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Equivalence of "generalized" solutions for nonlinear parabolic equations with variable exponents and diffuse measure data

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Abstract. We prove the equivalence of suitably defined weak solutions of a nonhomogeneous initial-boundary value problem for a class of nonlinear parabolic equations. We also develop the notion of both "*renormalized*" and "*entropy*" solutions with respect to the "generalized" $p(\cdot)$-capacity, initial datum, and diffuse measure data (which does not charge the set of null $p(\cdot)$-capacity). Conditions, under which "generalized weak" solutions of the nonhomogeneous problem are in fact well-defined, are also given.

Résumé. Equivalence entre les solutions "généralisées" des équations non-linéaires paraboliques à exposant variable et des données measures diffuses. Nous montrons l'équivalence entre les solutions faibles d'un problème non-homogène aux limites d'une classe d'équations non-linéaires paraboliques. Nous dévellopons aussi la notion des solutions "renormalisées" et "entropiques" par rapport à la $p(\cdot)$-capacité "généralisée", la donnée initiale et la mesure diffuse (qui ne charge pas les ensembles de capacité nulle). Les conditions, sous lesquelles les solutions "faibles généralisées" du problème non-homogène sont définies, sont aussi étudiées.

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1. Introduction & historical context

This paper is the second one of a series where we address "*generalized weak*" solutions of the initial boundary value problem whose model example is

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \mu \text{ in } (0,T)\times\Omega, \\ u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \quad u(0,x) = u_0(x) \text{ in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open domain of $\mathbb{R}^N$ with lipshitz boundary $\partial\Omega$, $N \geq 2$, $T > 0$ is any positive constant, $p(x) : \overline{\Omega} \mapsto [1, +\infty)$ is a continuous, real-valued function (the variable exponent) with $p_- = \min_{x\in\overline{\Omega}} p(x)$, $1 < p_- < \infty$, $u_0 \in L^1(\Omega)$ is an integrable function, $u \mapsto -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplace operator, and μ is a measure with bounded variation over $Q = (0,T)\times\Omega$ which does not charge sets of zero $p(\cdot)$-capacity in accordance with Definition 2.6 (we suppose that μ depends on time variable t). The content of this paper is an extension of the joint result [2] with, respectively, *S. Ouaro*[1] & *U. Traoré*[1], where we study the existence of generalized solutions of (1) for every diffuse measure, and in particular the link between the

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parabolic $p(\cdot)$-capacity and the absolutely continuous measures which is needed to have existence of solutions, and were we extend the theory of capacity to generalized Sobolev spaces in order to study some nonlinear parabolic equations (we define and give some properties of renormalized solutions and, as a consequence, we show the existence and uniqueness of solutions). The used main technical tools include estimates, compactness and convergence results. More precisely, we study the nonlinear parabolic problem

$$(P_\mu^Q) \quad \begin{cases} u_t - \operatorname{div}(a(t,x,u,\nabla u)) = \mu \text{ in } Q = (0,T)\times\Omega \\ u = u_0 \text{ on } \{0\}\times\Omega \quad u = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases}$$

based on the operator $u_t + A(u)$, where $A(u) = -\operatorname{div}(a(t,x,u,\nabla u))$ is a *Leray-Lions* operator. Since *Kovacik & Rákosník* systemically studied the variable Lebesgue and Sobolev spaces in [63], the interest in variable exponent spaces is increasing from year to year. Indeed, many variable exponent spaces have appeared such as "*Bessel potential*" spaces with variable exponent, "*Morrey*" and "*Hardy*" spaces with variable exponent. Variable exponent spaces have many applications in "*electrorheological*" fluid theory [86], in differential equations [54], in image restoration [30, 55, 67], and in variable order pseudo-differential equations [87]. The study of PDEs with measure data arouses much interest with the development of the concept of the thermal capacity and the classical potential theory, we refer the readers to [38, 66, 89]. Capacities defined in terms of function-spaces can be regarded as a very important class of parabolic capacities. There are already numerous results for such kind of problems, see [7, 40, 58, 75, 93], where the functional spaces to deal with these problems are the constant/exponent Lebesgue/Sobolev spaces.

The concept of parabolic capacities of generalized Sobolev spaces with variable exponent was studied by [74] in order to deal with existence of "*generalized*" solutions. The monumental work [86] contains plenty of motivations for studying much kind of spaces, see also [30, 90, 5, 6], where much of these nonlinear parabolic problems are non-homogeneous; consequently, they are more complicated than the classical parabolic case. The main motivation for using the notion of capacity is that it gives optimal results for boundary regularity, then it is reasonable to work with "*entropy*" solutions or "*renormalized*" solutions, which need less regularity than the usual weak solutions; we recall that the notion of renormalized solution was introduced in [24] by *DiPerna & Lions* in their studies on *Boltzmann* equations. This notion was adapted, by *Boccardo, Diaz, Giachetti & Murat* [12], and *Lions & Murat* [64], to study some nonlinear elliptic problems with Dirichlet boundary conditions; later, it was extended to more general problems of elliptic, parabolic and hyperbolic types, see [24, 77]. At the same time the notion of *entropy* solution has been proposed by *Bénilan & all*, in [8], for nonlinear elliptic problems. This framework was also extended to related parabolic problems, see [18, 83]. In the two former papers [88, 25], they have already studied the renormalized and entropy solutions for elliptic problems with variable exponents and arbitrary L^1-data, and recently *Zhan & Zhou*, see [95], have established the existence and uniqueness of entropy solution via the difference and variation methods. In [73], *Ouaro & al.* have combined the ideas of [94, 95] to prove the equivalence between entropy and renormalized solutions. Recalling that, for the study of existence of entropy/renormalized solutions for stationary problems with right-hand side measure, the authors use in general the techniques of measure's decomposition

In the context of constant exponent, the authors prove, in [18], that every diffuse measure μ (i.e, a measure which does not charge sets of null p-capacity) belongs to $L^1(\Omega)+W^{1,p'}(\Omega)$. A similar approach, in the context of variable exponent, is used in [71] for elliptic problems where μ is a diffuse measure. Nonlinear parabolic problems with measure data was studied in the context of constant exponent by many authors; for example in [13], the authors proved existence of a "*weak*" solution by approximating the measure μ with regular data (for more studies on entropy/renormalized solutions, see [8, 9, 39, 32, 39, 30]). Note that, the authors, in [74], develop a notion of parabolic $p(\cdot)$-capacity based on the operator $u_t + A(u)$, where A is a *Leray-Lions* operator; they worked with the space

$$W_{p(\cdot)} = \Big\{ u \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega)), \nabla u \in L^{p(\cdot)}(Q)^N, \\ u_t \in L^{(p_-)'}(0,T; (W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega))') \Big\}, \tag{2}$$

in order to obtain a "*representation theorem*" for measures that are zero on parabolic subsets of null capacity (denoted by $\mathcal{M}_0(Q)$); precisely, they prove the following result:

Theorem 1.1. *Let $\mu \in \mathcal{M}_0(Q)$, then there exists (f, F, g_1, g_2) with $f \in L^1(Q)$, $F \in L^{p'(\cdot)}(Q)^N$, $g_1 \in L^{(p_-)'}(0,T; W^{-1,p'(\cdot)}(\Omega))$ and $g_2 \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega))$ such that*

$$\int_Q \varphi d\mu = \int_Q f\varphi dxdt + \int_Q F \cdot \nabla\varphi dxdt + \int_0^T \langle g_1, \varphi\rangle dt - \int_0^T \langle \varphi_t, g_2\rangle dt, \tag{3}$$

for every $\varphi \in C_c^\infty([0,T]\times\Omega)$ (the quadruplet (f, F, g_1, g_2) is called a "decomposition" of μ).

More generally, in this work, we are concerned with the proof of equivalence of renormalized and entropy solutions, using the well-known results from the theory of generalized capacities. Motivated by the previous papers, our aim is to prove under which conditions we obtain the equivalence between the two formulations for evolution problems with variable exponent and diffuse measure data, see also [84] for some possible extensions for possibly generalized porous medium equations.

Our paper is organized as follows. In Section 2 we summarize several results we need about variable exponent spaces and measures of bounded variations, we give the definitions of *renormalized* and *entropy* solutions for problem $(\mathcal{P}_\mu^Q)$ and we state our main result. The Section 3 is devoted to prove the equivalence between both notions of solutions (the proof is inspired from the nonlinear compactness theory).

Notations. Throughout the paper, we assume that $p(x) \in C_+(\overline{\Omega})$ satisfies the *log-Hölder continuity* condition (also called as *Dini-Lipschitz*, *weak-Lipschitz* or 0-*Hölder* conditions): we say that $p(\cdot)$ is log-Hölder continuous if $p(\cdot) : \Omega \mapsto \mathbb{R}$ is a measurable function such that

$$\begin{gathered}\exists C>0 \text{ s.t } |p(x)-p(y)| \leq \frac{C}{-\ln|x-y|} \quad \text{for} \quad |x-y|<\frac{1}{2}, \\ 1<\operatorname*{ess\,inf}_{x\in\Omega} p(x) \leq \operatorname*{ess\,sup}_{x\in\Omega} p(x)<N.\end{gathered} \tag{4}$$

This condition has emerged as the right one to guarantee regularity of variable exponent Lebesgue spaces, and to obtain several regularity results for Sobolev spaces with variable exponents (in particular, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$). We denote by T_k the truncation function at level $k \geq 0$ and $\Theta_k : \mathbb{R} \to \mathbb{R}^+$ its primitive function defined by

$$\begin{cases} T_k(r) = \min\{k, \max\{r,-k\}\} = \begin{cases} -k & \text{if } r \leq -k, \\ r & \text{if } |r|<k, \\ k & \text{if } r \geq k, \end{cases} \\ \Theta_k(r) = \displaystyle\int_0^r T_k(s)ds = \begin{cases} \frac{r^2}{2} & \text{if } |r| \leq k, \\ k|r| - \frac{k^2}{2} & \text{if } |r| \geq k. \end{cases} \end{cases} \tag{5}$$

It is obvious that $\Theta_k(r) \geq 0$ and $\Theta_k(r) \leq k|r|$. We introduce the space $\widetilde{W}$ by

$$\begin{aligned}\widetilde{W} =&\big\{u \in L^{p_-}(0,T;W^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q), \nabla u \in (L^{p(\cdot)}(Q))^N; \\ & u_t \in L^{(p_-)'}(0,T;W^{-1,p'(\cdot)}(\Omega)) + L^1(Q)\big\}.\end{aligned} \tag{6}$$

We also need to define the very weak gradient of a measurable function u (where the proof follows from [8, Lemma 2.1] due to the fact that $W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p_-}(\Omega)$) as follows:

Proposition 1.2. *For every measurable function u, there exists a unique measurable function $v : Q \mapsto \mathbb{R}^N$, which we call the very weak gradient of u and we denote $v = \nabla u$, such that*

$$\nabla T_k(u) = v\chi_{|u|<k}, \quad \textit{almost everywhere (a.e.) in } Q \textit{ and for every } k>0, \tag{7}$$

where χ_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $L^1(0,T;W_0^{1,1}(\Omega))$, then v coincides with the weak gradient of u.

2. Preliminaries

In this section, we first state some elementary results for the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the generalized Lebesgue-Sobolev spaces $W^{m,p(x)}(\Omega)$. The basic properties of these spaces can be found in [46, 47], we also introduce the notion of nonlinear parabolic capacity with exponent variable $\text{cap}_{p(\cdot)}$ and then investigate the relationships between time-space dependent measures and generalized capacities.

2.1. Variable exponent Lebesgue/Sobolev spaces. We recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. We refer to *Fan & Zhao* [46, 47] for further properties on variable exponent spaces. The same spaces appear also in the study of variational integrals with non-standard growth, see [4, 31, 91]. Another area where these spaces have found applications is the study of electrorheological fluids, see the papers by *Diening* alone [35] and with Růžička [42] on the role of variable exponent in this context. To this aim, we start with a brief overview of the state of the art concerning elliptic spaces with variable exponent and parabolic spaces modeled upon them. First of all, let us introduce the following notations

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x),$$

and given a bounded measurable function $p(\cdot) : \Omega \mapsto \mathbb{R}$, the critical Sobolev exponent and the conjugate of $p(\cdot)$ are, respectively, defined by

$$p^\star(\cdot) = \frac{Np(\cdot)}{N - p(\cdot)} \quad \text{and} \quad p'(\cdot) = \frac{p(\cdot)}{p(\cdot) - 1}.$$

We define the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \mapsto \mathbb{R}$ for which the convex modular $\rho_{p(\cdot)}(\Omega) = \int_\Omega |u|^{p(x)}dx$ is finite, i.e.,

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \mapsto \mathbb{R},\ u \text{ is measurable with } \int_\Omega |u(x)|^{p(x)}dx < \infty \right\}.$$

If the exponent is bounded, i.e., if $p_+ < \infty$, we define a norm in $L^{p(\cdot)}(\Omega)$, called "*the Luxembourg norm*", by the formula

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0,\ \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) dx = \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\}.$$

The following inequality will be used later

$$\min \left\{ \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)} \right\} \le \int_\Omega |u(x)|^{p(x)}dx \le \max \left\{ \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)} \right\}. \tag{8}$$

The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}})$ is a separable *Banach* space. Moreover, if $p_- > 1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the following *Hölder's inequality*

$$\int_\Omega |uv|dx \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)},\ \forall u \in L^{p(\cdot)}(\Omega),\ \forall v \in L^{p'(\cdot)}(\Omega) \tag{9}$$

holds true. One central property on $L^{p(\cdot)}(\Omega)$ is that the norm and the modular topology coincide, i.e., $\rho_{p(\cdot)}(u_n) \to 0$ if and only if $\|u_n\|_{L^{p(\cdot)}} \to 0$.

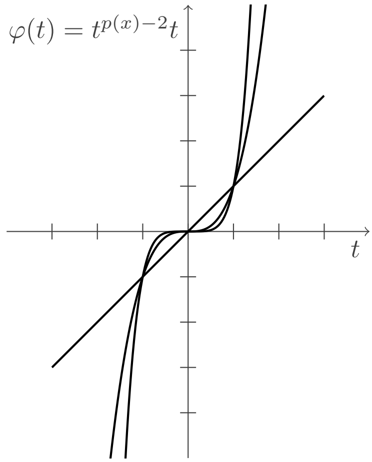

FIGURE 1. The function $t^{p(x)-2}t$ for $p(x) = 2, 4, 6$

We also define the variable Sobolev space by

$$W^{1,p(\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega),\ |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

which is a *Banach* space equipped with one of the following equivalent norms

$$\begin{cases} \|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \\ \|u\|_{W^{1,p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0,\ \int_\Omega \left(\left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} + \left| \frac{u(x)}{\lambda} \right|^{p(x)} \right) dx \leq 1 \right\}. \end{cases}$$

By $W_0^{1,p(\cdot)}(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, that is,

$$W_0^{1,p(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}.$$

Assuming $p^- > 1$, the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive *Banach* spaces and the space $W^{-1,p'(\cdot)}(\Omega)$ denotes the dual of $W_0^{1,p(\cdot)}(\Omega)$. For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega})$ and $p^- \geq 1$, the *Poincaré inequality* holds, see [53], for some constant C which depends on Ω and the function $p(\cdot)$. The proofs of the following Propositions can be found, respectively, in [46, 63, 45] (see also [34] for more details).

Proposition 2.1 (*The $p(\cdot)$-Poincaré inequality*). *Let Ω be a bounded open set and let $p(\cdot) : \Omega \mapsto [1, \infty)$ satisfy (4). Then there exists a constant C, depending on $p(\cdot)$ and Ω, such that the inequality*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \tag{10}$$

holds for every $u \in W_0^{1,p(\cdot)}(\Omega)$.

Note that the following inequality $\int_\Omega |u|^{p(x)} dx \leq C \int_\Omega |\nabla u|^{p(x)} dx$ does not hold in general, see [46].

Proposition 2.2 (1^{st} *Sobolev embedding*)**.** *Let Ω be a bounded open set, with a Lipschitz boundary, and let $p(\cdot) : \Omega \mapsto [1, \infty)$ satisfy (4). Then we have the following continuous embedding*

$$W^{1,p(\cdot)}(\Omega) \underset{cont}{\hookrightarrow} L^{p^*(\cdot)}(\Omega) \text{ with } p^*(\cdot) = \frac{Np(\cdot)}{N - p(\cdot)}.$$

Proposition 2.3 (2^{nd} *Sobolev embedding*)**.** *For $p(\cdot) \in C(\overline{\Omega})$ with $1 < p^- \le p(x) \le p^+ < N$, the Sobolev embedding*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega), \tag{11}$$

hold, for every measurable function $r(\cdot) : \Omega \mapsto [1, +\infty)$ such that

$$\underset{x\in\Omega}{\text{ess-inf}} \left(\frac{Np(x)}{N - p(x)} - r(x) \right) > 0.$$

For $Q := (0, T) \times \Omega$ with $T > 0$ and by extending the variable exponent $p(\cdot) : \overline{\Omega} \mapsto [1, +\infty)$ to $\overline{Q} = [0, T] \times \overline{\Omega}$ (by setting $p(t, x) := p(x)$ for all $(t, x) \in \overline{Q}$), one can define the generalized Lebesgue space (which, of course, shares the same type of properties as $L^{p(\cdot)}(\Omega)$) by

$$L^{p(\cdot)}(Q) = \left\{ u : Q \mapsto \mathbb{R} \text{ measurable with } \int_Q |u(t, x)|^{p(x)}\, dxdt < \infty \right\},$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(Q)} = \inf \left\{ \lambda > 0,\ \int_0^T \int_\Omega \left| \frac{u(t, x)}{\lambda} \right|^{p(x)} dxdt \le 1 \right\}.$$

Moreover, if $p(\cdot)$ is log-Hölder continuous in Ω, so it is in Q. Indeed, if $p(\cdot)$ satisfies the log-Hölder continuity condition in Ω, according to (4), there exists a non-decreasing function $\omega : (0, \infty) \mapsto \mathbb{R}$ such that $\limsup\limits_{t\to 0^+} \omega(t)\ln(\frac{1}{t}) < +\infty$ and

$$|p(t, x) - p(s, y)| = |p(x) - p(y)| < \omega(|x - y|) \le \omega(|(t, x) - (s, y)|),$$

holds for all $((t, x), (s, y)) \in \overline{Q} \times \overline{Q}$ such that $|(t, x) - (s, y)| < 1$. Now, If V is a *Banach* space, we will also use the standard notations for *Bochner* spaces, that is to say, if $1 \le q \le \infty$ and $T > 0$ then $L^q(0, T; V)$ denotes the space of strongly measurable functions $u : (0, T) \mapsto V$ such that $t \mapsto \|u(t)\|_V \in L^q(0, T)$. Moreover, $\mathcal{C}([0, T]; V)$ denotes the space of continuous functions $u : [0, T] \mapsto V$ endowed with the norm $\|u\|_{C([0,T];V} = \max\limits_{t\in[0,T]} \|u(t)\|_V$. The following interesting density result will be used in the study our parabolic problem.

Proposition 2.4. *Let $V = L^p(\Omega)$ (or $V = W^{1,p}(\Omega)$) and $1 \le p < \infty$. Then, $\mathcal{D}((0, T) \times \Omega)$ is dense in $L^q(0, T; V)$ for any $1 \le q < \infty$.*

Proof. From [41, Corollary 1.3.1], it follows that

$$Z := \left\{ \sum_{i=1}^n \phi_i(x)\psi_i(t),\ n \ge 1,\ \phi_i \in \mathcal{D}(\Omega),\ \psi_i \in \mathcal{D}(0, T) \right\} \subset \mathcal{D}((0, T) \times \Omega)$$

is dense in $L^q(0, T; V)$ for any Banach space V such that $\mathcal{D}(\Omega)$ is dense in V and $1 \le q < \infty$. $\square$

Let $p(\cdot) : \overline{\Omega} \mapsto [1, \infty)$ be a continuous variable exponent and $T > 0$, the two *Bochner* spaces $L^{p^+}(0, T; L^{p(\cdot)}(\Omega))$ and $L^{p_-}(0, T; L^{p(\cdot)}(\Omega))$ will be important in our study. In the following we identify a function like $v \in L^{p_-}(0, T; L^{p(\cdot)}(\Omega))$ with the real-valued function v defined by $v(t, x) = v(t)(x)$ for almost all $t \in (0, T)$ and a.e. $x \in \Omega$. In the same way we associate to any function $v \in L^{p(\cdot)}(Q)$ a function $v : (0, T) \mapsto L^{p(\cdot)}(\Omega)$ by setting $v(t) := v(t, \cdot)$ for a.e. $t \in (0, T)$.

Lemma 2.5. *We have the following continuous dense embeddings*

$$L^{p_+}(0, T; L^{p(\cdot)}(\Omega)) \overset{dense}{\hookrightarrow} L^{p(\cdot)}(Q) \overset{dense}{\hookrightarrow} L^{p_-}(0, T; L^{p(\cdot)}(\Omega)). \tag{12}$$

Proof. For $v \in L^{p(\cdot)}(Q)$, the corresponding function $v : (0, T) \mapsto L^{p(\cdot)}(\Omega)$ is strongly *Bochner* measurable by the *Dunford-Pettis* Theorem (we recall that *Dunford-Pettis* Theorem ensures that a sequence in $L^1(D)$, D any bounded open subset of $\mathbb{R}^N$, is weakly convergence in $L^1(D)$ if and only if it is equi-integrable), and since it is weakly measurable and $L^{p(\cdot)}(\Omega)$ is separable. Moreover, using the fact that

$$\begin{aligned}
\int_0^T \|v(t)\|^{p_-}_{L^{p(\cdot)}(\Omega)} dt &\le \int_0^T \max\left[\int_\Omega |v(t,x)|^{p(x)}dx,\ \left(\int_\Omega |v(t,x)|^{p(x)}dx\right)^{\frac{p_-}{p_+}}\right] dt \\
&\le \int_0^T \int_\Omega |v(t,x)|^{p(x)}dxdt + T^{1-\frac{p_-}{p_+}} \left(\int_0^T \int_\Omega |v(t,x)|^{p(x)}dxdt\right)^{\frac{p_-}{p_+}} \\
&\le \max\left[|v|^{p_-}_{L^{p(\cdot)}(Q)}, |v|^{p_+}_{L^{p(\cdot)}(Q)}\right] + T^{1-\frac{p_-}{p_+}} \max\left[\|v\|^{\frac{(p_-)^2}{p_+}}_{L^{p(\cdot)}(Q)}, \|v\|^{p_-}_{L^{p(\cdot)}(Q)}\right],
\end{aligned} \tag{13}$$

the embedding of $L^{p(\cdot)}(Q)$ into $L^{p_-}(0, T; L^{p(\cdot)}(\Omega))$ is continuous.
Now, if $u \in L^{p_+}(0, T; L^{p(\cdot)}(\Omega))$ and from the fact that $L^{p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$ it follows that $u \in L^{p_+}(0, T; L^1(\Omega))$; hence, according to [41, Proposition 1.8.1], the corresponding real-valued function $u : (0, T) \times \Omega \mapsto \mathbb{R}$ is measurable and using the same arguments as above we find the continuous embedding of $L^{p_+}(0, T; L^{p(\cdot)}(\Omega))$ into $L^{p(\cdot)}(Q)$ (it is left to prove that both embeddings are dense). Now, we consider the first embedding and we fix $u \in L^{p(\cdot)}(Q)$, then, since $\mathcal{D}(Q)$ is dense $L^{p(\cdot)}(Q)$, we find a sequence $(u_n) \subset \mathcal{D}(Q)$ converging to u in $L^{p(\cdot)}(Q)$ as $n \to \infty$. According to Proposition 2.4 we have $\mathcal{D}(Q)$ is densely embedded into $L^{p_+}(0, T; L^{p_+}(\Omega))$, therefore $u_n \in L^{p_+}(0, T; L^{p(\cdot)}(\Omega))$ for all $n \in \mathbb{N}$. To prove the density of the second embedding, we fix $v \in L^{p_-}(0, T; L^{p(\cdot)}(\Omega))$, and taking a standard sequence of mollifiers $(\rho_n)_n \subset \mathcal{D}(\mathbb{R})$ and extending v by zero onto $\mathbb{R}$, from [41, Proposition 1.7.1], it follows that the regularized (in time) function

$$(\rho_n * v)(\cdot) := \int_{\mathbb{R}} \rho_n(\cdot - s)v(s)ds \tag{14}$$

is in $L^{p_+}(\mathbb{R}, L^{p(\cdot)}(\Omega))$ for each $n \in \mathbb{N}$, hence in $L^{p(\cdot)}(Q)$ and converges to v in $L^{p_-}(0, T; L^{p(\cdot)}(\Omega))$ (see also [41, Theorem 1.7.1]). $\square$

Now, we state two embedding theorems that will play a central role in our work; the fist one is the well-known "*Gagliardo-Nirenberg*" generalized embedding that we state in a form general enough to our purpose.

Lemma 2.6 (*Gagliardo-Nirenberg generalized inequality*)**.** *Let v be a function in $W_0^{1,q(\cdot)}(\Omega)\cap L^{\rho(\cdot)}(\Omega)$ with q and ρsatisfy the log-Hölder continuity condition (4), $1 < q^- \le q(x) \le q^+ \le N$, $1<\rho^- \le \rho(x) \le \rho^+ \le N$. Then, there exists a positive constant C, depending on N, $q(x)$ and $\rho(x)$, such that*

$$\|v\|_{L^{\gamma(\cdot)}(\Omega)} \le C\|\nabla v\|^{\theta}_{(L^{q(\cdot)})^N}\|v\|^{1-\theta}_{L^{\rho(\cdot)}(\Omega)}$$

for every θ and $\gamma(\cdot)$ satisfying $0\le\theta\le 1$, $1\le\gamma(\cdot)\le+\infty$ and $\frac{1}{\gamma(\cdot)} = \theta(\frac{1}{q(\cdot)}-\frac{1}{N})+\frac{1-\theta}{\rho(\cdot)}$.

Proof. The proof follows the same lines as the proof for the case of constant exponent, see [72, Lecture II] (see also [23, Page 147]). □

The second one is a consequence of the previous result where we give here for completeness.

Corollary 2.7. *Let $v\in L^{q^-}((0,T),W_0^{1,q(\cdot)}(\Omega))\cap L^\infty((0,T),L^2(\Omega))$, with $q(x)$ satisfies the log-Hölder continuity condition (4) and $1<q^-\le q(x)\le N$. Then $v\in L^{\sigma(\cdot)}(\Omega)$ with $\sigma(\cdot)=q(\cdot)\frac{N+2}{N}$ and*

$$\int_Q |v|^{\sigma(x)}dxdt \le C\max\left(\|v\|^{\frac{2q^+}{N}}_{L^\infty(0,T;L^2(\Omega))};\|v\|^{\frac{2q^-}{N}}_{L^\infty(0,T;L^2(\Omega))}\right)$$
$$\times\max\left(\left(\int_Q|\nabla v|^{q(x)}dxdt\right)^{\frac{q^+}{q^-}};\left(\int_Q|\nabla v|^{q(x)}dxdt\right)^{\frac{q^-}{q^+}}\right).$$

Proof. See [44, Corollary A.1]. □

2.2. Variable exponent capacities & diffuse measures. The notion of $p(\cdot)$-capacity plays the expected role in the potential theory and in the study of Sobolev functions in the variable exponent setting, see [51, 52, 53, 57]. In general, the $p(\cdot)$-capacity is used to measure finite properties of functions and sets. Then $p(\cdot)$-capacity enjoys the usual fine properties of capacity when $1<p_-\le p(x)\le p_+<\infty$, see [52, 33] (some of the properties remain still open for the case $p_-=1$). In this part, we study Lebesgue points and quasi-continuity of Sobolev functions in the variable exponent setting. In [50] (these are extensions of the classical results of [56]), the authors proved that every Sobolev function has Lebesgue points outside of a set of $p(\cdot)$-capacity zero and that the precise pointwise representative of a Sobolev function is $p(\cdot)$ quasi-continuous.

To continue, we need to introduce some basic tools that we need in our study.

Definition 2.1. Let $p(\cdot):\Omega\mapsto[1,\infty)$ be a variable exponent, the $p(\cdot)$-capacity of a set $E\subset\mathbb{R}^N$ is defined as

$$C_{p(\cdot)}(E)=\inf\left\{\int_{\mathbb{R}^N}|u|^{p(x)}+|\nabla u|^{p(x)}dx\right\},$$

where the infimum is taken over admissible functions $u\in S_{p(\cdot)}(E)$, where

$$S_{p(\cdot)}(E)=\{u\in W^{1,p(\cdot)}(\mathbb{R}^N): u\ge 1 \text{ in an open set containing } E\}.$$

It is easy to see that if we restrict these admissible functions $S_{p(\cdot)}(E)$ to the case $0\le u\le 1$, we get the same capacity.

Definition 2.2. We say that a claim holds $p(\cdot)$ quasi-everywhere ($p(\cdot)$-q-e) if it holds everywhere except in a set of $p(\cdot)$-capacity zero. A function $u : \Omega \mapsto \mathbb{R}$ is said to be $p(\cdot)$ quasi-continuous ($p(\cdot)$-q-c) if for every $\epsilon > 0$ there exists an open U with $C_{p(\cdot)}(U) < \epsilon$ such that u restricted to $\Omega \backslash U$ is continuous.

A variable exponent version of the relative $p(\cdot)$-capacity of the condenser has been used in [51]. This alternative capacity of a set is taken relative to a surrounding open subset of $\mathbb{R}^N$. Suppose that $p_+ < \infty$ and $p(x)$ satisfies the log-Hölder continuity condition (4) and let K be a compact subset of Ω, the relative $p(\cdot)$-capacity of K in Ω is the number

$$\operatorname{cap}_{p(\cdot)}(K,\Omega) = \inf \left\{ \int_\Omega |\nabla \varphi|^{p(x)} dx : \varphi \in C_0^\infty(\Omega) \text{ and } \varphi \geq 1 \text{ in } K \right\}.$$

We define (for different types of sets)

$$\begin{cases} \operatorname{cap}_{p(\cdot)}(U,\Omega) = \sup \left\{ \operatorname{cap}_{p(\cdot)}(K,\Omega) : K \subset U \text{ compact} \right\} \text{ for an open set } U \subset \Omega, \\ \operatorname{cap}_{p(\cdot)}(E,\Omega) = \inf \left\{ \operatorname{cap}_{p(\cdot)}(U,\Omega) : U \supset E \text{ open} \right\} \text{ for an arbitrary } E \subset \Omega, \\ \operatorname{cap}_{p(\cdot)}(E,\Omega) = \sup \left\{ \operatorname{cap}_{p(\cdot)}(K,\Omega) : K \supset E \text{ compact} \right\} \text{ for all Borel sets } E \subset \Omega. \end{cases} \tag{15}$$

Let us recall some quasi-properties on the function u with respect to the $p(\cdot)$-capacity.

Definition 2.3. We say that $u : \Omega \to \overline{\mathbb{R}}$ is $p(\cdot)$ quasi-continuous ($p(\cdot)$-q.c) if for $\epsilon > 0$ there exists an open set $A \subset \Omega$ with $\operatorname{cap}_{p(\cdot)}(A,\Omega) \leq \epsilon$ such that $u_{(\Omega \backslash A)}$ is continuous. Every $u \in W^{1,p(\cdot)}(\Omega)$ has a $p(\cdot)$ quasi-continuous representative ($p(\cdot)$-q.c.r), always, denoted by u and which is essentially unique.

Similarly, and since we are interested in properties of solutions, we shall mainly work with capacities of compact sets and we shall restrict our attention on some specific results relying generalized capacities and measure spaces (more especially, the set of bounded measures on Q which are absolutely continuous with respect to the $p(\cdot)$-parabolic capacity), but first let us state the following (general) definition.

Definition 2.4. If $U \subset Q$ is an open set, we define the generalized parabolic capacity of U as

$$\operatorname{cap}_{p(\cdot)}(U) = \inf \left\{ \|u\|_{W_{p(\cdot)}(0,T)},\ u > \chi_U \text{ a.e. in } Q \right\}, \tag{16}$$

where $W_{p(\cdot)}(0,T)$ is defined in (2) (we will use the convention that $\inf \emptyset = +\infty$).

Remark 2.1. For any Borelian subset $B \subset Q$, the definition of the capacity can be extended by setting

$$\operatorname{cap}_{p(\cdot)}(B) = \inf \left\{ \operatorname{cap}_{p(\cdot)}(U),\ U \text{ open subset of } Q,\ B \subset U \right\}. \tag{17}$$

The following definition gives a characterization of the parabolic capacity of compact sets of Q.

Definition 2.5. Let K be a compact subset of Q, the capacity of K is defined as

$$\operatorname{cap}(K) = \inf \left\{ \|u\|_{W_{p(\cdot)}(0,T)},\ u \in C_c^\infty(Q)(0,T]) \times \Omega),\ u > \chi_K \right\}. \tag{18}$$

Now, one can deduce the following results from the previous definitions.

Proposition 2.8. *As a consequence:*

(i) *The parabolic capacity satisfies the "sub-additivity" property, i.e.,*

$$cap(E) \leq \sum_{i=1}^{\infty} cap(E_i) \text{ with } E = \cup_{i=1}^{\infty} E_i,\ E_i = 1, 2, \cdots, \text{ are arbitrary subsets of } Q. \tag{19}$$

(ii) *The parabolic capacity is a "monotonic" set function, i.e.,*

$$cap(E_1) \leq cap(E_2) \text{ if } E_1 \subset E_2. \tag{20}$$

(iii) *The parabolic capacity satisfies the standard "limiting" result, i.e.,*

$$\begin{cases} \lim\limits_{i\to\infty} cap(E_i) = cap(E), \ \text{with } E = \cup_{i=1}^{\infty} E_i,\ E_1 \subset E_2 \subset, \cdots, \\ \lim\limits_{i\to\infty} cap(E_i) = cap(E), \ \text{with } E = \cap_{i=1}^{\infty} E_i,\ E_1 \supset E_2 \supset, \cdots. \end{cases} \tag{21}$$

In the next, we denote by $\mathcal{M}_b(Q)$ the space of bounded measures on the σ-algebra of Borelian subsets of Q, by $\mathcal{M}_b^+(Q)$ the subsets of non-negative measures of $\mathcal{M}_b(Q)$, and by $\mathcal{M}_0(Q)$ the space of bounded measures not charging sets of null $p(\cdot)$-capacity.

Definition 2.6. We define $\mathcal{M}_0(Q)$ as:

$$\mathcal{M}_0(Q) = \left\{ \mu \in \mathcal{M}_b(Q) : \mu(E) = 0 \text{ for every subset } E \subset Q \text{ s.t. } \mathrm{cap}_{p(\cdot)}(E) = 0 \right\}. \tag{22}$$

The nonnegative measures in $\mathcal{M}_0(Q)$ will be said to belong to $\mathcal{M}_0^+(Q)$.

In order to better specify the nature of measures in $\mathcal{M}_0(Q)$, we need to detail the structure of the dual space $(W_{p(\cdot)}(0,T))'$.

Lemma 2.9. *Let $g \in (W_{p(\cdot)}(0,T))'$, then there exists $g_1 \in L^{(p_-)'}(0,T;W^{-1,p'(\cdot)}(\Omega))$, $g_2 \in L^{p_-}(0,T;V)$, $F \in (L^{p(\cdot)}(Q))^N$ and $g_3 \in L^{(p_-)'}(0,T;L^2(\Omega))$ such that*

$$\langle g, u\rangle = \int_0^T \langle g_1, u\rangle dt + \int_0^T \langle u_t, g_2\rangle + \int_Q F\cdot\nabla u dxdt + \int_Q g_3 u dxdt, \quad \forall u \in W_{p(\cdot)}(0,T), \tag{23}$$

where $V = W_0^{1,p(\cdot)}(\Omega)\cap L^2(\Omega)$. Moreover, one can choose (g_1, g_2, F, g_3) such that

$$\begin{aligned} &\|g_1\|_{L^{(p_-)'}(0,T;W^{-1,p'(\cdot)}(\Omega))} + \|g_2\|_{L^{p_-}(0,T;V)} + \|F\|_{L^{p'(\cdot)}(Q)} + \|g_3\|_{L^{(p_-)'}(0,T;L^2(\Omega))} \\ &\leq C\|g\|_{(W_{p(\cdot)}(0,T))'}. \end{aligned}$$

Proof. See [74, Lemma 4.2]. □

Before stating, in a suitable way, the decomposition theorem of elements of $\mathcal{M}_0(Q)$, let us first make a remark on a basic decomposition of these measures.

Remark 2.2. Let $\mu \in \mathcal{M}_0(Q)$, then there exist $g \in (W_{p(\cdot)}(0,T))'$ and $h \in L^1(Q)$ such that $\mu = g + h$ in the sense that

$$\int_Q \varphi d\mu = \langle g, \varphi\rangle + \int_Q h\varphi dxdt, \tag{24}$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$, see [74, Lemma 4.4].

The next result is a consequence of Lemma 2.9 and Remark 2.2.

Theorem 2.10. *Let $\mu \in \mathcal{M}_0(Q)$, then there exists (f, F, g_1, g_2) such that $f \in L^1(Q)$, $F \in (L^{p'(\cdot)}(\Omega))^N$, $g_1 \in L^{(p_-)'}(0, T; W^{-1,p'(\cdot)}(\Omega))$, $g_2 \in L^{p_-}(0, T; V)$ such that*

$$\int_Q \varphi d\mu = \int_Q f\varphi dxdt + \int_Q F \cdot \nabla\varphi dxdt + \int_0^T \langle g_1, \varphi\rangle dt - \int_0^T \langle \varphi_t, g_2\rangle dt, \tag{25}$$

for every $\varphi \in C_c^\infty([0, T] \times \Omega)$ (the quadruplet (f, F, g_1, g_2) will be called a "decomposition" of μ).

Proof. The proof is a combination of the proofs of Lemma 2.9 and Remark 2.2. □

The definitions used in this paper are not limited to the case of generalized spaces with exponent variable but they can be considered (with lack of "homogeneity") in *Orlicz-Sobolev* (or *Musielak-Orlicz*) setting. Some possible extensions involving replacement of the space $L^{p(\cdot)}(Q)$ with more general space $L_\Phi(Q)$ in which the role played by the convex function $t^{p(\cdot)}$ is assumed by more general convex functions $\Phi(t)$ (the spaces $L_\Phi(Q)$ called *Orlicz* spaces are studied in depth in the monograph [62] by *Krasnosel'skii & Rutickii*, and also in the doctoral thesis by *Luxemburg* [68]; for a more complete development, we refer to the books by *Adams* [1], *Adams & Hedberg* [3], *Musielak* [70], to the Monograph of *Rao & Ren* [85], and to the papers by *Gossez* [59, 61, 60], *Gossez & Benkirane* [17], *Benkirane & Elmahi* [15, 16], and *Elmahi* [43]). More recently in [28, 29, 27] and for a class of stationary problems different to the one we will discuss, the authors investigate the notion of generalized capacity and diffuse measures in the framework of weight Sobolev spaces and Orlicz/Musielak spaces.

2.3. Generalized solutions and main result. It is worth pointing that problem $(\mathcal{P}_\mu^Q)$ has two main features: firstly; since the standard subjectivity theorem of *Leray-Lions* operators cannot be applied, we should reason by means of the approximate theory, introduced in [37, 69, 76, 82], by using truncations of solutions in order to get a pseudo-monotone and a coercive differential operator in $L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega))$, then establish some a priori estimates on u, $T_k(u)$ and ∇u. Thus, a technical result on the a.e. convergence of gradients leads to pass to the limit. Secondly, the right-hand side μ of problem $(\mathcal{P}_\mu^Q)$, which contains a measure term, is not an element of the dual space $L^{p'_-}(0, T; W^{-1,p'(\cdot)}(\Omega))$, therefore the solution cannot be expected to belong to the energy space $L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega))$, so it is necessary to change the functional setting in order to prove the equivalence result; to overcome this problem, a concept of "*generalized*" solution should be considered in this specific class. Then, we should specify what we mean by "*generalized solution*"; let us recall that for equations with nonregular datum (say $L^1(Q)$, or more in general, measures), several notions of solutions have been introduced. A notion of "*renormalized*" solution when μ is a diffuse measure was introduced in [40], and in the same paper, the existence and uniqueness of such a solution are proved. In [39], a similar notion of "*entropy*" solution is also defined and proved to be equivalent to that of renormalized solution. A new definition of "*renormalized-entropy*" solution which, in contrast with the previous ones, is closer to the one used for conservation laws in [10] and to the one existing in the elliptic case in [36], is established in [80, 81]. The case of general measure in established in a similar way in [76, 79]. Recently, these frameworks was extended to related problems with variable exponent and measures as data in [73], where *Ouaro & Ouédraogo* studied a parabolic problem involving a $p(x)-$Laplacian type operator and obtained the existence/uniqueness of entropy solutions for L^1-data, as well as, the integrability results for the solution and its gradient (the proofs rely crucially on the semigroup theory). Besides, *Bendahmane et al.* proved the existence/uniqueness of renormalized solutions of the same problem in [26] using a priori estimates in *Marcinkiewicz* spaces with variable exponents, *Zhang & Zhou* [95] uses a different method to prove the equivalence of the two notions. The main idea in such formulations is to move the attention from the solution u to its truncations $T_k(u)$ and to use a "*truncated*" version of the equation. This is advantageous both in order to obtain a priori estimates and because requiring $T_k(u)$ to belong to the energy space allows to get informations about the solution. Observe that the above problem does not admit, in general, a solution in the sense of distributions since we cannot expect to have the fields $a(t,x,u,\nabla u)$ in $L^1_{\text{loc}}(Q)^N$. Indeed some assumptions are in fact crucial in order to obtain *renormalized/entropy* solutions that allow a priori estimates to hold true. To make precise these assumptions, we assume that $a : Q \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfies the following structural assumptions[1]:

(1) $(t,x) \mapsto a(t,x,s,\zeta)$ is measurable for every $(s,\zeta) \in \mathbb{R} \times \mathbb{R}^N$;
(2) $(s,\zeta) \mapsto a(t,x,s,\zeta)$ is continuous for every $(t,x) \in Q$;
(3) there exist constants $0 < \alpha \leq \beta < \infty$ such that for every $(s,\zeta) \in \mathbb{R} \times \mathbb{R}^N$ and for a.e. $(t,x) \in Q$, we have

$$\begin{cases} a(t,x,s,\zeta) \cdot \zeta \geq \alpha|\zeta|^{p(x)} - \Lambda(t,x) & (26) \\ |a(t,x,s,\zeta)| \leq \beta(b(t,x) + |s|^{\nu(x)} + |\zeta|^{p(x)-1}), & (27) \end{cases}$$

(4) "a" satisfies the monotonicity condition

$$(a(t,x,s,\zeta) - a(t,x,s,\eta)) \cdot (\zeta - \eta) > 0, \tag{28}$$

for every $s \in \mathbb{R}$ and for every $\zeta, \eta \in \mathbb{R}^N$ (with $\zeta \neq \eta$).

Although this class of equations is relevant for all $p \in C(\overline{\Omega})$ with $1 < p^- \leq p(x) \leq p^+ < N$, we shall only consider the case

$$p^- > \frac{2N}{N+1}, \tag{29}$$

the same lower bound for p^- appears also in the regularity theory of parabolic equations of the $p(x)$-Laplacien type, see [2, 74]. To make precise our notions of solutions we need the following definitions.

Definition 2.7. Assume that $u_0 \in L^2(\Omega)$ and $f \in L^{(p_-)'}(0,T;W^{-1,p'(\cdot)}(\Omega))$. A function $u \in C([0,T],L^2(\Omega))$ such that $u_t \in L^{(p_-)'}(0,T;W^{-1,p'(\cdot)}(\Omega))$ and $\nabla u \in L^{p(\cdot)}(Q)^N$ is a *weak* solution of

$$\begin{cases} u_t - \operatorname{div}(a(t,x,u,\nabla u)) = f \text{ in } (0,T) \times \Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega, \ u(t,x) = 0 \text{ on } (0,T) \times \partial\Omega, \end{cases} \tag{30}$$

in Q, if it holds that

$$-\int_\Omega u_0\varphi(0,x)dx + \int_Q \left[-u\varphi_t + a(t,x,u,\nabla u) \cdot \nabla\varphi\right] dxdt = \int_Q f\varphi dxdt, \tag{31}$$

for every $\varphi \in C^1(\overline{Q})$ with $\varphi(\cdot,T) = 0$.

[1] "a" is said to be a Carathéodory function if assumptions $(1) - (2)$ are satisfied.

Proof. See [95, Lemma 2.5]. □

We are naturally led to introduce the functional space

$$X = \{u : \overline{\Omega} \times (0,T) \mapsto \mathbb{R} \text{ is measurable such that } T_k(u) \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega)), \\ \text{with } |\nabla T_k(u)| \in L^{p(\cdot)}(Q)^N, \text{ for every } k > 0\}, \tag{32}$$

which, endowed with the norm (or, the equivalence norm)

$$\|u\|_X := \|\nabla u\|_{L^{p(\cdot)}(Q)} \quad (\text{or, } \|u\|_X := \|u\|_{L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \|\nabla u\|_{L^{p(\cdot)}(Q)}),$$

is a separable and reflexive *Banach* space (the equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(\cdot)}(Q) \underset{\text{cont}}{\hookrightarrow} L^{p_-}(0,T; L^{p(\cdot)}(\Omega))$ and the Poincaré's inequality). The notion of the very weak gradient allows us to give the following definition of a *renormalized* solution for problem (P_μ^Q).

Definition 2.8. A measurable function u is a *renormalized* solution of problem (P_μ^Q) if the following conditions are satisfied:

(i) $u - g_2 \in L^\infty(0,T; L^1(\Omega))$; $\quad T_k(u-g_2) \in X$ for every $k > 0$;

(ii) $\lim\limits_{n\to\infty} \int_{\{(t,x)\in Q: n \le |u-g_2|(t,x) \le n+1\}} |\nabla u|^{p(x)} dxdt = 0$;

(iii) for every function $\varphi \in C^1(\overline{Q})$ with $\varphi(\cdot,T) = 0$ and $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 such that S' has a compact support,

$$-\int_\Omega S(u_0)\varphi(x,0)dx - \int_Q S(u-g_2)\varphi_t dxdt + \int_Q S'(u-g_2)a(t,x,u,\nabla u)\cdot\nabla\varphi dxdt \\ + \int_Q S''(u-g_2)a(t,x,u,\nabla u)\cdot\nabla(u-g_2)\varphi dxdt = \int_Q S'(u-g_2)\varphi dxdt \tag{33}$$

holds.

(iv) $S(u-g_2)(0) = S(u_0)$ in $L^1(\Omega)$.

Here is our definition of entropy solution for problem (P_μ^Q).

Definition 2.9. A measurable function u is an *entropy* solution of problem (P_μ^Q) if:

(a) $T_k(u-g_2) \in X$ for every $k > 0$;

(b) $t \in [0,T] \mapsto \int_\Omega \Theta_k(u-g_2-\phi)(t,x)dx$ is a continuous function for all $k \ge 0$ and all $\phi \in \widetilde{W}$;

(c)

$$\int_\Omega \Theta_k(u-g_2-\phi)(t,x)dx - \int_\Omega \Theta_k(u-g_2-\phi)(0,x))dx \\ + \int_0^T \langle \phi_t, T_k(u-g_2-\phi)\rangle dt + \int_Q a(t,x,u,\nabla u)\cdot\nabla T_k(u-g_2-\phi)dxdt \\ \le \int_Q fT_k(u-g_2-\phi)dxdt + \int_\Omega G_1\cdot\nabla(T_k(u-g_2-\phi))dxdt, \tag{34}$$

for all $k \ge 0$ and $\phi \in \widetilde{W}$ with $\phi/\Gamma = 0$.

Our main result is the following:

Theorem 2.11. *Assume that condition (29) holds. Then, the renormalized solution of problem (P_μ^Q) is equivalent to the entropy solution of the same problem.*

Remark 2.3. Thanks to the decomposition result (25), if μ is absolutely continuous with respect to the generalized $p(\cdot)$-capacity (these are called soft measures), it admits a splitting $(f, F, g_1 = -\mathrm{div}(G_1), g_2)$ in the sense of distributions, for some $f \in L^1(Q)$, $F \in L^{p'(\cdot)}(Q)^N$, $g_1 \in L^{p'_-}(0,T;W_0^{1,p'(\cdot)}(\Omega))$ and $g_2 \in L^{p_-}(0,T;V)$ (recall that this decomposition is not uniquely determined and the renormalization argument can be applied to the difference $u - g_2$).

3. The proof of the main result

Now we are ready to prove the main result. Some of the reasoning is based on the ideas developed in [82, 83, 95].

Proof. *Step 1. The renormalized solution u is also an entropy solution.* We first introduce some essential regularity results following the equation in the sense of distribution (33); notice that, thanks to our regularity assumptions and the choice of S, all terms in (33) are well defined since $T_k(u-g_2)$ belongs to X, for every $k>0$, and since S' has compact support. Indeed by taking M such that Supp $S' \subset]-M, M[$, since $S'(u-g_2) = S''(u-g_2) = 0$ as soon as $|u-g_2| \geq M$, we can replace, in (33), $\nabla(u-g_2)$ by $\nabla T_M(u-g_2) \in (L^{p(\cdot)}(Q))^N$ and ∇u by $\nabla(T_M(u-g_2)) + \nabla g_2 \in (L^{p(\cdot)}(Q))^N$ (recall that $\nu(x) \leq p(x)-1$). Moreover, according to the assumption (27) and the definition of ∇u (i.e., $\nabla u = \nabla(u-g_2) + \nabla g_2$), we have $|a(t,x,u,\nabla u)|\chi_{\{|u-g_2\}|<M} \in L^{p'(x)}(Q)$. We also have, for all S as above, $S(u-g_2) = S(T_M(u-g_2)) \in X$, and

$$\begin{cases} S'(u-g_2)f \in L^1(Q); \\ S'(u-g_2)G_1 \in L^{p'(\cdot)}(Q); \\ S'(u-g_2)a(t,x,u,\nabla u) \in (L^{p'(\cdot)}(Q))^N; \\ S''(u-g_2)a(t,x,u,\nabla u)\cdot\nabla(u-g_2) \in L^1(Q); \\ S''(u-g_2)G_1\cdot\nabla(u-g_2) \in L^1(Q). \end{cases} \tag{35}$$

Thus, by equation (33), $(S(u-g_2))_t$ belongs to the space $X' + L^1(Q)$, and therefore $S(u-g_2)$ belongs to $C([0,T];L^1(\Omega))$ (see [25, Lemma 3.2] which is inspired from the result [77, Theorem 1.1]) then one can say that the initial datum is achieved in a weak sense, that is, $S(u-g_2)(0) = S(u_0)$ in $L^1(\Omega)$ for every renormalization S. Note also that, since $S(u-g_2)_t \in X' + L^1(Q)$, we can use in (33) not only test functions in $C_0^\infty(Q)$ but also in $X \cap L^\infty(Q)$. In the following, we make a constant use of the function $S_n : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$S_n(s) = \int_0^s h_n(r)dr, \text{ with } h_n(s) = 1 - |T_1(s - T_n(s))|, \tag{36}$$

and which satisfies

$$\begin{cases} S_n(r) = S_n(T_{n+1}(r)), \quad \|S'_n\|_{L^\infty(\mathbb{R})} \leq 1, & (37) \\ \text{Supp } S'_n \subset [-(n+1), n+1], \quad S''_n = \chi_{[-n-1;-n]} - \chi_{[n;n+1]}. & (38) \end{cases}$$

Now, one can use $T_k(v-\varphi)\theta_\epsilon$ as test function in (33) for $k>0$, where $v=u-g_2$, $\varphi\in\widetilde{W}$, $\theta_\epsilon(t)=1-(\frac{(t-t_1)^+}{\epsilon})$ with $t_1\in]0,T]$ and $S=S_n$. We note that, if $M:=k+\|\varphi\|_{L^\infty(Q)}$, then

$$T_k(v-\varphi)\theta_\epsilon=T_k(T_M(v)-\varphi)\theta_\epsilon\in X\cap L^\infty(Q), \tag{39}$$

and

$$\begin{aligned}
&\int_0^T\langle S_n(v)_t,T_k(v-\varphi)\theta_\epsilon\rangle dt+\int_Q S_n''(v)a(t,x,u,\nabla u)\cdot\nabla vT_k(v-\varphi)\theta_\epsilon dxdt\\
&+\int_Q S_n'(v)a(t,x,u,\nabla u)\cdot\nabla(T_k(v-\varphi))\theta_\epsilon dxdt=\int_Q S_n'(v)fT_k(v-\varphi)\theta_\epsilon dxdt\\
&+\int_Q S_n''(v)F\cdot\nabla vT_k(v-\varphi)\theta_\epsilon dxdt+\int_Q S_n'(v)F\cdot\nabla(T_k(v-\varphi))\theta_\epsilon dxdt\\
&+\int_Q S_n''(v)G_1\cdot\nabla vT_k(v-\varphi)\theta_\epsilon dxdt+\int_Q S_n'(v)G_1\cdot\nabla(T_k(v-\varphi))\theta_\epsilon dxdt.
\end{aligned}\tag{40}$$

Since $S_n''(s)=0$ for $|s|\notin[n;n+1]$, one can write

$$\begin{aligned}
&S_n''(v)a(t,x,u,\nabla u)\cdot\nabla vT_k(v-\varphi)\\
&\quad=S_n''(v)a(t,x,u,\nabla u)\cdot\nabla(T_{n+1}(v))T_k(v-\varphi)\in L^1(Q),
\end{aligned}\tag{41}$$

and

$$S_n''(v)G_1\cdot\nabla vT_k(v-\varphi)=S_n''(v)G_1\cdot\nabla(T_{n+1}(v))T_k(v-\varphi)\in L^1(Q). \tag{42}$$

Since $\theta_\epsilon\to\chi_{[0,t_1]}$ and is bounded by 1 as $\epsilon\to0$, using Lebesgue dominated convergence theorem in equality (40), we obtain

$$\begin{aligned}
&\int_0^{t_1}\int_\Omega(S_n(v))_t,T_k(v-\varphi)dxdt+\int_0^{t_1}\int_\Omega S_n'(v)a(t,x,u,\nabla u)\cdot\nabla(T_k(v-\varphi))dxdt\\
&\quad+\int_0^{t_1}\int_\Omega S_n''(v)a(t,x,u,\nabla u)\cdot\nabla vT_k(v-\varphi)dxdt\\
&=\int_0^{t_1}\int_\Omega fS_n'(v)T_k(v-\varphi)dxdt+\int_0^{t_1}\int_\Omega S_n'(v)F\cdot\nabla(T_k(v-\varphi))dxdt\\
&\quad+\int_0^{t_1}\int_\Omega S_n''(v)F\cdot\nabla vT_k(v-\varphi)dxdt+\int_0^{t_1}\int_\Omega S_n'(v)G_1\cdot\nabla(T_k(v-\varphi))dxdt\\
&\quad+\int_0^{t_1}\int_\Omega S_n''(v)G_1\cdot\nabla vT_k(v-\varphi)dxdt.
\end{aligned}\tag{43}$$

For n large enough ($n\ge M$) we have $T_k(v-\varphi)=T_k(S_n(v)-\varphi)$ (since $S_n(s)=s$ on $[-M,M]$, $|S_n(s)|\ge M$ and $\text{sign}(S_n(s))=\text{sign}(s)$ outside $[-M,M]$), $(S_n(v))(t_1)-\varphi(t_1,\cdot)\to v(t_1,\cdot)-\varphi(t_1,\cdot)$ in $L^1(\Omega)$, $S_n(u_0)\to u_0$ in $L^1(\Omega)$ and $S_n'(v)\to1$ a.e. in Q as $n\to+\infty$. So that thanks to the following Lemma.

Lemma 3.1 (*Integration by parts formula*). *Let $f:\mathbb{R}\mapsto\mathbb{R}$ be a continuous piecewise C^1-function such that $f(0)=0$ and f' is zero outside a compact set of $\mathbb{R}$. Let us denote $F(s)=\int_0^s f(r)dr$ and if $u\in X$ is such that $u_t\in X'+L^1(Q)$ and $\psi\in C^\infty(\overline{Q})$, then we have*

$$\int_0^T\langle u_t,f(u)\psi\rangle=\int_\Omega F(u(T))\psi(T)dx-\int_\Omega F(u(0))\psi(0)dx-\int_Q\psi_tF(u)dxdt,$$

where $\langle \cdot, \cdot \rangle$ *denotes the duality pairing between* $X' + L^1(Q)$ *and* $X \cap L^\infty(Q)$.

Proof. The proof follows the same lines of [39, Lemma 7.1] (see also [78, Lemma 6.10]). $\square$

Remark 3.1. A similar generalization of the integration by parts formula can be found in [21] and it plays a crucial role in order to find more estimates for entropy solutions and to get useful a priori estimates of approximate solutions to the equation (43) below.

By using the integration by parts formula formula, we get

$$\begin{aligned}
&\int_\Omega \Theta_k(S_n(v)(t_1) - \varphi(t_1))dx - \int_\Omega \Theta_k(S_n(u_0) - \varphi(0))dx \\
&+ \int_0^{t_1}\int_\Omega \varphi_t T_k(v-\varphi)dxdt + \int_0^{t_1}\int_\Omega S_n'(v)a(t,x,u,\nabla u)\cdot\nabla(T_k(v-\varphi))dxdt \\
&+ \int_0^{t_1}\int_\Omega S_n''(v)a(t,x,u,\nabla u)\cdot\nabla v T_k(v-\varphi)dxdt = \int_0^{t_1}\int_\Omega f S_n'(v)T_k(v-\varphi)dxdt \\
&+ \int_0^{t_1}\int_\Omega S_n'(v)F\cdot\nabla(T_k(v-\varphi))dxdt + \int_0^{t_1}\int_\Omega S_n''(v)F\cdot\nabla v T_k(v-\varphi)dxdt \\
&+ \int_0^{t_1}\int_\Omega S_n'(v)G_1\cdot\nabla(T_k(v-\varphi))dxdt + \int_0^{t_1}\int_\Omega S_n''(v)G_1\cdot\nabla v T_k(v-\varphi)dxdt.
\end{aligned} \tag{44}$$

Since $|S_n''(s)| \le 1$ and $S_n''(s) \neq 0$ only if $|s| \in [n, n+1]$, using (27) one can write

$$\begin{aligned}
&\left|\int_0^{t_1}\int_\Omega S_n''(v)a(t,x,u,\nabla u)\cdot\nabla v T_k(v-\varphi)dxdt\right| \le k\int_{\{n\le|v|\le n+1\}} |a(t,x,u,\nabla u)\cdot\nabla v|dxdt \\
&\le k\int_{\{n\le|v|\le n+1\}} \beta\left(b(t,x) + |u|^{\nu(x)} + |\nabla u|^{p(x)-1}\right)|\nabla v|dxdt \\
&\le k\Big[\int_{\{n\le|v|\le n+1} \frac{p(x)-1}{p(x)}\left(|b(t,x)|^{p'(x)} + |u|^{\nu(x)p'(x)} + |\nabla u|^{(p(x)-1)p'(x)}\right)dxdt \\
&\quad + \int_{\{n\le|v|\le n+1} \frac{1}{p(x)}|\nabla v|^{p(x)}dxdt\Big] \\
&\le k\Big[\int_{\{n\le|v|\le n+1\}} \frac{p_+-1}{p_-}\left(|b(t,x)|^{p'(x)} + |u|^{\nu(x)p'(x)} + |\nabla u|^{p(x)}\right)dxdt \\
&\quad + \int_{\{\{n\le|v|\le n+1\}} \frac{C}{p_-}(|\nabla u|^{p(x)} + |\nabla g|^{p(x)})dxdt\Big]
\end{aligned} \tag{45}$$

Observe that, thanks to (26) and of Definition 2.8(ii), and using Young's inequality one can easily show that there exists a positive constant M such that

$$\frac{1}{n}\int_{\{n\le|v|\le n+1\}} |\nabla u|^{p(x)}dxdt \le M. \tag{46}$$

On the other hand, using the definition of v, we get

$$\int_Q |\nabla T_k(v)|^{p(x)}dxdt \le C\int_{\{|v|\le k\}} |\nabla u|^{p(x)}dxdt + C. \tag{47}$$

Hence we have to control the first term on the right hand side of (47) ; using (46) we have

$$\begin{aligned}\int_{\{|v|<k\}} |\nabla u|^{p(x)} dxdt &\leq \sum_{n=0}^{[log_2 k]+1} \int_{\{2^n \leq |v| < 2^{n+1}\}} |\nabla u|^{p(x)} dxdt + \int_{\{0 \leq |v| < 1\}} |\nabla u|^{p(x)} dxdt \\ &\leq M \sum_{n=0}^{[log_2 k]+1} 2^n + C = M(2^{[log_2 k]+2} - 1) + C \\ &\leq C(k+1),\end{aligned} \tag{48}$$

which, together with (47), yields

$$\int_Q |\nabla T_k(v)|^{p(x)} dxdt \leq \widetilde{C}(k+1) \tag{49}$$

where $\widetilde{C}$ is a positive constant not depending on k. We can improve this kind of estimate by using the *Gagliardo-Nirenberg* inequality. Indeed, this way, we obtain

$$\int_Q |T_k(v)|^{p_- + \frac{p_-}{N}} dxdt \leq Ck \tag{50}$$

and so, we can write

$$k^{p_- + \frac{p_-}{N}} \operatorname{meas}\{|v| \geq k\} \leq \int_{\{|u| \geq k\}} |T_k(v)|^{p_- + \frac{p_-}{N}} dxdt \leq \int_Q |T_k(v)|^{p_- + \frac{p_-}{N}} dxdt \leq Ck; \tag{51}$$

then,

$$\operatorname{meas}\{|v| \geq k\} \leq \frac{C}{k^{p_- - 1 + \frac{p_-}{N}}}.$$

Therefore, v is uniformaly bounded in the Marcinkiewicz space $M^{p_- - 1 + \frac{P_-}{N}}(Q)$; that implies, since in particular $p_- > \frac{2N}{N+1}$, that v is uniformaly bounded in $L^{q(x)}(Q)$ for all $1 \leq q(x) < p_- - 1 + \frac{p_-}{N}$. Note that $\nu p'(x) < p_- - 1 + \frac{1}{N}$ then $|v|^{\nu p(x)} \in L^1(Q)$, and by the fact that $\operatorname{meas}(\{n \leq |v| < n+1\}) \to 0$, we have by (45)

$$\left| \int_0^{t_1} \int_\Omega S_n''(v) a(t,x,u,\nabla u) \cdot \nabla v T_k(v-\varphi) dxdt \right| \to 0 \text{ as } n \to +\infty, \tag{52}$$

and

$$\begin{aligned}&\left| \int_0^{t_1} \int_\Omega S_n''(v) G_1 \cdot \nabla v T_k(v-\varphi) dxdt \right| \leq k \int_{\{n \leq |v| \leq n+1\}} |G_1||\nabla v| \, dxdt \\ &\leq k \left[\int_{\{n \leq |v| \leq n+1\}} \frac{p_- - 1}{p_+} |G_1|^{p'(x)} dxdt + \int_{\{n \leq |v| \leq n+1\}} \frac{1}{p_-} |\nabla v|^{p(x)} dxdt \right]\end{aligned} \tag{53}$$

Similarly

$$\begin{aligned}&\left| \int_0^{t_1} \int_\Omega S_n''(v) F \cdot \nabla v T_k(v-\varphi) dxdt \right| \leq k \int_{\{n \leq |v| \leq n+1\}} |F||\nabla v| \, dxdt \\ &\leq k \left[\int_{\{n \leq |v| \leq n+1\}} \frac{p_- - 1}{p_+} |F|^{p'(x)} dxdt + \int_{\{n \leq |v| \leq n+1\}} \frac{1}{p_-} |\nabla v|^{p(x)} dxdt \right]\end{aligned} \tag{54}$$

Now, since meas$(\{n \leq |v| < n+1\}) \to 0$; this implies that

$$\begin{cases} \left| \int_0^{t_1} \int_\Omega S_n''(v) G_1 \cdot \nabla v T_k(v-\varphi) dx dt \right| \to 0 \text{ as } n \to +\infty, \\ \left| \int_0^{t_1} \int_\Omega S_n''(v) F \cdot \nabla v T_k(v-\varphi) dx dt \right| \to 0 \text{ as } n \to +\infty. \end{cases} \tag{55}$$

Passing to the limit in (43) as $n \to +\infty$, we obtain

$$\begin{aligned} &\int_0^{t_1} \int_\Omega v_t T_k(v-\varphi) dx dt + \int_0^{t_1} \int_\Omega a(t,x,u,\nabla u) \cdot \nabla(T_k(v-\varphi)) dx dt \\ &= \int_0^{t_1} \int_\Omega f T_k(v-\varphi) dx dt + \int_0^{t_1} \int_\Omega F \cdot \nabla(T_k(v-\varphi)) dx dt \\ &+ \int_0^{t_1} \int_\Omega G_1 \cdot \nabla(T_k(v-\varphi)) dx dt, \end{aligned} \tag{56}$$

for all $t_1 \in (0,T)$. □

Proof. (2) The entropy solution u is also a renormalized solution. Our aim here is to prove that the entropy solution is also a renormalized solution of (P_μ^Q). The proof of this result consist in two steps. First, we prove the behavior of the energy of u on the set where u is very large, secondly we obtain other properties on solutions. We need then to recall the following definition of a time-regularization of $T_k(v)$, which was first introduced in [65], then used in several papers afterwards (see in particular [14, 20]. Let z_ν be a sequence of functions such that

$$\begin{cases} z_\nu \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \quad \|z_\nu\|_{L^\infty(\Omega)} \leq k, \\ z_\nu \to T_k(u_0) \;\text{ a.e. on } \Omega \text{ as } \nu \text{ tends to infinity}, \\ \dfrac{1}{\nu}\|z_\nu\|_{W_0^{1,p(\cdot)}(\Omega)} \to 0 \text{ as } \nu \text{ tends to infinity}. \end{cases} \tag{57}$$

Then, for fixed $k > 0$ and $\nu > 0$, we denote by $T_k(v)_\nu$ the unique solution of the problem

$$\begin{cases} \dfrac{\partial T_k(v)_\nu}{\partial t} = \nu(T_k(v) - T_k(v)_\nu) \text{ in the sense of distributions}, \\ T_k(v)_\nu(0) = z_\nu \text{ in } \Omega. \end{cases} \tag{58}$$

Then $T_k(v)_\nu$ belongs to $X \cap L^\infty(Q)$ and $(T_k(v)_\nu)_t$ belongs to X, and it can be proved (see also [19]) that, up to a subsequence,

$$\begin{cases} T_k(v)_\nu \to T_k(v) \text{ strongly in } X \text{ and a.e.in } Q, \\ \|T_k(v)_\nu\|_{L^\infty(Q)} \leq k \quad \forall \nu > 0. \end{cases} \tag{59}$$

For $h > 0$, we use $(T_h(v))_\zeta$ defined by

$$\begin{cases} (T_h(v)_\zeta)_t = \zeta(T_h(v) - (T_h(v))_\zeta) \in L^\infty(Q) \subset X' + L^1(Q), \\ (T_h(v))_\zeta(0) = z_\zeta, \end{cases} \tag{60}$$

as test function in (34) to get

$$\begin{aligned}
&\int_\Omega \Theta_k(v-(T_h(v)_\zeta))(T)dx - \int_\Omega \Theta_k(u_0 - z_\zeta)dx \\
&\int_0^T \int_\Omega (T_h(v)_\zeta)_t T_k(v-(T_h(v)_\zeta))dt + \int_Q a(t,x,u,\nabla u)\cdot \nabla(T_k(v-(T_h(v)_\zeta)))dxdt \\
&\le \int_Q fT_k(v-(T_h(v)_\zeta))dxdt + \int_Q F\cdot\nabla(T_k(v-(T_h(v)_\zeta)))dxdt \\
&\quad + \int_Q G_1\cdot\nabla(T_k(v-(T_h(v)_\zeta)))dxdt
\end{aligned} \tag{61}$$

From the definition of Θ_k and the properties of $(T_h(v)_\zeta)$ and since

$$\text{sign }(v-(T_h(v)_\zeta)) = \text{sign }(T_h(v)-(T_h(v)_\zeta)) = \text{sign }((T_h(v)_\zeta)_t)$$

we obtain

$$\begin{cases} \displaystyle\int_\Omega (v-(T_h(v)_\zeta))(T)dx \ge 0, \\ \displaystyle\int_Q ((T_h(v)_\zeta))_t T_k(v-(T_h(v)_\zeta))dxdt \ge 0 \end{cases} \tag{62}$$

Moreover, since $(T_h(v)_\zeta)$ converges to $T_h(v)$ strongly in X and a.e. in Q as ζ tends to infinity, we have

$$\lim_{\nu\to\infty} T_k(v-(T_h(v)_\zeta)) = \lim_{\nu\to\infty} T_k(T_{k+h}(v)-(T_h(v)_\zeta)) = T_k(T_{k+h}(v)-T_h(v)),$$

then

$$T_k(v-(T_h(v)_\zeta)) \to T_k(v-T_h(v)) \text{ strongly in } X \text{ and a.e. in } Q. \tag{63}$$

By means of Lebesgue's theorem and by the fact that $|a(t,x,u,\nabla u)|\chi_{\{|v|<k+h\}} \in L^{p'(\cdot)}(Q)$, and since $\nabla T_k(v-(T_h(v)_\zeta)) = 0$ if $|v| > h+k$, we can conclude using (62)-(63) that

$$\begin{aligned}
&\int_Q a(t,x,u,\nabla u)\cdot\nabla T_k(v-(T_h(v)_\zeta))dxdt \\
&\le \int_\Omega \Theta_k(u_0 - T_h(u_0))dx + \int_Q fT_k(v-T_h(v))dxdt \\
&\quad + \int_Q G_1\cdot\nabla(T_k(v-T_h(v)))dxdt + \int_Q F\cdot\nabla(T_k(v-T_h(v)))dxdt
\end{aligned}$$

Next, we split the integral in the sets $\{|v| \leq h\}$ and $\{|v| > h\}$ and we obtain (by recalling that $v - T_h(v) = 0$ on $\{|v| \leq h\}$)

$$\begin{aligned}
&\int_{\{h\leq|v|\leq h+k\}} \alpha|\nabla u|^{p(x)} - \Lambda dxdt \leq \int_{\{h\leq|v|\leq h+k\}} a(t,x,u,\nabla u)\cdot\nabla v dxdt \\
&\leq \int_{\{h\leq|v|\leq h+k\}} a(t,x,u,\nabla u)\cdot\nabla v dxdt + \int_{\{h\leq|v|\leq h+k\}} a(t,x,u,\nabla u)\cdot\nabla g_2 dxdt \\
&\leq \int_Q a(t,x,u,\nabla u)\cdot\nabla T_k(v - T_h(v))dxdt + \int_{\{h\leq|v|\leq h+k\}} a(t,x,u,\nabla u)\cdot\nabla g_2 dxdt \\
&\leq k\int_\Omega |u_0 - T_h(u_0)|dx + k\int_{\{|u|\geq h\}} |f|dxdt \\
&\quad + \int_{\{h\leq|v|\leq h+k\}} |G_1||\nabla v|dxdt + \int_{\{h\leq|v|\leq h+k\}} |F||\nabla v|dxdt \\
&\quad + C\int_{\{h\leq|v|\leq h+k\}} (|b| + |u|^{\nu(x)} + |\nabla u|^{p(x)-1})|\nabla g_2|dxdt,
\end{aligned}$$

that is,

$$\begin{aligned}
\alpha &\int_{\{h\leq|v|\leq h+k\}} |\nabla u|^{p(x)} dxdt \leq k\int_{\{|u_0|\geq h\}} |u_0|dxdt + k\int_{\{|v|>h\}} |f|dxdt \\
&+ \int_{\{h\leq|v|\leq h+k\}} \frac{1}{p'_-}|G_1|^{p'(x)}dxdt + \int_{\{h\leq|v|\leq h+k\}} \frac{1}{p_-}|\nabla u + \nabla g_2|^{p(x)}dxdt \\
&+ \int_{\{h\leq|v|\leq h+k\}} \frac{1}{p'_-}|F|^{p'(x)}dxdt + \int_{\{h\leq|v|\leq h+k\}} \frac{1}{p_-}|\nabla u + \nabla g_2|^{p(x)}dxdt \\
&+ \frac{C}{p'_-}\int_{\{h\leq|v|\leq h+k\}} |b|^{p'(x)} + |u|^{\nu(x)p'(x)}dxdt + \frac{C}{p'_-}\int_{\{h\leq|v|\leq h+k\}} |\nabla u|^{p(x)} \\
&+ \int_{\{h\leq|v|\leq h+k\}} \frac{C}{p_-}|\nabla g_2|^{p(x)}dxdt + \int_{\{h\leq|v|\leq h+k\}} \Lambda dxdt.
\end{aligned}$$

Using the log-Hölder criterion and Young's inequality, we have

$$\begin{aligned}
&\int_{\{h\leq|v|\leq h+k\}} \alpha|\nabla u|^{p(x)}dxdt \leq k\int_{\{|u_0|\geq h\}} |u_0|dx + k\int_{\{|v|\geq h\}} |f|dxdt \\
&+ C\left(\int_{\{h\leq|v|\leq h+k\}} |G_1|^{p'(x)} \int_{\{h\leq|v|\leq h+k\}} |F|^{p'(x)} + |\nabla g_2|^{p(x)} + |b|^{p'(x)} + |u|^{\nu(x)p'(x)}dxdt\right) \\
&+ \frac{\alpha}{2}\int_{\{h\leq|v|\leq h+k\}} |\nabla u|^{p(x)}dxdt + \int_{\{h\leq|v|\leq h+k\}} \Lambda dxdt.
\end{aligned}$$

We have to prove that $|u|^{\nu(x)p'(x)} \in L^1(Q)$. To this aim, we use the definition of entropy solution with $k = 1$ and $\varphi = 0$ to get (where $\Theta_1(s) \geq |s| - 1$)

$$t \in [0,T] \to \int_\Omega \Theta_1(u - g_2)(t,x) \text{ is a continuous function,}$$

then the integral satisfies

$$\int_\Omega |v(t,x)|dx \leq \int_\Omega (1 + \Theta_1(v))(t,x)dx \leq \text{meas}(\Omega) + C.$$

Therefore, we conclude that $v \in L^{\infty}(0,T;L^1(\Omega))$. Now, again by taking $\varphi = 0$ as a test function in (34) we obtain

$$\begin{aligned}\int_Q a(t,x,u,\nabla u)\nabla T_k(v)dxdt \leq \int_\Omega \Theta_k(u_0)dx + \int_Q fT_k(v)dxdt \\ + \int_Q G_1 \cdot \nabla T_k(v)dxdt + \int_Q F \cdot \nabla T_k(v)dxdt.\end{aligned} \tag{64}$$

By definition of Θ_k, we have $\Theta_k(s) \leq k|s|$, using $|T_k(s)| \leq k$, inequality (64) becomes

$$\begin{aligned}\int_Q a(t,x,u,\nabla u)\nabla T_k(v)dxdt \leq & k\|u_0\|_{L^1(\Omega)} + k\|f\|_{L^1(Q)} \\ & + C\left[\|G_1\|_{L^{p'(x)}(Q)} + \|F\|_{L^{p'(x)}(Q)}\right]\|\nabla T_k(v)\|_{L^{p(x)}(Q)}.\end{aligned}$$

Since

$$\begin{aligned}\int_{\{|v|\leq k\}} a(t,x,u,\nabla u)\nabla u dxdt = \int_{\{|v|\leq k\}} a(t,x,u,\nabla u)\cdot\nabla T_k(v)dxdt \\ + \int_{\{|v|\leq k\}} a(t,x,u,\nabla u)\cdot\nabla g_2 dxdt,\end{aligned}$$

and using assumptions (26)-(27) and Poincarré's inequality, this yields

$$\begin{aligned}&\alpha\int_{\{|v|\leq k\}} |\nabla u|^{p(x)}dxdt - \int_{\{|v|\leq k\}} \Lambda(t,x)dxdt \\ &\leq \int_Q a(t,x,u,\nabla u)\cdot\nabla T_k(v)dxdt + \int_{\{|v|\leq k\}} \beta(b(t,x) + |u|^{\nu(x)} + |\nabla u|^{p(x)-1})|\nabla g_2|dxdt \\ &\leq \int_Q a(t,x,u,\nabla u)\nabla T_k(v)dxdt \\ &\quad + \beta\Big(\|b(t,x)\|_{L^{p'(x)}(Q)} + \|u\|^{p(x)-1}_{L^{p(x)}(\{|v|\leq k\})} + \|\nabla u\|^{p(x)-1}_{L^{p(x)}(\{|v|\leq k\})}\Big)\|\nabla g_2\|_{L^{p(x)}(Q)} \\ &\leq \int_Q a(t,x,u,\nabla u)\nabla T_k(v)dxdt + \beta C\Big(\|b\|_{L^{p'(x)}(Q)} + \|T_k(v)\|^{p(x)-1}_{L^{p'(x)}(Q)} \\ &\quad + \|g_2\|^{p(x)-1}_{L^{p(x)}(Q)} + \|\nabla T_k(v)\|^{p(x)-1}_{L^{p(x)}(Q)} + \|\nabla g_2\|^{p(x)-1}_{L^{p(x)}(Q)}\Big)\|\nabla g_2\|_{L^{p(x)}(Q)} \\ &\leq \int_Q a(t,x,u,\nabla u)\cdot\nabla T_k(v)dxdt + \beta C\Big(\|b\|_{L^{p'(x)}(Q)} + C'\|\nabla T_k(v)\|^{p(x)-1}_{L^{p(x)}(Q)} \\ &\quad + \|g_2\|^{p(x)-1}_{L^{p(x)}(Q)} + \|\nabla T_k(v)\|^{p(x)-1}_{L^{p(x)}(Q)}\Big)\|\nabla g_2\|_{L^{p(x)}}\end{aligned}$$

Now, let us come back to (64), for $C' = C(\beta, C, \|b\|_{L^{p'(x)}(Q)}, \|g_2\|_{L^{p(x)}(Q)}, \|\nabla g_2\|_{L^{p(x)}(Q)})$, we have

$$\begin{aligned}&\int_Q |\nabla T_k(v)|^{p(x)}dxdt \leq C\int_{\{|v|\leq k\}} |\nabla u|^{p(x)}dxdt + C\int_Q |\nabla g_2|^{p(x)}dxdt \\ &\leq C\int_Q a(t,x,u,\nabla u)\cdot\nabla T_k(v)dxdt + C\left(\|\nabla T_k(v)\|^{p(x)-1}_{L^{p(x)}(Q)} + C'\right) \\ &\leq C\left[k\|u_0\|_{L^1(\Omega)} + K\|f\|_{L^1(Q)} + C\|G_1\|_{L^{p'(x)}(Q)}\|\nabla T_k(v)\|_{L^{p(x)}(Q)}\right] \\ &\quad + C\left(\|\nabla T_k(v)\|^{p(x)-1}_{L^{p(x)}(Q)} + 1\right) \leq C(k+1).\end{aligned} \tag{65}$$

According to the *Gagliardo-Nirenberg* result, see Lemma 2.6, we deduce from (65) that $|u|^{\nu(x)p(x)} \in L^1(Q)$, note also that

$$\frac{\alpha}{2}\int_{\{h\le|v|\le h+k\}} |\nabla u|^{p(x)}dxdt \le k\int_{\{|u_0|\ge h\}}|u_0|dx + k\int_{\{|v|\ge h\}}|f|dxdt \\ + C\int_{\{h\le|v|\le h+k\}}\left(|G_1|^{p'(x)} + F|^{p'(x)} + |\nabla g_2|^{p(x)} + |b|^{p(x)} + |u|^{\nu(x)p(x)} + \Lambda\right)dxdt$$

Since meas$(\{|v|\ge h\}) \to 0$ as $h\to\infty$, then

$$\int_{\{h\le|v|\le h+k\}}|\nabla u|^{p(x)}dxdt \to 0 \text{ as } h\to\infty.$$

□

Now, we are ready to prove that the entropy solution u satisfies all other properties of the renormalized solution. First we introduce the functions $\tilde{a}(t,x,\zeta) = a(t,x,u(t,x),\zeta)$ and $\tilde{b} = b + |u|^{\nu(x)}$ such that

$$\tilde{b}\in L^{p'(x)}(Q), \qquad |\tilde{a}(t,x,\zeta)| \le \beta(\tilde{b} + |\zeta|^{p(x)-1}).$$

Then, we consider the following auxiliary problems

$$\begin{cases} \tilde{u}_t - \operatorname{div}(\tilde{a}(t,x,\nabla\tilde{u})) = \mu & \text{in }]0,T[\times\Omega,\\ \tilde{u}(t,x) = 0 & \text{on }]0,T[\times\partial\Omega,\\ \tilde{u}(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{66}$$

By employing the arguments in [2], we easily find a renormalized solution $\tilde{u}$ of problem (66). Our aim is to prove the uniqueness of $\tilde{u}$ (i.e. $u = \tilde{u}$). We will divide the proof into several steps where some of the reasoning is based on the ideas developed in [2]. Let $\tilde{v} = \tilde{u} - g_2$, since $\tilde{u}$ is a renormalized solution of (66), we have $S_n(\tilde{v}) \in E$. Choosing $T_k(v - S_n(\tilde{v}))$ as a test function in (34) and (66) and using the fact that $T_k(T_{k+n+1}(v) - S_n(\tilde{v})) \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))\cap L^\infty(Q)$, we get

$$\int_\Omega \Theta_k(v - S_n(v))(T)dx - \int_\Omega \Theta_k(u_0 - S_n(u_0))dx \\ + \int_0^T \langle (S_n(\tilde{v}))_t, T_k(v - S_n(\tilde{v}))\rangle\, dt + \int_Q \tilde{a}(t,x,\nabla u)\cdot\nabla(T_k(v - S_n(\tilde{v})))dxdt \\ \le \int_Q fT_k(v - S_n(\tilde{v}))dxdt + \int_Q G_1\cdot\nabla(T_k(v - S_n(\tilde{v})))dxdt + \int_Q F\cdot\nabla(T_k(v - S_n(\tilde{v})))dxdt \tag{67}$$

and

$$\begin{aligned}
&\int_0^T \langle (S_n(\tilde v))_t, T_k(v - S_n(\tilde v))\rangle \, dt \\
&= \int_Q f S_n'(\tilde v) T_k(v - S_n(\tilde v)) dxdt + \int_Q S_n'(\tilde v) G_1 \cdot \nabla(T_k(v - S_n(\tilde v))) dxdt \\
&\quad + \int_Q S_n''(\tilde v) G_1 \cdot \nabla \tilde v T_k(v - S_n(\tilde v)) dxdt + \int_Q S_n'(\tilde v) F \cdot \nabla(T_k(v - S_n(\tilde v))) dxdt \\
&\quad + \int_Q S_n''(\tilde v) F \cdot \nabla \tilde v T_k(v - S_n(\tilde v)) dxdt - \int_Q S_n''(\tilde v) \tilde a(t, x, \nabla \tilde u) \nabla \tilde v T_k(v - S_n(\tilde v)) dxdt \\
&\quad - \int_Q S_n'(\tilde v) \tilde a(t, x, \nabla \tilde u) \nabla(T_k(v - S_n(\tilde v))) dxdt,
\end{aligned}$$

which implies, since $S_n''(s) = 0$ if $s \notin [n, n+1]$ and $|S_n''| \le 1$, that

$$\begin{aligned}
&\left| \int_Q S_n''(\tilde v) G_1 \cdot \nabla \tilde v T_k(v - S_n(\tilde v)) dxdt - \int_Q S_n''(\tilde v) \tilde a(t, x, \nabla \tilde u) \cdot \nabla \tilde v T_k(v - S_n(\tilde v)) dxdt \right| \\
&\le Ck \int_{\{n \le |\tilde v| \le n+1\}} \left(|G_1|^{p'(x)} + |F|^{p'(x)} + |\nabla g_2|^{p(x)} + |\nabla \tilde u|^{p(x)} \right) \\
&\quad + Ck \int_{\{n \le |\tilde v| \le n+1\}} \left(\tilde b^{p'(x)} + |\nabla g_2|^{p(x)} + |\nabla \tilde u|^{p(x)} \right) dxdt \\
&\le \omega_1(n) \underset{n \to +\infty}{\rightarrow} 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^T \langle (S_n(\tilde v))_t, T_k(v - S_n(\tilde v))\rangle \, dt \\
&\ge -\omega_1(n) + \int_Q f S_n'(\tilde v) T_k(v - S_n(\tilde v)) dxdt + \int_Q S_n'(\tilde v) G_1 \cdot \nabla(T_k(v - S_n(\tilde v))) dxdt \\
&\quad + \int_Q S_n'(\tilde v) F \cdot \nabla(T_k(v - S_n(\tilde v))) dxdt - \int_Q S_n'(\tilde v) \tilde a(t, x, \nabla \tilde u) \cdot \nabla(T_k(v - S_n(\tilde v))) dxdt.
\end{aligned}$$

Now, since Θ_k is nonnegative and using (67) we have

$$\begin{aligned}
&\int_Q (\tilde a(t, x, \nabla u) - S_n'(\tilde v) \tilde a(t, x, \nabla \tilde u)) \cdot \nabla(T_k(v - S_n(\tilde v))) dxdt \\
&\le \int_Q (1 - S_n'(\tilde v)) f T_k(v - S_n(\tilde v)) dxdt + \int_Q (1 - S_n'(\tilde v)) G_1 \cdot \nabla(T_k(v - S_n(\tilde v))) dxdt \\
&\quad + \int_Q (1 - S_n'(\tilde v)) F \cdot \nabla(T_k(v - S_n(\tilde v))) dxdt + \int_\Omega \Theta_k(u_0 - S_n(u_0)) dx + \omega_1(n).
\end{aligned}$$

The left hand side can be split as

$$\begin{aligned}
&\int_Q (\tilde{a}(t,x,\nabla u) - S_n'(\tilde{v})\tilde{a}(t,x,\nabla\tilde{u}))\cdot\nabla(T_k(v) - S_n(\tilde{v})))dxdt\\
&= \int_{\{|\tilde{v}|\le n\}} (\tilde{a}(t,x,\nabla u) - S_n'(\tilde{v})\tilde{a}(t,x,\nabla\tilde{u}))\cdot\nabla(T_k(v - S_n(\tilde{v})))dxdt\\
&\quad + \int_{\{|\tilde{v}|>n\}} \tilde{a}(t,x,\nabla u)\nabla(T_k(v - S_n(\tilde{v})))dxdt\\
&\quad - \int_{\{|\tilde{v}|>n\}} S_n'(\tilde{v})\tilde{a}(t,x,\nabla\tilde{u})\cdot\nabla(T_k(v - S_n(\tilde{v})))dxdt,
\end{aligned}$$

and for every positive integer n, we know that on the set $\{|\tilde{v}|\le n\}$:

$$S_n(\tilde{v}) = \tilde{v}, \quad S_n'(\tilde{v}) = 1$$

and

$$\begin{aligned}
\nabla(T_k(v - S_n(\tilde{v}))) &= \chi_{\{|v-S_n(\tilde{v})|\le k\}}(\nabla v - S_n'(\tilde{v})\nabla\tilde{v})\\
&= \chi_{\{|v-\tilde{v}|\le k\}}(\nabla u - \nabla\tilde{u})\chi_{\{|\tilde{v}|\le n\}} = \chi_{\{|u-\tilde{u}|\le k\}}(\nabla u - \nabla\tilde{u})\chi_{\{|\tilde{v}|\le n\}},
\end{aligned}$$

then

$$\begin{aligned}
&\int_{\{|\tilde{v}|\le n\}} (\tilde{a}(t,x,\nabla u) - S_n'(\tilde{v})\tilde{a}(t,x,\nabla\tilde{u}))\cdot\nabla(T_k(v - S_n(\tilde{v})))dxdt\\
&= \int_{\{|\tilde{v}|\le n\}} \chi_{\{|u-\tilde{u}|\le k\}}(\tilde{a}(t,x,\nabla u) - \tilde{a}(t,x,\nabla\tilde{u}))\cdot(\nabla u - \nabla\tilde{u})dxdt
\end{aligned}$$

Recalling that on the set $\{|\tilde{v}|\ge n\}$ we have $n\le |S_n(\tilde{v})|\le n+1$, and if $|v - S_n(\tilde{v})|\le k$, then
$|v|\le k + |S_n(\tilde{v})|\le k+n+1$ and $|v|\ge |S_n(\tilde{v})| - k\ge n-k$. Now, since $S_n' = 0$ outside $[-n-1, n+1]$ and $|S_n'|\le 1$, we get

$$\begin{aligned}
&\left|\int_{\{|\tilde{v}|>n\}} \tilde{a}(t,x,\nabla u)\nabla(T_k(v - S_n(\tilde{v})))dxdt\right| = \left|\int_{\{|\tilde{v}|>n, |v-S_n(\tilde{v})|\le k\}} \tilde{a}(t,x,\nabla u)(\nabla v - S_n'(\tilde{v})\nabla\tilde{v})dxdt\right|\\
&\le \left|\int_{\{|\tilde{v}|>n, |v-S_n(\tilde{v})|\le k\}} \tilde{a}(t,x,\nabla u)\cdot\nabla v dxdt\right| + \left|\int_{\{|\tilde{v}|>n, |v-S_n(\tilde{v})|\le k\}} S_n'(\tilde{v})\tilde{a}(t,x,\nabla u)\cdot\nabla\tilde{v}dxdt\right|\\
&\le \beta\int_{\{n-k\le|v|\le n+k+1\}} \left(\tilde{b} + |\nabla u|^{p(x)-1}\right)|\nabla v| + \beta\int_{\{n\le|\tilde{v}|\le n+1, n-k\le|v|\le n+k+1\}} (\tilde{b} + |\nabla u|^{p(x)-1})|\nabla\tilde{u}|dxdt\\
&\le C\int_{\{n-k\le|v|\le n+k+1\}} \left(\tilde{b}^{p'(x)} + |\nabla u|^{p(x)} + |\nabla g_2|^{p(x)}\right)dxdt + C\int_{\{n<|\tilde{v}|\le n+1\}} |\nabla\tilde{u}|^{p(x)}dxdt\\
&\quad = \omega_2(n) \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

By the same procedures and for every S_n and S'_n be defined as before, we have

$$\begin{aligned}
&\left|\int_{\{|\tilde v|>n\}} S'_n(\tilde v)\tilde a(t,x\nabla\tilde u)\cdot\nabla(T_k(v-S_n(\tilde v)))dxdt\right|\\
&\le \beta\int_{\{n<|\tilde v|\le n+1,|v-S_n(\tilde v)|\le k\}}\left(\tilde b+|\nabla\tilde u|^{p(x)-1}\right)(|\nabla v|+|\nabla\tilde v|)\,dxdt\\
&\le C\int_{\{n<|\tilde v|\le n+1,n-k\le|v|\le n+k+1\}}\left(\tilde b^{p'(x)})+|\nabla\tilde u|^{p(x)}+|\nabla g_2|^{p(x)}+|\nabla u|^{p(x)}\right)dxdt\\
&\quad=\omega_3(n)\underset{n\to\infty}{\to}0.
\end{aligned}$$

Then we have (with $\omega_4(n)\underset{n\to\infty}{\to}0$)

$$\begin{aligned}
&\int_Q\left(\tilde a(t,x,\nabla u)-S'_n(\tilde v)\tilde a(t,x,\nabla\tilde u)\right)\nabla(T_k(v-S_n(\tilde v))dxdt\\
&\ge\int_{\{|\tilde v|\le n\}}\chi_{\{|\tilde v|\le n\}}\chi_{\{|u-\tilde u|\le k\}}\left(\tilde a(t,x,\nabla u)-\tilde a(t,x,\nabla\tilde u)\right)(\nabla u-\nabla\tilde u)dxdt-\omega_4(n).
\end{aligned}$$

Noting that equality (with $\omega_5(n)\underset{n\to\infty}{\to}0$)

$$\begin{aligned}
&\int_{\{|\tilde v|\le n\}}\chi_{\{|u-\tilde u|\le n\}}\left(\tilde a(t,x,\nabla u)-\tilde a(t,x,\nabla\tilde u)\right)\cdot(\nabla u-\nabla\tilde u)dxdt\\
&\le\omega_5(n)+k\int_Q|f||1-S'_n(\tilde v)|dxdt+\int_Q|G_1||1-S'_n(\tilde v)||\nabla(T_k(v-S_n(\tilde v))|dxdt\\
&\quad+\int_Q|F||1-S'_n(\tilde v)||\nabla(T_k(v-S_n(\tilde v))|dxdt+k\int_\Omega|u_0-S_nu_0)|dx.
\end{aligned}$$

Recall that $S_n(s)\to s$ as $n\to\infty$, $|S_n(s)|\le|s|$, $S'_n\to1$ and $|S'_n|\le1$, so that the dominated convergence theorem gives

$$\int_\Omega|u_0-S_n(u_0)|dx+\int_Q|f|(1-S'_n(\tilde v))dxdt\to0\text{ as }n\to\infty.$$

Moreover, $S'_n=1$ on $[-n,n]$, $S'_n=0$ outside $[-n-1,n+1]$ and $0\le S'_n\le1$ imply that

$$\begin{aligned}
&\int_Q|G_1||1-S'_n(\tilde v)||\nabla(T_k(v-S_n(\tilde v)))|dxdt\\
&\le\int_{\{|\tilde v|>n\}}|G_1||\nabla(T_k(v-S_n(\tilde v))|dxdt\\
&\le\int_{\{|\tilde v|>n,|v-S_n(\tilde v)|\le k\}}(|G_1||\nabla v|+|G_1||S'_n(\tilde v)||\nabla\tilde v|dxdt\\
&\le C\int_{\{n-k\le|v|\le n+k+1\}}(|G_1|^{p'(x)}+|\nabla u|^{p(x)}+|\nabla g_2|^{p(x)})\\
&\quad+C\int_{\{n<|\tilde v|\le n+1\}}(|G_1|^{p'(x)}+|\nabla\tilde u|^{p(x)}+|\nabla g_2|^{p(x)}dxdt\\
&=\omega_6(n)\underset{n\to\infty}{\to}0.
\end{aligned}$$

It follows that

$$\int_Q \chi_{\{|\tilde{v}|\le n\}}\chi_{\{|u-\tilde{u}|\le k\}}(\tilde{a}(t,x,\nabla u)-\tilde{a}(t,x,\nabla\tilde{u}))\cdot(\nabla u-\nabla\tilde{u})dxdt$$
$$\le \omega_7(n) \underset{n\to\infty}{\to} 0.$$

Since $\chi_{\{|u-\tilde{u}|\le k\}}(\tilde{a}(t,x,\nabla\tilde{u})-\tilde{a}(t,x,\nabla\tilde{u}))\cdot(\nabla u-\nabla\tilde{u})$ is nonnegative and applying Fatou's lemma, we obtain

$$\begin{cases} \displaystyle\int_Q \chi_{\{|u-\tilde{u}|\le n\}}(\tilde{a}(t,x,\nabla u)-\tilde{a}(t,x,\nabla\tilde{u}))\cdot(\nabla u-\nabla\tilde{u})dxdt \le 0, \\ \displaystyle\int_Q \chi_{\{|u-\tilde{u}|\le k\}}\left(\tilde{a}(t,x,\nabla u)-\tilde{a}(t,x,\nabla\tilde{u})\right)(\nabla u-\nabla\tilde{u})dxdt = 0 \text{ a.e. on } Q. \end{cases}$$

Then,

$$(\tilde{a}(t,x,\nabla u)-\tilde{a}(t,x,\nabla\tilde{u}))\cdot(\nabla u-\nabla\tilde{u}) = 0 \text{ a.e. on } Q,$$

which implies that

$$\nabla u = \nabla\tilde{u} \text{ a.e. on } Q,$$

that is,

$$\nabla v = \nabla\tilde{v} \text{ a.e. on } Q.$$

To conclude, we shall prove that $u=\tilde{u}$. To this aim, let us consider $W_n = T_1(T_k(v)-T_k(\tilde{v})) \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$. We have $\nabla w_n = \chi_{\{|T_k(v)-T_k(\tilde{v})|\le 1\}}(\chi_{\{|v|\le n\}}\nabla v - \chi_{\{|\tilde{v}|\le n\}}\nabla\tilde{v})$ and that

$$\nabla w_n = \begin{cases} 0 & \text{on } \{|v|\le n, |\tilde{v}|\le n\}\cup\{|v|>n, |\tilde{v}|>n\}, \\ \chi_{\{|v-T_k(\tilde{v})|\le 1\}}\nabla v & \text{on } \{|v|\le n, |\tilde{v}|>n\}, \\ -\chi_{\{|\tilde{v}-T_k(v)|\le 1\}}\nabla v & \text{on } \{|v|>n, |\tilde{v}|\le n\}. \end{cases}$$

But, if $|s|>n$, $|z|\le n$ and $|z-T_1(s)|\le 1$ then $n-1\le|z|\le n$, which implies that

$$\begin{aligned} \int_Q |\nabla w_n|^{p(x)}dxdt &\le \int_{\{n-1\le|v|\le n\}} |\nabla v|^{p(x)}dxdt + \int_{\{n-1\le|\tilde{v}|\le n\}} |\nabla\tilde{v}|^{p(x)}dxdt \\ &\le C\int_{\{n-1\le|v|\le n\}} (|\nabla u|^{p(x)} + |\nabla g_2|^{p(x)})dxdt \\ &\quad + C\int_{\{n-1\le|\tilde{v}|\le n\}} (|\nabla\tilde{u}|^{p(x)} + |\nabla g_2|^{p(x)})dxdt. \end{aligned}$$

Since $W_n \to 0$ in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and in $D'(Q)$, then using the a.e. convergence w_n to $T_1(v-\tilde{v})$, we conclude that

$$W_n \to T_1(v-\tilde{v}) \text{ in } D'(Q).$$

Then

$$T_1(v-\tilde{v}) = 0 \text{ i.e., } v=\tilde{v} \text{ on } Q.$$

Then $u=\tilde{u}$, that is u is a renormalized solution of the parabolic problem (2), and this concludes the proof of Theorem 1.1.

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