$

and putting $z = z_0$ in the expression $\zeta_0 z + \overline{\zeta_0}\,\overline{z}$, we have

$$\zeta_0 z_0 + \overline{\zeta_0}\,\overline{z_0} = \zeta_0 \left(\frac{\overline{\zeta_0}}{|\zeta_0|^2} \right) + \overline{\zeta_0} \overline{\left(\frac{\overline{\zeta_0}}{|\zeta_0|^2} \right)} = \frac{\zeta_0 \overline{\zeta_0}}{|\zeta_0|^2} + \frac{\overline{\zeta_0} \zeta_0}{|\zeta_0|^2} = 2 > 1.$$

Therefore,

$$L^{-1}\left(\overline{Ext\,(\Gamma_\Omega)} \right) = \left\{ z : \zeta_0 z + \overline{\zeta_0}\,\overline{z} \le 1 \right\}.$$

Now using the properties of Möbius Transformation, we have

$$\begin{aligned} L^{-1}(\Omega) &= L^{-1}\left(\overline{Int\,(\Gamma_D)} \right) \cap L^{-1}\left(\overline{Ext\,(\Gamma_\Omega)} \right) \\ &= \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2 \right)} \right| \ge \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2 \right)} \right|, \zeta_0 z + \overline{\zeta_0}\,\overline{z} \le 1 \right\}. \end{aligned}$$

(v) For $t_0 t_0' > 1$ and $t_0 = |\zeta_0| \, (\ne 0)$, the images of

$$\Gamma_D : |\zeta - \zeta_0| = t_0 \text{ and } \Gamma_\Omega : |\zeta - \zeta_0| = \frac{1}{t_0'}$$

under $z = L^{-1}(\zeta)$ are given by

$$L^{-1}(\Gamma_D) = \Lambda : \zeta_0 z + \overline{\zeta_0}\,\overline{z} = 1 \;\; (\text{as } t_0 = |\zeta_0|)$$

and

$$L^{-1}(\Gamma_\Omega) = \Delta : \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2 \right)} \right| = \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2 \right)} \right|$$

$$\left(\text{as } t_0 = |\zeta_0| > \frac{1}{t_0'}, \; \frac{1}{t_0'} \ne |\zeta_0| \right)$$

respectively. Now the image of $\zeta_0 \in Int\,(\Gamma_D)$ under $z = L^{-1}(\zeta)$ goes to $z_0 = \frac{\overline{\zeta_0}}{|\zeta_0|^2}$ and

$$\zeta_0 z_0 + \overline{\zeta_0}\,\overline{z_0} = \zeta_0 \left(\frac{\overline{\zeta_0}}{|\zeta_0|^2} \right) + \overline{\zeta_0} \overline{\left(\frac{\overline{\zeta_0}}{|\zeta_0|^2} \right)} = \frac{\zeta_0 \overline{\zeta_0}}{|\zeta_0|^2} + \frac{\overline{\zeta_0} \zeta_0}{|\zeta_0|^2} = 2 > 1,$$

which imply

$$L^{-1}\left(\overline{Int\,(\Gamma_D)} \right) = \left\{ z : \zeta_0 z_0 + \overline{\zeta_0}\,\overline{z_0} \ge 1 \right\}.$$

Again $0 \in Ext\,(\Gamma_\Omega)$ and the image of 0 under $z = L^{-1}(\zeta)$ goes to ∞ which give

$$L^{-1}\left(\overline{Ext\,(\Gamma_\Omega)} \right) = \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2 \right)} \right| \ge \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2 \right)} \right| \right\},$$

and therefore,

$$\begin{aligned} L^{-1}(\Omega) &= L^{-1}\left(\overline{Int\,(\Gamma_D)} \right) \cap L^{-1}\left(\overline{Ext\,(\Gamma_\Omega)} \right) \\ &= \left\{ z : \zeta_0 z_0 + \overline{\zeta_0}\,\overline{z_0} \ge 1, \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2 \right)} \right| \ge \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2 \right)} \right| \right\}. \end{aligned}$$

(vi) For $t_0 t_0' > 1$ and $\frac{1}{t_0'} > |\zeta_0| \, (\neq 0)$, the images of

$$\Gamma_D : |\zeta - \zeta_0| = t_0 \text{ and } \Gamma_\Omega : |\zeta - \zeta_0| = \frac{1}{t_0'}$$

under $z = L^{-1}(\zeta)$ are

$$L^{-1}(\Gamma_D) = \Lambda : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| = \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \ \left(\text{as } t_0 > \frac{1}{t_0'}, \, t_0 \neq |\zeta_0|\right)$$

and

$$L^{-1}(\Gamma_\Omega) = \Delta : \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| = \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \ \left(\text{as } \frac{1}{t_0'} \neq |\zeta_0|\right)$$

respectively. In this case, we see that 0 belongs to $Int(\Gamma_D)$ as well as $Int(\Gamma_\Omega)$ whose image under $z = L^{-1}(\zeta)$ goes to ∞. Therefore,

$$L^{-1}\left(\overline{Int(\Gamma_D)}\right) = \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \geq \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \right\}$$

and

$$L^{-1}\left(\overline{Ext(\Gamma_\Omega)}\right) = \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \leq \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \right\},$$

and hence,

$$L^{-1}(\Omega)$$
$$= L^{-1}\left(\overline{Int(\Gamma_D)}\right) \cap L^{-1}\left(\overline{Ext(\Gamma_\Omega)}\right)$$
$$= \left\{ z : \left| z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)} \right| \geq \left| \frac{t_0}{(t_0^2 - |\zeta_0|^2)} \right|, \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \leq \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \right\}.$$

(vii) For $t_0 t_0' > 1$ and $\frac{1}{t_0'} < |\zeta_0| \, (\neq 0) < t_0$, the images of

$$\Gamma_D : |\zeta - \zeta_0| = t_0 \text{ and } \Gamma_\Omega : |\zeta - \zeta_0| = \frac{1}{t_0'}$$

under $z = L^{-1}(\zeta)$ are

$$L^{-1}(\Gamma_D) = \Lambda : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| = \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \ (\text{as } t_0 \neq |\zeta_0|),$$

and

$$L^{-1}(\Gamma_\Omega) = \Delta : \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| = \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \ \left(\text{as } \frac{1}{t_0'} \neq |\zeta_0|\right),$$

respectively. Since $0 \in Int\left(\Gamma_D\right)$ and the image of 0 under $z = L^{-1}(\zeta)$ goes to ∞, it imply

$$L^{-1}\left(\overline{Int\left(\Gamma_D\right)}\right) = \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \geq \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \right\}.$$

Again, $0 \in Ext\left(\Gamma_\Omega\right)$ and the image of 0 under $z = L^{-1}(\zeta)$ goes to ∞, which give

$$L^{-1}\left(\overline{Ext\left(\Gamma_\Omega\right)}\right) = \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \geq \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \right\},$$

and so

$$\begin{aligned} &L^{-1}(\Omega) \\ &= L^{-1}\left(\overline{Int\left(\Gamma_D\right)}\right) \cap L^{-1}\left(\overline{Ext\left(\Gamma_\Omega\right)}\right) \\ &= \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \geq \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right|, \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \geq \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \right\}. \end{aligned}$$

$(viii)$ For $t_0 t_0' > 1$ and $t_0 < |\zeta_0| \, (\neq 0)$, the images of

$$\Gamma_D : |\zeta - \zeta_0| = t_0 \text{ and } \Gamma_\Omega : |\zeta - \zeta_0| = \frac{1}{t_0'}$$

under $z = L^{-1}(\zeta)$ are

$$L^{-1}\left(\Gamma_D\right) = \Lambda : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| = \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \quad (\text{as } t_0 \neq |\zeta_0|)$$

and

$$L^{-1}\left(\Gamma_\Omega\right) = \Delta : \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| = \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \quad \left(\text{as } \frac{1}{t_0'} \neq |\zeta_0|\right)$$

respectively. Clearly 0 belongs to $Ext\left(\Gamma_D\right)$ as well as $Ext\left(\Gamma_\Omega\right)$ and its image under $z = L^{-1}(\zeta)$ goes to ∞. Therefore,

$$L^{-1}\left(\overline{Int\left(\Gamma_D\right)}\right) = \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \leq \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \right\}$$

and

$$L^{-1}\left(\overline{Ext\left(\Gamma_\Omega\right)}\right) = \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \geq \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \right\}.$$

Consequently, we get

$$\begin{aligned} L^{-1}(\Omega) &= L^{-1}\left(\overline{Int\left(\Gamma_D\right)}\right) \cap L^{-1}\left(\overline{Ext\left(\Gamma_\Omega\right)}\right) \\ &= \left\{ z : \left| z + \frac{\overline{\zeta_0}}{\left(t_0^2 - |\zeta_0|^2\right)} \right| \leq \left| \frac{t_0}{\left(t_0^2 - |\zeta_0|^2\right)} \right|, \left| z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \geq \left| \frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)} \right| \right\}. \end{aligned}$$

(ix) For $t_0 t_0' > 1$ and $|\zeta_0| = 0$, the images of

$$\Gamma_D : |\zeta| = t_0 \text{ and } \Gamma_\Omega : |\zeta| = \frac{1}{t_0'}$$

under $z = L^{-1}(\zeta)$ are

$$L^{-1}(\Gamma_D) = \Lambda : |z| = \frac{1}{t_0} \quad (\text{as } t_0 \neq |\zeta_0|),$$

and

$$L^{-1}(\Gamma_\Omega) = \Delta : |z| = t_0' \quad \left(\text{as } \frac{1}{t_0'} \neq |\zeta_0|\right)$$

respectively.
As 0 belongs to $Int(\Gamma_D)$ as well as $Int(\Gamma_\Omega)$ and the image of 0 under $z = L^{-1}(\zeta)$ goes to ∞, it imply

$$L^{-1}\left(\overline{Int(\Gamma_D)}\right) = \left\{z : |z| \geq \frac{1}{t_0}\right\}$$

and

$$L^{-1}\left(\overline{Ext(\Gamma_\Omega)}\right) = \{z : |z| \leq t_0'\}$$

respectively and so,

$$\begin{aligned} L^{-1}(\Omega) &= L^{-1}\left(\overline{Int(\Gamma_D)}\right) \cap L^{-1}\left(\overline{Ext(\Gamma_\Omega)}\right) \\ &= \left\{z : \frac{1}{t_0} \leq |z| \leq t_0'\right\}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 1.3. Clearly, the derivative of $P(z)$ with respect to z is given by
$P'(z) = na_0 z^{n-1} + (n-\lambda) a_\lambda z^{n-\lambda-1} + (n-\lambda-1) a_{\lambda+1} z^{n-\lambda-2} \cdots + 2a_{n-2} z + a_{n-1}$
Using Theorem 1.2 on $P'(z)$, we can easily say that $P'(z)$ has no zeros in the following region:

(i) $\left\{z : |z| < \frac{1}{t_0}\right\}$, if $\zeta_0 = 0$,

(ii) $\left\{z : \zeta_0 z + \overline{\zeta_0}\,\overline{z} < 1\right\}$, if $t_0 = |\zeta_0| \, (\neq 0)$,

(iii) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| < \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$, if $t_0 > |\zeta_0| \, (\neq 0)$,

(iv) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| > \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$, if $t_0 < |\zeta_0| \, (\neq 0)$,

where t_0 is an unique positive root of the equation

$$f(t) \equiv |a_{n-1}| t^{n-1} - |b_2| t^{n-3} - |b_3| t^{n-4} - \cdots - |b_{n-2}| t - |b_{n-1}| = 0,$$

$$\begin{aligned} b_p &= \binom{n-1}{p} a_{n-1} \zeta_0^p + 2\binom{n-2}{p-1} a_{n-2} \zeta_0^{p-1} + \cdots + (j+1)\binom{n-j-1}{p-j} a_{n-j-1} \zeta_0^{p-j} \\ &\quad + \cdots + p\binom{n-p}{1} a_{n-p} \zeta_0 + (p+1)\binom{n-p-1}{0} a_{n-p-1}, \; p = 2, \ldots, n-1; \end{aligned}$$

$$a_k = 0, \; k = 1, 2, \ldots, \lambda - 1; \; \zeta_0 = -\frac{2a_{n-2}}{(n-1) a_{n-1}} \text{ with } \binom{0}{0} = 1.$$

Moreover, if $|b_{n-1}| \neq 0$, then $P'(z)$ has no zeros in the region:

(i) $\left\{z : |z| > \frac{1}{t_0}\right\} \cup \left\{z : |z| < \frac{1}{t_0}\right\}$, if $t_0 t_0' = 1, |\zeta_0| = 0$,

(ii) $\left\{z : \zeta_0 z + \overline{\zeta_0}\overline{z} > 1\right\} \cup \left\{z : \zeta_0 z + \overline{\zeta_0}\overline{z} < 1\right\}$, if $t_0 t_0' = 1, t_0 = |\zeta_0| \, (\neq 0)$,

(iii) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| > \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\} \cup \left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| < \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$,
if $t_0 t_0' = 1, t_0 \neq |\zeta_0| \, (\neq 0)$,

(iv) $\left\{z : \zeta_0 z + \overline{\zeta_0}\overline{z} > 1\right\} \cup \left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| < \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$,
if $t_0 t_0' > 1, \frac{1}{t_0'} = |\zeta_0| \, (\neq 0)$,

(v) $\left\{z : \zeta_0 z + \overline{\zeta_0}\overline{z} < 1\right\} \cup \left\{z : \left|z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right| < \left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right|\right\}$,
if $t_0 t_0' > 1, t_0 = |\zeta_0| \, (\neq 0)$,

(vi) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right| > \left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right|\right\} \cup \left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| < \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$,
if $t_0 t_0' > 1, \frac{1}{t_0'} > |\zeta_0| \, (\neq 0)$,

(vii) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right| < \left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right|\right\} \cup \left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| < \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\}$,
if $t_0 t_0' > 1, \frac{1}{t_0'} < |\zeta_0| \, (\neq 0) < t_0$,

(viii) $\left\{z : \left|z + \frac{\overline{\zeta_0}}{(t_0^2 - |\zeta_0|^2)}\right| > \left|\frac{t_0}{(t_0^2 - |\zeta_0|^2)}\right|\right\} \cup \left\{z : \left|z + \frac{\overline{\zeta_0}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right| < \left|\frac{\frac{1}{t_0'}}{\left(\frac{1}{t_0'^2} - |\zeta_0|^2\right)}\right|\right\}$,
if $t_0 t_0' > 1, t_0 < |\zeta_0| \, (\neq 0)$,

(ix) $\left\{z : |z| < \frac{1}{t_0}\right\} \cup \left\{z : |z| > t_0'\right\}$, if $t_0 t_0' > 1, |\zeta_0| = 0$,

where t_0' is an unique positive root of the equation

$$f'(t) \equiv |b_{n-1}| t^{n-1} - |b_{n-2}| t^{n-2} - |b_{n-3}| t^{n-3} - \cdots - |b_2| t^2 - |a_{n-1}| = 0.$$

Consequently, we conclude that $P(z)$ has no multiple zeros in the above region and this gives the desired result. □

3. Some application of our result

In this section, we present some application of our result. To illustrate this, we consider two polynomials in which one having no gaps and the other to be a lacunary type. At first we consider a polynomial

$$p(z) = z^6 - 2z^5 - 93z^4 + 484z^3 + 2219z^2 - 18330z + 38025$$

having no gaps. Here,

$$n = 6, a_0 = 1, a_1 = -2, a_2 = -93, a_3 = 484,$$
$$a_4 = 2219, a_5 = -18330, a_6 = 38025.$$

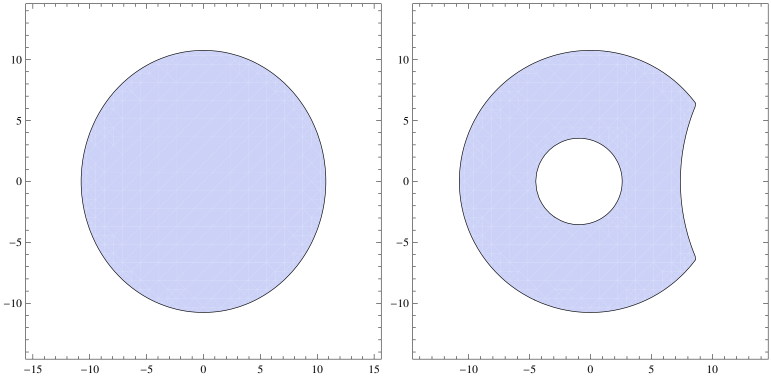

FIGURE 3. The left side region obtain by [11, Theorem 3.1] whereas the right side region obtain by applying Theorem 1.2 on it.

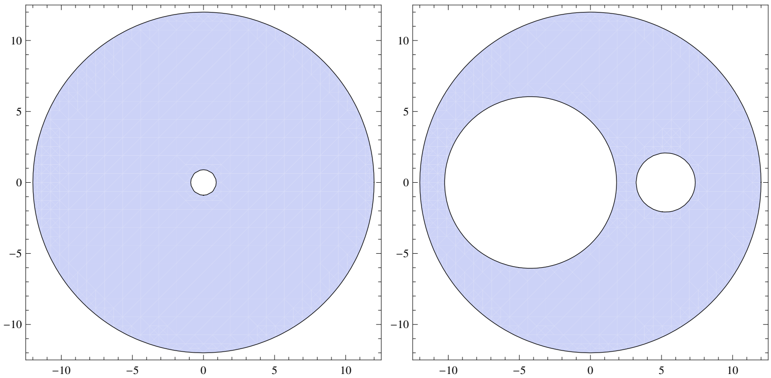

FIGURE 4. The left side region obtain by 1.1 or [3, Theorem 3] whereas the right side region obtain by applying Theorem 1.2 on it.

For estimating the region with the help of Theorem 1.2, we calculate

$$b_1 = 0, b_2 = -1462.6667, b_3 = 408.3358, b_4 = 38.3024, b_5 = -6.0230,$$
$$b_6 = 0.5313, \zeta_0 = \overline{\zeta_0} = 0.0803, t_0 = 0.3035, t_0' = 18.1416, \frac{1}{t_0'} = 0.0551.$$

Using Theorem 1.2, all the zeros of $p(z)$ lie in the region

$$\{z : |z - 23.5166| \geq 16.1345, |z + 0.9381| \geq 3.5434\}.$$

Also, the regions obtain by previous well-known results are as follows:
(i) $|z| \leq 35.5587$, by Sun and Hsieh [13, Theorem 1]
(ii) $|z| \leq 17.1240$, by Jain [8, Theorem 1]
(iii) $|z| \leq 12.2624$, by Rahmanand and Schmeisser [12, Theorem 8.3.1]
(iv) $|z| \leq 10.7572$, by Melman [11, Theorem 3.1] (which is smallest)
(v) $|z| \leq 17.4951$, by Batra, Mignotte and Stefanescu [1, Theorem 3.1]
(vi) $1.6125 \leq |z| \leq 13.3208$, by Theorem 1.1

(vii) $1.4816 \leq \left|z - \frac{1}{3}\right| \leq 12.3111$, by Das [3, Theorem 3].
Among them the best and distinct regions were given by
$1.6125 \leq |z| \leq 13.3208$, by Theorem 1.1,
$1.4816 \leq \left|z - \frac{1}{3}\right| \leq 12.3111$, by Das [3, Theorem 3],
$|z| \leq 10.7572$, by Melman [11, Theorem 3.1]
respectively. Now if we apply our result on the regions obtained from Theorem 1.1, [3, Theorem 3] and Melman's result, then the regions of zeros of $p(z)$ are found to be
$\{z : |z - 23.5166| \geq 16.1345, |z + 0.9381| \geq 3.5434, 1.6125 \leq |z| \leq 13.3208\}$,
$\left\{z : |z - 23.5166| \geq 16.1345, |z + 0.9381| \geq 3.5434, 1.4816 \leq \left|z - \frac{1}{3}\right| \leq 12.3111\right\}$,
$\{z : |z - 23.5166| \geq 16.1345, |z + 0.9381| \geq 3.5434, |z| \leq 10.7572\}$
respectively. The effect of the above results on the regions of zeros of $p(z)$ are shown in the picture (see Fig 1, Fig 2 and Fig 3 respectively marked by shaded area).
Now we consider a Lacunary type polynomial

$$P(z) = z^5 - 87z^3 + 564z^2 - 1340z + 1200.$$

Here,

$$n = 5, a_0 = 1, a_1 = 0, a_2 = -87, a_3 = 564, a_4 = -1340, a_5 = 1200.$$

For finding the region with the help of Theorem 1.2, we calculate

$$b_1 = 0, b_2 = -34.5333, b_3 = 23.5351, b_4 = 0.7529, b_5 = 0.2763,$$
$$\zeta_0 = \overline{\zeta_0} = 0.2233, t_0 = 0.3209, t_0' = 11.4078, \frac{1}{t_0'} = 0.0877.$$

Since $\frac{1}{t_0'} < |\zeta_0| < t_0$, using Theorem 1.2, all the zeros of $P(z)$ lie in the region

$$\{z : |z - 5.2931| \geq 2.0776, |z + 4.2071| \geq 6.0446\}.$$

Now we find the regions obtain by other well-known results as follows:
(i) $|z| \leq 12.8349$, by Sun and Hsieh [13, Theorem 1]
(ii) $|z| \leq 12.0051$, by Jain [8, Theorem 1]
(iii) $|z| \leq 12$, by Rahmanand and Schmeisser [12, Theorem 8.3.1, pp. 253]
(iv) $|z| \leq 17.5895$, by Batra, Mignotte and Stefanescu [1, Theorem 3.1]
(v) $|z| \leq 12$, by Melman [11, Theorem 3.1] (smallest one)
(vi) $0.6802 \leq |z| \leq 12$, by Das [3, Theorem 3] or Theorem 1.1.
Among these regions we see that the best and distinct regions are $0.6802 \leq |z| \leq 12$ (by Theorem 1.1 or [3, Theorem 3]) and $|z| \leq 12$ (by Rahman and Schmeisser or Melman) respectively. Again if we apply our result in Theorem 1.1 or [3, Theorem 3], and the result obtained by Rahman and Schmeisser or Melman, then the regions of zeros of $P(z)$ become
$\{z : |z - 5.2931| \geq 2.0776, |z + 4.2071| \geq 6.0446, 0.6802 \leq |z| \leq 12\}$,
$\{z : |z - 5.2931| \geq 2.0776, |z + 4.2071| \geq 6.0446, |z| \leq 12\}$
respectively. Here we also show the effect of the above results on the regions of zeros of $P(z)$ in the picture (see also Fig 4 and Fig 5 respectively).

4. Conclusion

Our basic aim was to refine the smallest possible region for the zeros of a polynomial. Clearly the region obtained from Theorem 1.2 is neither circular nor annular except

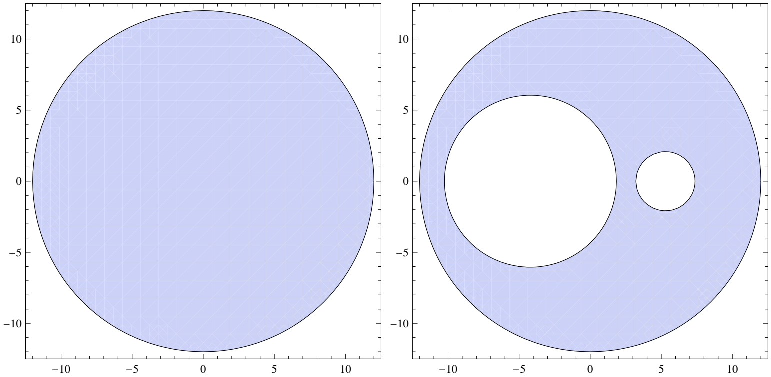

FIGURE 5. The left side region obtain by [3, Theorem 3] whereas the right side region obtain by applying Theorem 1.2 on it.

in some particular cases. So if we applying our result in Theorem 1.2 on the regions obtained from many well-known results then the region of zeros may be reduced.

References

[1] P. Batra, M. Mignotte, and D. Stefanescu, Improvements of Lagrange's bound for polynomial roots, *Journal of Symbolic Computation* **82** (2017), 19–25. DOI: 10.1016/j.jsc.2016.10.001

[2] A.L. Cauchy, *Exercise de Mathematiques: Iv Annee de Bure Freres*, Paris, 1829.

[3] S. Das, Annular bounds for the zeros of polynomials, *Bull. Math. Soc. Sci. Math. Roumanie* **63** (2020), no. 4, 335–347.

[4] B. Datt and N.K. Govil, On the location of the zeros of a polynomial, *J. Approx. Theory* **24** (1978), 78–82.

[5] M. Dehmer and A. Mowshowitz, Bounds on the moduli of polynomial zeros, *Appl. Maths. Comp.* **218** (2011), 4128–4137. DOI: 10.1016/j.amc.2011.09.043

[6] V.K. Jain, On Cauchy's bound for zeros of a polynomial, *Approx. Theory Appl.* **6** (1990), 18–24.

[7] V.K. Jain, On Cauchy's bound for zeros of a polynomial, *Glasnik Mathematicki* **31** (1996), no. 2, 245–252.

[8] V.K. Jain, On Cauchy's bound for zeros of a polynomial, *Turk. J. Math.* **30** (2006), 95–100.

[9] J.L. Lagrange, Sur la resolution des équations numeriques, *Mémoires de l'Academie royale des Sciences et Belles-lettres de Berlin* **XXIII** (1769), 539–578.

[10] M. Marden, *The Geometry of polynomial*, AMS Math Surveys 3, New York, 1966.

[11] A. Melman, Improved Cauchy radius for scalar and matrix polynomials, *Proc. Amer. Math. Soc.* **146** (2018), no. 2, 613–624. DOI: 10.1090/proc/13826

[12] Q.I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, Oxford Univ. Press, 2002.

[13] Y.J. Sun and J.G. Hsieh, A note on circular bound of polynomial zeros, *IEEE Trans. Circuits Syst.* **43** (1996), no. 1, 476–478.

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