Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 50(1), 2023, Pages 53–59, DOI: 10.52846/ami.v50i1.1618
ISSN: 1223-6934

Some fixed point results with integral type $(H, \psi)_F$-contraction

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Abstract. Recently, Jleli et. al. [9] introduced by the concept of a (H, ϕ)-fixed point, and they establish some fixed point results for various classes of operators defined on a metric space (M, d). This study is devoted to investigate the problem whether the existence and uniqueness of integral type $(H, \psi)_F$-contraction mappings on complete metric space. At the end, we give an illustrative example.

2010 Mathematics Subject Classification. 47H10, 54C20, 54H25.
Key words and phrases. fixed point, metric space,integral type.

1. Introduction and preliminaries

Fixed point theory plays a significative role in applications to different fields of mathematics. Survey on fixed points of generalized contraction mappings has been the basis of potent research attempts in fixed point theory. One of the elementary and the most widely executed results in metric fixed point theory is Banach contraction principle due to Banach in 1922 [3]. Banach's Contraction Principle says that, whenever (M, d) is complete, then any contraction selfmap of M has a unique fixed point. This classical fixed point theorem has been generalized by many researchers ın various vays. One of them is integral type contraction. So, the study of fixed point theorems of mappings satisfying contractive conditions of integral type has been a very interesting field of research activity after the establishment of a theorem by A. Branciari [4]. Afterwards, many authors obtained some fixed point theorems for several classes of contractive mappings of integral type; see [2, 6, 7, 8, 10, 11].

In 2012, Wardowski [17] originated the idea of $F-$contractions and produced a fixed point theorem involving such mappings. On the other, Jleli et. al. [9] introduced by the concept of a (H, ϕ)-fixed point, and they establish some fixed point results for various classes of operators defined on a metric space (M, d). Following this direction, Vetro [15] and Acar [1] proved some fixed point results using (H, ϕ)-contraction with F-contraction.

This study is devoted to investigate the problem whether the existence and uniqueness of integral type $(H, \psi)_F$-contraction mappings on complete metric space. To do this, we first recall some fundamental definitions and notations of corresponding mappings and space.

Let $\mathcal{F}$ be the family of all functions $F : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F_1) F is nondecreasing;

Received February 9, 2022. Accepted August 9, 2022.

(F_2) for every sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to+\infty}\alpha_n = 0$ if and only if $\lim_{n\to+\infty} F(\alpha_n) = -\infty$;
(F_3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha) = 0$. ([17])

Definition 1.1 ([17]). Let (M,d) be a metric space and f be a self-mapping on M. Then f is an F-contraction, if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that

$$a, b \in M,\ d(fa, fb) > 0 \Rightarrow \tau + F(d(fa, fb)) \le F(d(a,b)). \tag{1}$$

Also, we can say that every F-contraction f is a contractive mapping,so F-contraction is a continuous mapping .

Theorem 1.1 ([17]). *Let (M,d) be a complete metric space (C.M.S). If f is an F-contraction, then f has a unique fixed point in M.*

In [9], the authors introduced a family $\mathcal{H}$ of functions $H : [0,+\infty[^3 \to [0,+\infty[$ satisfying the following conditions:
(H_1) $\max\{\alpha,\beta\} \le H(\alpha,\beta,\gamma)$ for all $\alpha,\beta,\gamma \in [0,+\infty[$;
(H_2) $H(0,0,0) = 0$;
(H_3) H is continuous.

In the following, you can see some examples from the family of $\mathcal{H}$.
(i) $H(\alpha,\beta,\gamma) = \alpha+\beta+\gamma$ for all $\alpha,\beta,\gamma \in [0,+\infty[$;
(ii) $H(\alpha,\beta,\gamma) = \max\{\alpha,\beta\}+\gamma$ for all $\alpha,\beta,\gamma \in [0,+\infty[$;
(iii) $H(\alpha,\beta,\gamma) = \alpha+\beta+\alpha\beta+\gamma$ for all $\alpha,\beta,\gamma \in [0,+\infty[$.

Then the authors of [9] gave the following notion of (H,ϕ)-contraction.

Definition 1.2 ([9]). Let (M,d) be a metric space, $\phi : M \to [0,+\infty[$ be a given function and $H \in \mathcal{H}$. Then, $f : M \to M$ is called a (H,ϕ)-contraction with respect to the metric d if and only if

$$H(d(fa, fb), \phi(fa), \phi(fb)) \le kH(d(a,b), \phi(a), \phi(b)) \quad \text{for all } a, b \in M,$$

for some constant $k \in]0,1[$.

Now, we set

$$Z_\phi := \{a \in M : \phi(a) = 0\},$$
$$F_f := \{a \in M : fa = a\}.$$

Theorem 1.2 ([9]). *Let (M,d) be a complete metric space, $\phi : M \to [0,+\infty[$ is a function and $H \in \mathcal{H}$. Assume that the following conditions hold:*
(A_1) *ϕ is lower semi-continuous(l.s.c.);*
(A_2) *$f : M \to M$ is a (H,ϕ)-contraction with respect to the metric d.*

Then
(i) *$F_f \subset Z_\phi$;*
(ii) *f is a ϕ-Picard operator;*
(iii) *for all $a \in M$ and for all $n \in \mathbb{N}$ we have*

$$d(f^n a, \varsigma) \le \frac{k^n}{1-k} H(d(fa, a), \phi(fa), \phi(a)),$$

where $\{\varsigma\} = F_f \cap Z_\phi = F_f$.

Recently, Vetro ([15]) generalized Theorem1.2 by combining H-functions with F-contraction and show that every F-contraction is an F-H-contraction such that $H \in \mathcal{H}$ defined by $H(a,b,c) = a+b+c$ for all $a,b,c \in [0,+\infty[$, and $\phi : M \to [0,+\infty[$ defined by $\phi(a) = 0$ for all $a \in M$. After Vetro [15], Acar [1] introduced the concept of rational type F-H-contraction and proved that on a complete metric space, rational type F-H-contraction has a unique fixed point.

2. Main results

In this section, firstly, we defined integral type $(H,\psi)_F$-contraction and then give some results for this type mapping. At the ent, we give an illustrative example.

Definition 2.1. Let (M,d) be a metric space and $f : M \to M$ be a mapping. We say that f is an integral type $(H,\psi)_F$-contraction if there exist $F \in \mathcal{F}$, $H \in \mathcal{H}$, $\tau > 0$ and $\psi : M \to [0,+\infty[$ such that

$$\tau + F\left(\int_0^{H(d(fa,fb),\psi(fa),\psi(fb))} \varphi(t)dt\right) \le F\left(\int_0^{H(d(a,b),\psi(a),\psi(b))} \varphi(t)dt\right) \quad (2)$$

for all $a,b \in M$ with

$$H(d(fa,fb),\psi(fa),\psi(fb)) > 0$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesque integrable mapping which is summable, nonnegative and $\int\limits_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$.

Lemma 2.1. *Let (M,d) be a metric space and f be an integral type $(H,\psi)_F$-contraction. If $\{a_n\}$ is a Picard sequence starting at $a_0 \in M$, then*

$$\lim_{n\to+\infty} H(d(a_{n-1},a_n),\psi(a_{n-1}),\psi(a_n)) = 0,$$

and hence

$$\lim_{n\to+\infty} d(a_{n-1},a_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \psi(a_n) = 0.$$

Proof. By replacing the contradiction in [15, (3.2)] with contradiction (2) and following the proof of [15, Lemma 3.1], we immediately have the desired result. □

Theorem 2.2. *Let (M,d) be a complete metric space and f be an integral type $(H,\psi)_F$-contraction with a lower semicontinuous function $\psi : M \to [0,+\infty[$ such that (2) holds. Then f has a unique fixed point ς such that $\varphi(\varsigma) = 0$.*

Proof. First, let's show the uniqueness of the fixed point. On the contrary, assume that the fixed point is not unique and $\varsigma, w \in M$ such that $\varsigma = f\varsigma$, $w = fw$ and $\varsigma \ne w$. Using (2), we obtain

$$F\left(\int_0^{H(d(\varsigma,w),\psi(\varsigma),\psi(w))} \varphi(t)dt\right) = F\left(\int_0^{H(d(f\varsigma,fw),\psi(f\varsigma),\psi(fw))} \varphi(t)dt\right)$$
$$\le F\left(\int_0^{H(d(\varsigma,w),\psi(\varsigma),\psi(w))} \varphi(t)dt\right) - \tau \;<\; F\left(\int_0^{H(d(\varsigma,w),\psi(\varsigma),\psi(w))} \varphi(t)dt\right).$$

So, this is a contradiction and then $w = \varsigma$. Hence, the fixed point is unique.

Let us now show the existence of the fixed point. Create the $\{a_n\}$ sequence starting at $a_0 \in M$. If $a_{k-1} = a_k$ for some $k \in \mathbb{N}$, then $\varsigma = a_{k-1} = a_k = fa_{k-1} = f\varsigma$, that is, ς is a fixed point of f such that $\varphi(\varsigma) = 0$. So, by Lemma 2.1 and the property (H_1) of the function H, we have $\varphi(\varsigma) = 0$. In that case, we assume that $a_{n-1} \neq a_n$ for every $n \in \mathbb{N}$.

Now, we prove that $\{a_n\}$ is a Cauchy sequence. By Lemma 2.1, we say that

$$0 \quad < \quad h_{n-1} = H(d(a_{n-1}, a_n), \varphi(a_{n-1}), \varphi(a_n)) \to 0 \qquad \text{as } n \to +\infty.$$

The property (F_3) of the function F ensures that there exists $k \in (0,1)$ such that $h_n^k F(h_n) \to 0$ as $n \to +\infty$. Using (2) with $a = a_{n-1}$ and $b = a_n$, we get

$$\begin{aligned} F\left(\int_0^{H(d(a_n,a_{n+1}),\psi(a_n),\psi(a_{n+1}))} \varphi(t)dt\right) &\leq F\left(\int_0^{H(d(a_{n-1},a_n),\psi(a_{n-1}),\psi(a_n))} \varphi(t)dt\right) - \tau \\ &\vdots \\ &\leq F\left(\int_0^{H(d(a_0,a_1),\psi(a_0),\psi(a_1))} \varphi(t)dt\right) - n\tau \end{aligned} \tag{3}$$

for all $n \in \mathbb{N}$. Denote

$$\delta_n \quad = \quad \int_0^{H(d(a_n,a_{n+1}),\psi(a_n),\psi(a_{n+1}))} \varphi(t)dt$$

for $n = 0, 1, 2, \cdots$. Then $\delta_n > 0$ for all n and bu using (3)

$$F(\delta_n) \quad \leq \quad F(\delta_{n-1}) - \tau \leq \cdots \leq F(\delta_0) - n\tau \qquad \text{for all } n \in \mathbb{N}.$$

So, we get

$$0 \quad = \quad \lim_{n\to+\infty} h_n^k F(\delta_n) \leq \lim_{n\to+\infty} h_n^k (F(\delta_0) - n\tau) \leq 0,$$

and then,

$$\lim_{n\to+\infty} \delta_n^k n \quad = \quad 0.$$

So, $\sum_{n=1}^{+\infty} \delta_n$ is convergent series. From (H_1), $\sum_{n=1}^{+\infty} d(a_n, a_{n+1})$ is convergent series. Hence $\{a_n\}$ is a Cauchy sequence. Then there exists $\varsigma \in M$ such that

$$\lim_{n\to+\infty} a_n \quad = \quad \varsigma.$$

Because (M, d) is complete. By (2), we have

$$0 \quad \leq \quad \psi(\varsigma) \leq \liminf_{n\to+\infty} \psi(a_n) = 0,$$

that is, $\psi(\varsigma) = 0$. Now, show that ς is a fixed point of f. Clearly, ς is a fixed point of f if there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} = \varsigma$ or $fa_{n_k} = f\varsigma$, for all $k \in \mathbb{N}$. For the rest, we can suppose that $a_n \neq \varsigma$ and $fa_n \neq f\varsigma$ for all $n \in \mathbb{N}$. Then, taking $a = a_n$ and $b = \varsigma$ and using (2), we obtain that

$$\tau + F\left(\int_0^{H(d(fa_n,f\varsigma),\psi(fa_n),\psi(f\varsigma))} \varphi(t)dt\right) \quad \leq \quad F\left(\int_0^{H(d(a_n,\varsigma),\psi(a_n),\psi(\varsigma))} \varphi(t)dt\right).$$

Since $\tau > 0$, this inequality leads to for all $n \in \mathbb{N}$

$$\int_0^{H(d(fa_n,f\varsigma),\psi(fa_n),\psi(f\varsigma))} \varphi(t)dt) \quad < \quad \int_0^{H(d(a_n,\varsigma),\psi(a_n),\psi(\varsigma))} \varphi(t)dt,$$

and so

$$H(d(fa_n, f\varsigma), \psi(fa_n), \psi(f\varsigma)) < H(d(a_n, \varsigma), \psi(a_n), \psi(\varsigma))$$

for all $n \in \mathbb{N}$. Then, we obtain

$$\begin{aligned} d(\varsigma, f\varsigma) &\leq d(\varsigma, a_{n+1}) + d(fa_n, f\varsigma) \\ &\leq d(\varsigma, a_{n+1}) + H(d(fa_n, f\varsigma), \psi(fa_n), \psi(f\varsigma)) \\ &< d(\varsigma, a_{n+1}) + H(d(a_n, \varsigma), \psi(a_n), \psi(\varsigma)) \end{aligned} \tag{4}$$

for all $n \in \mathbb{N}$. Once and for all, letting $n \to +\infty$ in 4 and using continuity of H in $(0,0,0)$, we get $d(\varsigma, f\varsigma) \leq H(0,0,0) = 0$, that is, $\varsigma = f\varsigma$. □

Example 2.1. Let $M = [0,1]$ endowed with the metric $d(a,b) = |a-b|$ for all $a,b \in M$. Consider the mapping $f : M \to M$ defined by

$$f(a) = \begin{cases} \frac{a}{2} & ; \quad a \in [0,1) \\ \frac{3}{4} & ; \quad a = 1 \end{cases}.$$

Then f is an integral type (H,ψ)-contraction with respect to the function $F(a) = \ln a$, $H(a,\beta,\gamma) = \alpha + \beta + \gamma$, $\psi(a) = a$ and $\varphi(t) = t$.Indeed,

Case 1. For $0 < a \leq b < 1$ or $0 = a < b < 1$, we have

$$\begin{aligned} F\left(\int_0^{H(d(fa,fb),\psi(fa),\psi(fb))} \varphi(t)dt\right) &= F\left(\int_0^{H(d(0,\frac{b}{2}),\psi(0),\psi(\frac{b}{2}))} tdt\right) \\ &= F\left(\int_0^{b} tdt\right) = \ln\left(\frac{b^2}{2}\right) \end{aligned}$$

and

$$\begin{aligned} F\left(\int_0^{H(d(a,b),\psi(a),\psi(b))} \varphi(t)dt\right) &= F\left(\int_0^{H(d(0,b),\psi(0),\psi(b))} tdt\right) \\ &= F\left(\int_0^{2b} tdt\right) = \ln\left(2b^2\right). \end{aligned}$$

Case 2. For $a \in [0,1]$ and $b = 1$, we have

$$F\left(\int_0^{H(d(fa,fb),\psi(fa),\psi(fb))} \varphi(t)dt\right) = F\left(\int_0^{\frac{3}{2}} tdt\right) = \ln\left(\frac{9}{8}\right)$$

and

$$F\left(\int_0^{H(d(a,b),\psi(a),\psi(b))} \varphi(t)dt\right) = F\left(\int_0^{2} tdt\right) = \ln\left(2\right).$$

This shows that f is integral type $(H,\psi)_F$-contraction with $0 < \tau < \ln\frac{16}{9}$ therefore, all conditions of theorem are satisfied and so f has a fixed point in M.

Corollary 2.3. *Let (M, d) be a complete metric space and let f be an integral type (H, ψ)-contraction with a lower semicontinuous function $\psi : M \to [0, +\infty[$ such that (2) holds, that is*

$$\int_0^{H(d(fa,fb),\psi(fa),\psi(fb))} \varphi(t)dt \quad \leq \quad k\int_0^{H(d(a,b),\psi(a),\psi(b))} \varphi(t)dt, k \in (0,1).$$

Then f has a unique fixed point ς such that $\psi(\varsigma) = 0$.

Corollary 2.4. *Let (M, d) be a complete metric space and let f be an (H, ψ)-contraction with a lower semicontinuous function $\psi : M \to [0, +\infty[$ such that (2) holds, that is*

$$H(d(fa, fb), \psi(fa), \psi(fb)) \quad \leq \quad kH(d(a, b), \psi(a), \psi(b)), k \in (0, 1).$$

Then f has a unique fixed point ς such that $\psi(\varsigma) = 0$.

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