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On the nonhomogeneous wavelet bi-frames for reducing subspaces of $H^s(K)$

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Abstract. Ahmad and Shiekh in *Filomat* 34: 6(2020), have constructed dual wavelet frames in Sobolev spaces on local fields of positive characteristic. We continued the study and provided the characterization of nonhomogeneous wavelet bi-frames. First of all we introduce the reducing subspaces of Sobolev spaces over local fields of prime characteristics and then provide the way to characterize the nonhomogeneous wavelet bi-frames over such fields. Our results are better than those established by Ahmad and Shiekh.

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1. Introduction

To start with it is to be noted that a refinable structure gives birth to the classical nonhomogeneous systems with some technical restrictions on them [16, 22, 23, 24]. The wavelet systems thus obtained have fast wavelet transform. However the correspondence between them is not exact. It was Han [22, 23] who showed that the nonstationary wavelets and nonhomogeneous wavelet systems are closely related. With these considerations in mind, our aim in this paper is to construct and characterize nonhomogeneous wavelet bi-frames (NWBFs) on Sobolev spaces over local fields of positive characteristic.

Moving to the side of frames it is here worth to mention that Duffin and Schaeffer [21] introduced frames in non-harmonic Fourier series in 1952. They were again studied in 1986 by Daubechies and the process continued. Frames and the dual frames have an important role to play in the characterization of signal, image and video processing, function spaces, sampling theory and many more. Mathematically a frame is defined in the following manner. A sequence of functions $\{f_k\}_{k=1}^{\infty}$ of Hilbert space $\mathbb{H}$ is called a *frame* for $\mathbb{H}$ if there exist constants $\mathfrak{A}, \mathfrak{B} > 0$ such that for all $f \in \mathbb{H}$,

$$\mathfrak{A}\|f\|_2^2 \leq \sum_{k=1}^{\infty} \left|\langle f, f_k\rangle\right|^2 \leq \mathfrak{B}\|f\|_2^2,$$

where $\mathfrak{A}$ is lower bound and $\mathfrak{B}$ is the upper one. If $\mathfrak{A} = \mathfrak{B}$, then we have a *tight frame*. If $\mathfrak{A} = \mathfrak{B} = 1$, then we end up with a *normalized tight frame*. For more about frames, we refer to [19, 18, 20] and the references therein.

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From 1980 a sustainable and progressive growth has occurred int the construction wavelets and its associates on local fields of positive characteristic. Local fields are broadly divided into the fields of zero and positive characteristics. Although their topology is similar, their MRA (multiresolution analysis) and wavelet theories are quite different. The construction of wavelets, wavelet frames, MRA and other related works on local fields of positive characteristics (LFPC)have been studied by Benedetto, Behera and Jahan, Ahmad and Shah, Jiang, Li and Ji, Shukla and Mittal, Bhat, etc in the series of works [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 25, 26, 27]. However the research on local fields is still in its infancy. Recently Ahamd and Shiekh [1] have studied dual wavelet frames in Sobolev spaces on local fields of positive characteristic. Continuing our study of frames on local fields, we introduce a comprehensive characterization of nonhomogeneous wavelet bi-frames in Sobolev spaces our local fields of positive characteristic. First of all we introduce the reducing subspaces of Sobolev spaces over local fields of prime characteristics and then characterize the nonhomogeneous wavelet bi-frames over such fields.

The rest of the paper is tailored as follows. In Section 2, we recall some basic Fourier analysis on local fields and also some results which are required in the subsequent sections. In Section 3, we prove the results that are required in the characterization of NWBFs over Local Fields.

2. Preliminaries on local fields

It is well that a field K is local if has the properties of locally compactness, non-discreteness and totally disconnectedness. Local fields are broadly divided into the fields of zero and positive characteristics. For characteristic zero, it becomes a field of p-adic numbers $\mathbb{Q}_p$ or its finite extension. When K has positive characteristic, it is a formal Laurent series over a finite field $GF(p^c)$. We define the ring of integers over local fields as $\mathfrak{D} = \{x \in K : |x| \leq 1\}$. Due to the locally compactness, we have Haar measure dx for K^+. For the ring defined above, we have the prime ideal like $\mathfrak{B} = \{x \in K : |x| < 1\}$. The residue space thus formed $\mathfrak{D}/\mathfrak{B}$ will be isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Due to the disconnectedness of local fields we have a prime element $\mathfrak{p}$ of K such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. We closely follow the results and notations of the Taibleson's book [28]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

For any function $f \in L^1(K)$, the classical Fourier transform denoted by $\widehat{f}(\omega)$ is defined as

$$\mathcal{F}\{f(x)\} = \widehat{f}(\omega) = \int_K f(x)\overline{\chi_\omega(x)}\, dx.$$

Note that

$$\widehat{f}(\omega) = \int_K f(x)\, \overline{\chi_\omega(x)} dx = \int_K f(x)\chi(-\omega x)\, dx.$$

Definition 2.1. For $k \in \mathbb{N}_0$, the *translation operator* $T_{u(k)} : L^2(K) \to L^2(K)$ is defined by

$$T_{u(k)}\psi(\cdot) = \psi\left(\cdot - u(k)\right)$$

and the *dilation operator* $D\psi(\cdot) : L^2(K) \to L^2(K)$ by

$$D\psi\left(\cdot\right) = \psi\left(\mathfrak{p}^{-1}\cdot\right).$$

For $s \in K$, we define the Sobolev space $H^s(k)$ as the space of all tempered distributions f such that

$$\|f\|^2_{H^s(k)} = \int_K |\hat{f}(\omega)|^2 \left(1 + \|\omega\|^2\right)^2 d\omega < \infty,$$

where $\|\cdot\|$ denotes the Euclidean norm on K. The inner product for $f, g \in H^s(k)$ is given by

$$\langle f, g\rangle_{H^s(k)} = \int_K \hat{f}(\omega)\overline{\hat{g}(\omega)} \left(1 + \|\omega\|^2\right)^2 d\omega,$$

It is to be noted that for each $f \in H^s(k)$ and $g \in H^{-s}(k)$, we have

$$\langle f, g\rangle_{H^s(k)} = \int_K \hat{f}(\omega)\overline{\hat{g}(\omega)} d\omega.$$

The spaces $H^s(k)$ and $H^{-s}(k)$ form a pair of dual spaces over local fields of positive characteristic.

For a distribution $f,\ j \in \mathbb{N}_0, s, k \in K$, we can write

$$f_{j,k} = q^{j/2} f\left(\mathfrak{p}^{-j}\omega - u(k)\right) \quad \textit{and} \quad f_{j,k} = q^{-js/2} f\left(\mathfrak{p}^{-j}\omega - u(k)\right)$$

Given $L \in \mathbb{N}$, let us suppose that $\psi_0 \in H^s(K)$ be a tempered distribution and $\Psi = \{\psi_1, \cdots, \psi_L\} \subset H^s(K)$ be a finite set of tempered distributions. We denote the homogeneous wavelet system $X^s(\Psi)$ and the nonhomogeneous wavelet system $X^s(\psi_0, \Psi)$ in $H^s(K)$ respectively by

$$X^s(\Psi) = \left\{\psi^s_{\ell,j,k} : j \in \mathbb{N}_0, k \in K, 1 \le \ell \le L\right\}$$

and

$$X^s(\psi_0, \Psi) = \{\psi_{0,0,k} : k \in K\} \left\{\psi^s_{\ell,j,k} : j \in \mathbb{N}_0, k \in K, 1 \le \ell \le L\right\}$$

Ahamd and Shiekh [1] have constructed dual wavelet frames in Sobolev spaces on local fields. We use the lemmas and results proved by them to obtain the characterization of NWBFs over local fields of positive characteristic.

3. Characterization of NWBFs over local field

In this section, we provide the characterization of NWBFs in $(FH^s(\Omega), FH^{-s}(\Omega))$. Firstly we need following two lemmas.

Lemma 3.1. *Given $s \in K$, let $\{T_{u(k)}\psi_0 : k \in \mathbb{N}_0\} \cup \{T_{u(k)}\psi_\ell : k \in \mathbb{N}_0,\ 1 \le \ell \le L\}$ be a Bessel sequence in $H^s(K)$. Then*

$$
\begin{aligned}
&\sum_{k\in\mathbb{N}_0} |\langle g,\psi_{0,0,k}\rangle|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} \left|\langle g,\psi^s_{\ell,j,k}\rangle\right|^2 \\
&= \int_K |\hat{g}(\omega)|^2 \left(\left|\hat{\psi}_0(\omega)\right|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} m^{-2js}\left|\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)\right|^2 \right) d\omega \\
&\quad + \int_K \overline{\hat{g}(\omega)} \sum_{k\in\mathbb{N}_0} \hat{g}\left(\omega+u(k)\right) \\
&\quad \times \left(\hat{\psi}_0(\omega)\overline{\hat{\psi}_0\left(\omega+u(k)\right)} + \sum_{\ell=1}^{L}\sum_{j=0}^{k} m^{-2js}\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)\overline{\hat{\psi}_\ell\left(\omega+u(k)\right)} \right) \qquad (1)
\end{aligned}
$$

for $g\in\mathcal{D}$

Proof. From [1], we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{N}_0} |\langle g,\psi_{0,0,k}\rangle|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} \left|\langle g,\psi^s_{\ell,j,k}\rangle\right|^2 \\
&= \int_{\mathfrak{D}} \left| \sum_{k\in\mathbb{N}_0} \hat{g}\left(\omega+u(k)\right)\overline{\hat{\psi}_0\left(\omega+u(k)\right)} \right|^2 d\omega \\
&\quad + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} m^{j(d-2s)} \int_{\mathfrak{D}} \left| \sum_{k\in\mathbb{N}_0} \hat{g}\left(\mathfrak{p}^j(\omega+u(k))\right)\overline{\hat{\psi}_\ell\left(\omega+u(k)\right)} \right|^2 d\omega \\
&= \int_{\mathfrak{D}} \left(\sum_{k\in\mathbb{N}_0} \hat{\psi}_0\left(\omega+u(k)\right)\overline{\hat{g}\left(\omega+u(k)\right)} \right)\left(\sum_{k\in\mathbb{N}_0} \hat{g}\left(\omega+u(k)\right)\overline{\hat{\psi}_0\left(\omega+u(k)\right)} \right) d\omega \\
&\quad + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} m^{j(d-2s)} \int_{\mathfrak{D}} \left(\sum_{k\in\mathbb{N}_0} \hat{\psi}_\ell\left(\omega+u(k)\right)\overline{\hat{g}\left(\mathfrak{p}^j(\omega+u(k))\right)} \right) \\
&\quad \times \left(\sum_{k\in\mathbb{N}_0} \hat{g}\left(\mathfrak{p}^j(\omega+u(k))\right)\overline{\hat{\psi}_\ell\left(\omega+u(k)\right)} \right) \\
&= \int_{\mathfrak{D}} \left(\sum_{k\in\mathbb{N}_0} \hat{\psi}_0\left(\omega+u(k)\right)\overline{\hat{g}\left(\omega+u(k)\right)} \right) E_0(\omega)d\omega \\
&\quad + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} m^{j(d-2s)} \int_{\mathfrak{D}} \left(\sum_{k\in\mathbb{N}_0} \hat{\psi}_\ell\left(\omega+u(k)\right)\overline{\hat{g}\left(\mathfrak{p}^j(\omega+u(k))\right)} \right) E_{\ell,j}d\omega \\
&= R_1 + R_2, \qquad (2)
\end{aligned}
$$

where $E_0(\cdot) = \sum_{k\in\mathbb{N}_0} \hat{g}\left(\omega+u(k)\right)\overline{\hat{\psi}_0\left(\omega+u(k)\right)}$ and
$E_{\ell,j}(\cdot) = \sum_{k\in\mathbb{N}_0} \hat{g}\left(\mathfrak{p}^j(\omega+u(k))\right)\overline{\hat{\psi}_\ell\left(\omega+u(k)\right)}$. Since $\{T_{u(k)}\psi_0 : k\in\mathbb{N}_0\}$ is a Bessel sequence in $H^s(K)$ and $g\in\mathcal{D}$, it follows that $|E_0(\cdot)| \le [\hat{g},\hat{g}]^{\frac{1}{2}}_{-s}(\cdot)[\hat{\psi}_0,\hat{\psi}_0]^{\frac{1}{2}}_{-s}(\cdot) < \infty$

by [1]. Therefore

$$\int_{\mathfrak{D}}\left|\sum_{k\in\mathbb{N}_0}\hat{g}\left(\omega+u(k)\right)\overline{\hat{\psi}_0\left(\omega+u(k)\right)}E_0(\omega)\right|\leq ||E_0||_{L^{\infty}(\mathfrak{D})}\int_{\mathfrak{D}}[\hat{g},\hat{g}]_{-s}^{\frac{1}{2}}(\omega)[\hat{\psi}_0,\hat{\psi}_0]_{-s}^{\frac{1}{2}}(\omega)<\infty,$$

hence by Fubini-Tonelli theorem

$$\begin{aligned}&\int_{\mathfrak{D}}\left(\sum_{k\in\mathbb{N}_0}\hat{\psi}_0\left(\omega+u(k)\right)\overline{\hat{g}\left(\omega+u(k)\right)}\right)\left(\sum_{k\in\mathbb{N}_0}\hat{g}\left(\omega+u(k)\right)\overline{\hat{\psi}_0\left(\omega+u(k)\right)}\right)d\omega\\&=\int_{\mathfrak{D}}\hat{\psi}_0(\omega)\overline{\hat{g}(\omega)}\sum_{k\in\mathbb{N}_0}\hat{g}\left(\omega+u(k)\right)\overline{\hat{\psi}_0\left(\omega+u(k)\right)}d\omega\end{aligned}$$

Moreover

$$\begin{aligned}&\int_{K}|\hat{\psi}_0(\omega)\overline{\hat{g}(\omega)}|\sum_{k\in\mathbb{N}_0}|\hat{g}\left(\omega+u(k)\right)\overline{\hat{\psi}_0\left(\omega+u(k)\right)}|d\omega\\&\leq\int_{supp(\hat{g})}\left(\sum_{k\in\mathbb{N}_0}|\hat{g}\left(\omega+u(k)\right)\overline{\hat{\psi}_0\left(\omega+u(k)\right)}\right)^2d\omega\\&\leq\int_{supp(\hat{g})}[\hat{g},\hat{g}]_{-s}(\omega)[\hat{\psi}_0,\hat{\psi}_0]_s(\omega)d\omega\\&<\infty.\end{aligned}$$

Since $[\hat{g},\hat{g}]_{-s}(\omega)[\hat{\psi}_0,\hat{\psi}_0]_s(\cdot)$ is essentially bounded by [1], we have

$$\begin{aligned}R_1&=\int_{K}\hat{\psi}_0(\omega)\overline{\hat{g}(\omega)}\sum_{k\in\mathbb{N}_0}\hat{g}\left(\omega+u(k)\right)\overline{\hat{\psi}_0\left(\omega+u(k)\right)}d\omega\\&=\int_{K}|\hat{\psi}_0(\omega)|^2|\hat{g}(\omega)|^2d\omega\\&\quad+\int_{K}\hat{\psi}_0(\omega)\overline{g(\omega)}\sum_{k\in\mathbb{N}_0}\hat{g}\left(\omega+u(k)\right)\overline{\hat{\psi}_0\left(\omega+u(k)\right)}d\omega\end{aligned}\tag{3}$$

In order to complete the proof, we need to calculate R_2. Let us define $\tilde{g}$ by $\hat{\tilde{g}}(\cdot)=\hat{g}((\mathfrak{p})^j\cdot)$. Then as $g\in\mathcal{D}$ and [1], we have

$$[\hat{g}((\mathfrak{p})^j\cdot),\hat{g}((\mathfrak{p})^j\cdot)](\cdot)\leq C$$

Thus

$$|E_{i,j}(\cdot)|\leq[\hat{g}((\mathfrak{p})^j\cdot),\hat{g}((\mathfrak{p})^j\cdot)]_{-s}^{\frac{1}{2}}(\cdot)[\hat{\psi}_\ell,\hat{\psi}_\ell]_s^{\frac{1}{2}}<\infty.$$

Hence

$$R_2=\sum_{\ell=1}^{L}\sum j=0^{\infty}m^{j(d-2s)}\int_K\hat{\psi}_\ell(\omega)\overline{g(\mathfrak{p}^j\omega)}\sum_{k\in\mathbb{N}_0}\hat{g}\left(\mathfrak{p}^j(\omega+u(k))\right)\overline{\hat{\psi}_\ell\left(\omega+u(k)\right)}d\omega$$

Taking A a bounded set in K such that $supp(\hat{g})\subset A$. So by [1], we get

$$A\cap(A+\mathfrak{p}^ju(k))=\emptyset\quad\text{for }(j,k)\notin A_1\times A_2\quad\text{with }k\neq0,$$

where $A_1\subset\mathbb{N}_0$ and $A_2\subset\mathbb{N}_0\setminus\{0\}$ are two finite sets. Therefore

$$R_2=\sum_{\ell=1}^{L}\sum_{j\in A_1}m^{j(d-u(s))}\int_K\hat{\psi}_\ell(\omega)\overline{g(\mathfrak{p}^j\omega)}\sum_{k\in A_2}\hat{g}\left(\mathfrak{p}^j(\omega+u(k))\right)\overline{\hat{\psi}_\ell\left(\omega+u(k)\right)}d\omega$$

Denote $S = \bigcup_{k\in A_2\cap\{0\}}(\bigcup_{j\in A_1}\left(\mathfrak{p}^j A + u(k)\right))$. Therefore, for each $(j,k)\in A_1\times A_2$, we have

$$\begin{aligned}
&\int_K \left|\hat{\psi}_\ell(\omega)\overline{g(\mathfrak{p}^j\omega)}\hat{g}\left(\mathfrak{p}^j(\omega+u(k))\right)\overline{\hat{\psi}_\ell\left(\omega+u(k)\right)}\right| d\omega \\
&\le ||\hat{g}||^2_{L^\infty(K)}\int_{\mathfrak{p}^{-j}A}|\hat{\psi}_\ell(\omega)\hat{\psi}_\ell(\omega+u(k))|d\omega \\
&\le ||\hat{g}||^2_{L^\infty(K)}\left(\int_{\mathfrak{p}^{-j}A}|\hat{\psi}_\ell(\omega)|^2 d\omega\right)^{\frac{1}{2}}\left(\int_{\mathfrak{p}^{-j}A}|\hat{\psi}_\ell(\omega+u(k))|^2\right)^{\frac{1}{2}} d\omega \\
&\le ||\hat{g}||^2_{L^\infty(K)}\int_S|\hat{\psi}_\ell(\omega)|^2 d\omega
\end{aligned}$$

Note that $1\le(\max_{\omega\in A}(1+|\omega|^2)^{-s})(1+|\omega|^2)^s$ for $\omega\in A$. Thus

$$\begin{aligned}
&\int_K \left|\hat{\psi}_\ell(\omega)\overline{g(\mathfrak{p}^j\omega)}\hat{g}\left(\mathfrak{p}^j(\omega+u(k))\right)\overline{\hat{\psi}_\ell\left(\omega+u(k)\right)}\right| d\omega \\
&\le \left(\max_{\omega\in A}(1+|\omega|^2)^{-s}\right)||\hat{g}||^2_{L^\infty(K)}\int_A|\hat{\psi}_\ell(\omega)|^2(1+|\omega|^2)^s d\omega \\
&\le \left(\max_{\omega\in A}(1+|\omega|^2)^{-s}\right)||\hat{g}||^2_{L^\infty(K)}||\psi_\ell||^2_{H^s(K)} \\
&< \infty
\end{aligned}$$

On combining the formula given above, we get

$$\begin{aligned}
R_2 &= \int_K\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} m^{j(d-u(s))}|\hat{\psi}_\ell(\omega)|^2|\hat{g}(\mathfrak{p}^j\omega)|^2 d\omega \\
&\quad+\int_K\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} m^{j(d-u(s))}\overline{\hat{g}(\mathfrak{p}^j\omega)}\hat{\psi}_\ell(\omega)\sum_{k\in\mathbb{N}_0}\hat{g}\left(\mathfrak{p}^j(\omega+u(k))\right)\overline{\hat{\psi}_\ell\left(\omega+u(k)\right)}d\omega \\
&= \int_K\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} m^{-u(js)}|\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)|^2|\hat{g}(\omega)|^2 d\omega \\
&\quad+\int_K\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} m^{-u(js)}\overline{\hat{g}(\omega)}\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)\sum_{k\in\mathbb{N}_0}\hat{g}\left(\omega+\mathfrak{p}^{-j}u(k)\right)\overline{\hat{\psi}_\ell\left(\mathfrak{p}^{-j}\omega+u(k)\right)}d\omega \\
&= \int_K\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} m^{-u(js)}|\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)|^2|\hat{g}(\omega)|^2 d\omega \\
&\quad+\int_K\overline{\hat{g}(\omega)}\sum_{k\in\mathbb{N}_0}\hat{g}\left(\omega+u(k)\right)\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)} m^{-u(js)}\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)\overline{\hat{\psi}_\ell\left(\mathfrak{p}^{-j}\omega+u(k)\right)}d\omega \qquad (4)
\end{aligned}$$

using the definition of $\kappa(k)$. Hence using (2), (3) and (4), we get (1). This completes the proof of the lemma.

Lemma 3.2. *Given $s \in K$, let $X^s(\psi_0;\Psi)$ and $X^{-s}(\tilde{\psi}_0;\tilde{\Psi})$ be a Bessel sequences in $H^s(K)$ and $H^{-s}(K)$, respectively. Then for all $f, g \in \mathcal{D}$, we have*

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0}\langle f,\tilde{\psi}_{0,0,k}\rangle\langle\psi_{0,0,k},g\rangle+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0}\langle f,\tilde{\psi}^{-s}_{\ell,j,k}\rangle\langle\psi^{s}_{\ell,j,k},g\rangle\\
&\quad=\int_K\hat{f}(\omega)\overline{\hat{g}(\omega)}\left(\hat{\psi}_0(\omega)\overline{\hat{\tilde{\psi}}_0(\omega)}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\omega)}\right)d\omega\\
&\qquad+\int_K\overline{\hat{g}(\omega)}\sum_{k\in\mathbb{N}_0}\hat{f}(\omega+u(k))\\
&\qquad\times\left(\hat{\psi}_0(\omega)\overline{\hat{\tilde{\psi}}_0(\omega+u(k))}+\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}(\omega+u(k))}\right)d\omega \qquad (5)
\end{aligned}$$

Proof. As $X^s(\psi_0;\Psi)$ and $X^{-s}(\tilde{\psi}_0;\tilde{\Psi})$ are Bessel sequences in $H^s(K)$ and $H^{-s}(K)$, respectively, the expression in (5) is meaningful. Proceeding in a similar fashion as in Lemma 3.1, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0}\langle f,\tilde{\psi}_{0,0,k}\rangle\langle\psi_{0,0,k},g\rangle+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0}\langle f,\tilde{\psi}^{-s}_{\ell,j,k}\rangle\langle\psi^{s}_{\ell,j,k},g\rangle\\
&=\int_K\hat{\psi}_0(\omega)\overline{\hat{g}(\omega)}\sum_{k\in\mathbb{N}_0}\hat{f}(\omega+u(k))\overline{\hat{\tilde{\psi}}_0(\omega+u(k))}d\omega\\
&\quad+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^j\int_K\hat{\psi}_\ell(\omega)\overline{\hat{g}(\mathfrak{p}^j\omega)}\sum_{k\in\mathbb{N}_0}\hat{f}(\mathfrak{p}^j(\omega+u(k)))\overline{\hat{\tilde{\psi}}_\ell(\omega+u(k))}d\omega\\
&=I_1+I_2. \qquad (6)
\end{aligned}$$

Note that

$$|\hat{\psi}_0(\cdot)\overline{\hat{g}(\cdot)}|\sum_{k\in\mathbb{N}_0}|\hat{f}(\cdot+u(k))\overline{\hat{\tilde{\psi}}_0(\cdot+u(k))}|\le[\hat{f},\hat{f}]_s^{\frac12}(\cdot)[\hat{\tilde{\psi}}_0,\tilde{\psi}_0]_{-s}^{\frac12}(\cdot)[\hat{g},\hat{g}]_{-s}^{\frac12}(\cdot)[\hat{\psi}_0,\psi_0]_s^{\frac12}(\cdot),$$

is bounded due to Lemma 2.2. Hence

$$\begin{aligned}
&\int_K|\hat{\psi}_0(\omega)\overline{\hat{g}(\omega)}|\sum_{k\in\mathbb{N}_0}|\hat{f}(\omega+u(k))\overline{\hat{\tilde{\psi}}_0(\omega+u(k))}|d\omega\\
&\quad\le\int_{supp(\hat{g})}|\hat{\psi}_0(\omega)\overline{\hat{g}(\omega)}|\sum_{k\in\mathbb{N}_0}|\hat{f}(\omega+u(k))\overline{\hat{\tilde{\psi}}_0(\omega+u(k))}|d\omega<\infty
\end{aligned}$$

Therefore

$$I_1=\int_K\hat{f}(\omega)\overline{\hat{g}(\omega)}\hat{\psi}_0(\omega)\overline{\hat{\tilde{\psi}}_0(\omega)}+\int_K\hat{\psi}_0(\omega)\overline{\hat{g}(\omega)}\sum_{k\in\mathbb{N}_0}\hat{f}(\omega+u(k))\overline{\hat{\tilde{\psi}}_0(\omega+u(k))}d\omega. \qquad (7)$$

To complete the proof of the lemma, we need to discuss I_2. Let's break it into two parts, for $k=0$ and $k\neq 0$. Hence by Cauchy-Schwartz inequality and [1], we have

$$\begin{aligned}&\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}|\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\omega)}|\\ &\quad\leq\left(\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}m^{-2js}|\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)|^2\right)^{\frac{1}{2}}\left(\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}m^{2js}|\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\omega)|^2\right)^{\frac{1}{2}}\\ &\quad\leq B_1B_2. \end{aligned}\tag{8}$$

Therefore

$$\begin{aligned}&\int_K|\hat{f}(\omega)\overline{\hat{g}(\omega)}|\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}|\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\omega)}|\\ &\quad\leq B_1B_2\left|supp(\hat{f})\cap supp(\hat{g})\right|\,\|\hat{f}\|_{L^\infty(K)}\|\hat{g}\|_{L^\infty(K)}\\ &\quad<\infty\end{aligned}$$

Fix a compact set $A\in K$ such that $supp(\hat{f})\cap supp(\hat{g})\subset A$. Using [1], it follows that

$$A\cap(A+\mathfrak{p}^j u(k))=\emptyset\quad\text{for } (j,k)\notin A_1\times A_2\quad\text{with } k\neq 0\tag{9}$$

where $A_1\subset\mathbb{N}_0$ and $A_2\subset\mathbb{N}_0\setminus\{0\}$ are two finite sets. With the same argument as applied to R_2, we have

$$\begin{aligned}&\int_K|\overline{\hat{g}(\mathfrak{p}^j\omega)}\hat{f}(\mathfrak{p}^j(\omega+u(k)))\hat{\psi}_\ell(\omega)\overline{\hat{\tilde{\psi}}_\ell(\omega+u(k))}|d\omega\\ &\quad\leq\|\hat{g}\|_{L^\infty(K)}\|\hat{f}\|_{L^\infty(K)}\left(\int_{\mathfrak{p}^{-j}A}|\hat{\psi}_\ell(\omega)|^2\right)^{\frac{1}{2}}\left(\int_{\mathfrak{p}^{-j}A}|\hat{\tilde{\psi}}_\ell(\omega+u(k))|^2\right)^{\frac{1}{2}}\\ &\quad\leq\|\hat{g}\|_{L^\infty(K)}\|\hat{f}\|_{L^\infty(K)}\left(\int_{T}|\hat{\psi}_\ell(\omega)|^2\right)^{\frac{1}{2}}\left(\int_{T}|\hat{\tilde{\psi}}_\ell(\omega+u(k))|^2\right)^{\frac{1}{2}}\\ &\quad\leq\|\hat{g}\|_{L^\infty(K)}\|\hat{f}\|_{L^\infty(K)}\left(\max_{\omega\in T}(1+|\omega|^2)\right)^{-s/2}\\ &\qquad\times\left(\max_{\omega\in T}(1+|\omega|^2)\right)^{s/2}\|\psi_\ell\|_{H^s(K)}\|\tilde{\psi}_\ell\|_{H^{-s}(K)}\\ &\quad<\infty\end{aligned}\tag{10}$$

for $(j,k)\in A_1\times A_2$, where $T=\bigcup_{k\in A_2\cup\{0\}}(\bigcup_{j\in A_1}\mathfrak{p}^{-j}A+u(k))$. Using (8) and (10), we get

$$\begin{aligned}I_2&=\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\int_K\overline{\hat{g}(\omega)}\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)\sum_{k\in\mathbb{N}_0}\hat{f}(\omega+\mathfrak{p}^j u(k))\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}(\omega+u(k)))}d\omega\\ &=\int_K\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\overline{\hat{g}(\omega)}\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)\sum_{k\in\mathbb{N}_0}\hat{f}(\omega+\mathfrak{p}^j u(k))\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}(\omega+u(k)))}d\omega\\ &=\int_K\overline{\hat{g}(\omega)}\sum_{k\in\mathbb{N}_0}\hat{f}(\omega+u(k))\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)}\times\hat{\psi}_\ell(\mathfrak{p}^{-j}\omega)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}(\omega+u(k)))}d\omega\end{aligned}$$

On combining (6), (7) and (11), we get (5), which completes the proof of the lemma.

We now state the theorem which will characterize NWBFs in $(FH^s(\Omega), FH^{-s}(\Omega))$. We are working on the proof and will be provided in the subsequent articles

Theorem 3.3. *Given $s \in K$, let $FH^s(\Omega)$ and $FH^{-s}(\Omega)$ be reducing subspaces of $H^s(K)$ and $H^{-s}(K)$, respectively, $\psi_0 \in H^s(K)$, $\tilde{\psi}_0 \in H^{-s}(K)$ and $\Psi \in H^s(K)$, $\tilde{\Psi} \in H^{-s}(K)$. Suppose that $X^s(\psi_0, \Psi)$ and $X^{-s}(\tilde{\psi}_0, \tilde{\Psi})$ are Bessel sequences in $FH^s(\Omega)$ and $FH^{-s}(\Omega)$, respectively. Then $X^s(\psi_0, \Psi); X^{-s}(\tilde{\psi}_0, \tilde{\Psi})$ is an NWBFs in $(FH^s(\Omega), FH^{-s}(\Omega))$ if and only if*

$$\hat{\psi}_0(\cdot)\overline{\hat{\tilde{\psi}}_0(\cdot + u(k))} \sum_{\ell=1}^{L} \sum_{j=0}^{\kappa(k)} \hat{\psi}_\ell(\mathfrak{p}^{-j}\cdot)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}(\cdot + u(k)))} = \delta_{0,k} \quad a.e.\ on\ \Omega. \tag{11}$$

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