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Dedekind groupoids for posets

Gina Petre

Abstract. We associate with every poset $(P, \leq)$ a groupoid $(P, \circ)$ in such a way that $x \leq y \Longleftrightarrow x \circ y = x$ and obtain several other equivalences between properties of the poset and properties of the associated groupoid. In the case of chains our operation $\circ$ reduces to min.

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It is well known that Dedekind's definition of lattices as algebras with two operations satisfying commutativity, associativity and absorption, is equivalent to the definition of lattices as partially ordered sets (posets) in which every two elements have greatest lower bound (g.l.b. or inf) and least upper bound (l.u.b. or sup). As a matter of fact, this equivalence holds at the level of semilattices (see e.g. [1]–[3]). Recall that a *meet semilattice* is a poset $(S, \leq)$ such that every two elements have g.l.b. Then the operation

$$x \wedge y = \inf\{x, y\} \tag{1}$$

is commutative, associative and idempotent; we say that $(S, \wedge)$ is a *Dedekind semilattice.* Conversely, if $(S, \wedge)$ is a commutative and idempotent semigroup, i.e., a Dedekind semilattice, then by defining

$$x \leq y \Longleftrightarrow x \wedge y = x \ , \tag{2}$$

we obtain a meet semilattice $(S \leq)$. Similarly, the correspondence

$$x \vee y = \sup\{x, y\} \ , \tag{1'}$$

$$x \leq y \Longleftrightarrow x \vee y = y \ , \tag{2'}$$

establishes a bijection between join semilattices and Dedekind semilattices. As a matter of fact the above bijections can be lifted to the categorial level: the categories of meet semilattices, join semilattices and Dedekind semilattices are isomorphic; cf. [4].

In this note we suggest a Dedekind-like construction for arbitrary posets instead of meet semilattices, by associating with every poset a certain groupoid, which we call the *Dedekind groupoid* of the poset. The dual construction, having in mind join semilattices, is left to the reader. Our construction reduces to the conventional construction (1) only in the case of chains, but property (2) is generalized to arbitrary posets. We obtain several other equivalences between properties of posets and properties of the associated Dedekind groupoids.

Definition 1. Given a poset $(P, \leq)$, the binary operation $\circ$ is defined by

$$x \circ y = \begin{cases} x, & \text{if } x \leq y \ , \\ y, & \text{if } x \not\leq y \ . \end{cases} \tag{3}$$

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In the sequel $(P,\circ)$ will always stand for the groupoid defined by (3), which we call the *Dedekind groupoid* of the poset $(P,\leq)$. $\square$

Remark 1. Definition 1 is not a generalization of (1): the operation $\circ$ reduces to inf only if $(P,\leq)$ is a *chain*, i.e., a *totally ordered set*. In that case $x\circ y=\inf\{x,y\}=\min\{x,y\}$. $\square$

Remark 2. $(P,\circ)$ is an *idempotent* groupoid, that is, $x\circ x=x$ for all x. $\square$

Unlike property (1), property (2) is recaptured within our framework. We can say even more:

Proposition 1. *The following conditions are equivalent for two elements $x,y\in P$:*
(i) $x\leq y$;
(ii) $x\circ y=x$;
(iii) $x\circ y=y\circ x=x$.

Proof. (i)$\Longrightarrow$(iii): We have $x\circ y=x$ by Definition 1. If $x<y$ then $y\not\leq x$, hence $y\circ x=x$. If $x=y$ then $y\circ x=x\circ x=x$ by Remark 2.

(iii)$\Longrightarrow$(ii): Trivial.

(ii)$\Longrightarrow$(i): Suppose, by way of contradiction, that $x\not\leq y$ although $x\circ y=x$. Then $x\circ y=y$ by Definition 1, hence $x=y$; this implies $x\leq y$, a contradiction. $\square$

Corollary 1. $x\circ y=y\circ x\Longleftrightarrow x\leq y$ or $y\leq x$.

Proof. The element $x\circ y=y\circ x$ is x or y, so that we can apply twice Proposition 1. $\square$

Proposition 2. *The Dedekind groupoid $(P,\circ)$ is commutative if and only if $(P,\leq)$ is a chain.*

Proof. By Corollary 1. $\square$

Definition 2. We introduce the notation

$$x\#y\Longleftrightarrow x\not\leq y \text{ and } y\not\leq x\ . \tag{4}$$

If $x\#y$ we say that the elements x,y are *incomparable*. A poset is called a *totally unordered set* or an *antichain* if $x\#y$ for every x,y with $x\neq y$. $\square$

Note that $\#$ is a symmetric relation and $x\#y\Longrightarrow x\neq y$. An antichain is characterized by the property $x\leq y\Longleftrightarrow x=y$.

Lemma 1. *The poset $(P,\leq)$ is an antichain if and only if $x\not< y$ $\forall x$ $\forall y$.*

Proof. If P is an antichain, take $x,y\in P$. If $x=y$ then $x\not<y$ because $x\not<x$; if $x\neq y$ then $x\not\leq y$, hence $x\not<y$ again. If P is not an antichain, then by Definition 2 there exist $a,b\in P$ with $a\neq b$ and either $a\leq b$ or $b\leq a$, which implies $a<b$ or $b<a$. $\square$

Proposition 3. *The poset $(P,\leq)$ is an antichain if and only if $x\circ y=y$ identically.*

Proof. The following possibilities hold for two elements x,y: if $x=y$ then $x\circ y=y\circ y=y$; if $x<y$ then $x\circ y=x\neq y$; if $y<x$ then $x\circ y=y$; if $x\#y$ then $x\circ y=y$. Therefore $x<y\Longleftrightarrow x\circ y\neq y$, whence the desired result holds by Lemma 1. $\square$

Proposition 3 suggests the following natural question: is it possible that the property $x\circ y=x$ be satisfied identically? The answer is trivially affirmative if P is a singleton. Otherwise P has at least two distinct elements a and b. Since the relation $\leq$ is antisymmetric, we cannot have simultaneously $a\leq b$ and $b\leq a$. If e.g. $a\not\leq b$ then $a\circ b=b\neq a$. We thus obtain:

Remark 3. The equation $x\circ y=x$ holds identically if and only if P is a singleton. $\square$

Two other consequences of Proposition 1 are immediately obtained.

Corollary 2. *An element $e \in P$ is the unit of the groupoid $(P, \circ)$, i.e.,$x \circ e = e \circ x = x\ \forall x$, if and only if it is the greatest element of the poset $(P, \leq)$.* □

Corollary 3. *An element $o \in P$ is the zero (or annihilator) of the groupoid $(P, \circ)$, i.e., $x \circ o = o \circ x = o\ \forall x$, if and only if it is the least element of the poset $(P, \leq)$.*

Proof. By Proposition 1 with $x := o$ and $y := x$. □

We now wish to determine under what conditions the groupoid $(P, \circ)$ is a semigroup.

Lemma 2. *If the elements x,y,z are not pairwise distinct, then*

$$(x \circ y) \circ z = x \circ (y \circ z)\ . \tag{5}$$

Proof. If $x = y = z$ then (5) holds by idempotency. Now suppose card $\{x, y, z\} = 2$.

If $x = y$ then (5) reduces to $x \circ z = x \circ (x \circ z)$. But $x \circ z = x$ or $x \circ z = z$, which reduces the latter equality to $x = x \circ x$ or to $z = x \circ z$, respectively.

If $x = z$ then (5) reduces to $(x \circ y) \circ x = x \circ (y \circ x)$. Since $x \neq y$ we have the following possibilities. If $x < y$ then $y \not\leq x$ and

$$(x \circ y) \circ x = x \circ x = x \circ (y \circ x)\ .$$

If $y < x$ then $x \not\leq y$, hence

$$(x \circ y) \circ x = y \circ x = y \text{ and } x \circ (y \circ x) = x \circ y = y\ .$$

If $x \# y$ then

$$(x \circ y) \circ x = y \circ x = x \text{ and } x \circ (y \circ x) = x \circ x = x\ .$$

If $y = z$ then (5) reduces to $(x \circ y) \circ y = x \circ (y \circ y)$, that is, $(x \circ y) \circ y = x \circ y$. But $x \circ y = x$ or $x \circ y = y$, which reduces the latter equality to $x \circ y = x \circ y$ or $y \circ y = y$, respectively. □

Remark 4. If the elements *x,y,z* are pairwise distinct, then the Hasse diagram of the poset $\{x, y, z\}$ has one of the forms depicted in Figures 1–5 below. □

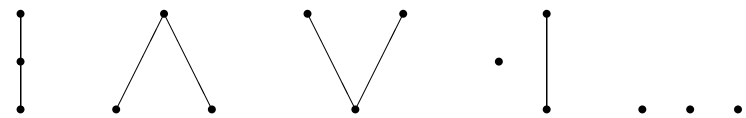

Fig. 1 Fig. 2 Fig. 3 Fig. 4 Fig. 5

Lemma 3. *If the Hasse diagram of the set $\{x, y, z\}$ has the form in* Fig.1*, then relation* (5) *holds.*

Proof. In this case the operation $\circ$ coincides with min, which is known to be associative. □

Lemma 4. *If the Hasse diagram of the set $\{x, y, z\}$ has the form in* Fig.2*, then relation* (5) *holds.*

Proof. The following three cases are possible.

If $x < z$, $y < z$ and $x \# y$, then $(x \circ y) \circ z = y \circ z = y$ and $x \circ (y \circ z) = x \circ y = y$.

If $x < y$, $z < y$ and $x \# z$, then $(x \circ y) \circ z = x \circ z = x \circ (y \circ z)$.

If $y < x$, $z < x$ and $y \# z$, then $(x \circ y) \circ z = y \circ z = z$ and $x \circ (y \circ z) = x \circ z = z$. □

Lemma 5. *If the Hasse diagram of the set $\{x, y, z\}$ has the form in* Fig.3*, then relation* (5) *holds.*

Proof. The following three cases are possible.
If $x < y$, $x < z$ and $y\#z$, then $(x \circ y) \circ z = x \circ z = x \circ (y \circ z)$.
If $y < x$, $y < z$ and $x\#z$, then $(x \circ y) \circ z = y \circ z = y$ and $x \circ (y \circ z) = x \circ y = y$.
If $z < x$, $z < y$ and $x\#y$, then $(x \circ y) \circ z = y \circ z = z$ and $x \circ (y \circ z) = x \circ z = z$.
□

Remark 5. If $x\#y$, $z\#y$ and $x < z$, then the Hasse diagram of the set $\{x, y, z\}$ has the form in Fig.4 and relation (5) fails. For $(x \circ y) \circ z = y \circ z = z$, while $x \circ (y \circ z) = x \circ z = x$. □

Lemma 6. *If the Hasse diagram of the set* $\{x, y, z\}$ *has the form in* Fig.5, *then relation* (5) *holds.*

Proof. For $(x \circ y) \circ z = y \circ z = z$ and $x \circ (y \circ z) = x \circ z = z$. □

We thus obtain a forbidden-configuration characterization of associativity.

Proposition 4. *The Dedekind groupoid* $(P, \circ)$ *is a semigroup if and only if the poset* $(P, \circ)$ *does not include a subset having the Hasse diagram in* Fig.4.

Proof. By Lemma 2, Remark 4, Lemmas 3–6 and Remark 5. □

Corollary 4. *The Dedekind groupoid is a monoid* $(P, \circ, e)$ *if and only if the poset* $(P, \leq)$ *has greatest element* e *and does not include a subset having the Hasse diagram depicted in* Fig.4.

Proof. By Corollary 2 and Proposition 4. □

Remark 6. The unique inversible element of the Dedekind monoid in Corollary 4 is the unit e. Indeed, suppose $a \circ b = e$. If $a \circ b = a$, then $a = e$. If $a \circ b = b$, then $b = e$, hence $a \circ e = e$; but $a \circ e = a$, therefore $a = e$ again. □

The case of lattices is remarkable.

Proposition 5. *The Dedekind groupoid* $(L; \circ)$ *of a lattice* L *is a semigroup if and only if the lattice is modular and has greatest element.*

Proof. Notice that if L includes a subset having the form depicted in Fig.4, then it also contains the supremum and the infimum of the elements in Fig.4; these five elements form the lattice known as the *pentagon*. Therefore, in view of Proposition 4, the groupoid L is a semigroup if and only if the lattice L with 1 does not include a pentagon, and it is well known that the latter condition is equivalent to the modularity of L. □

Concluding remarks

After this work was completed we discovered the paper [6], from which we learned that Neggers [5][1] introduced the operations (3) in dual form and provided an axiomatic characterization of the groupoids that occur in this way, which he called po-groupoids. Neggers and Kim [6] define a partial order in an arbitrary semigroup and relate it to po-groupoids. Among other results they find the present Proposition 4 (in another terminology). Their proof starts from the assumption that the elements x, y, z do not fulfil (5); this yields 6 cases relative to the possible values x, y, z of the two sides of (5), and the order configuration of x, y, z is established in each case. The modularity of a lattice which does not contain the configuration in Fig.4 (in our terminology) is also mentioned.

[1]Paper [5] is not available to us.

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(Gina Petre) Department of Mathematics, University of Craiova,
Al. I. Cuza street, 13, Craiova RO-200585, Romania, Tel/Fax: 40-251412673
E-mail address: gina_cernaianu@yahoo.fr