Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 31, 2004, Pages 79–84
ISSN: 1223-6934

About the generalization of Voronovskaja's theorem

Ovidiu T. Pop

Abstract. In this paper we will make some remearcks about the generalization of Voronovskaja's theorem.

2000 Mathematics Subject Classification. Primary 41A10; Secondary 41A36.
Key words and phrases. generalization of Voronovskaja's theorem and applications.

1. Introduction

Let m be a non zero natural number and $B_m : C([0,1]) \to C([0,1)]$ the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{2}$$

for any $x \in [0,1]$ and any $k \in \{0, 1, \dots, m\}$.

In 1932, E. Voronovskaja, proved the result contained in the following theorem (see [8]).

Theorem 1.1. *Let $f \in C([0,1])$ be a two times derivable function in the point $x \in [0,1]$. Then the equality*

$$\lim_{m\to\infty} m\left[(B_m f)(x) - f(x)\right] = \frac{x(1-x)}{2} f''(x). \tag{3}$$

holds.

In 1932, S. Bernstein, proved the following result (see[2]).

Theorem 1.2. *Let $f \in [0,1] \to \mathbb{R}$ be a four times derivable function in the point $x \in [0,1]$. Then the equality*

$$\lim_{m\to\infty} m^2\left[(B_m f)(x) - f(x) - \frac{x(1-x)}{2m} f''(x)\right] = \tag{4}$$
$$\frac{x(1-x)(1-2x)}{6} f'''(x) + \frac{x^2(1-x)^2}{8} f^{(IV)}(x)$$

holds.

Received: October 30, 2004.

Let m and i be a positive integers, $m \neq 0$ and $T_{m,i}(x)$ be the polynomials

$$T_{m,i}(x) = \sum_{k=0}^{m} (k - mx)^i p_{m,k}(x) \tag{5}$$

for any $x \in [0,1]$ (see [3] or [6]).

In [3] are the results contained in the following theorems.

Theorem 1.3. *If $f : [0,1] \to \mathbb{R}$ is a s times derivable function in the point $x \in [0,1]$, s is a natural number, s even, then*

$$\lim_{m\to\infty} m^{\frac{s}{2}} \left[(B_m f)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!} T_{m,i}(x) f^{(i)}(x) \right] = 0. \tag{6}$$

Theorem 1.4. *If i is a natural number, then*

$$\lim_{m\to\infty} \frac{T_{m,i}(x)}{m^{\left[\frac{i}{2}\right]}} = [x(1-x)]^{\left[\frac{i}{2}\right]} (a_i x + b_i), \tag{7}$$

for any $x \in [0,1]$, where

$$a_i = \begin{cases} 0, \ \textit{if} \ i \ \textit{is even or} \ i = 1 \\ -(i-1)!! \displaystyle\sum_{k=1}^{\left[\frac{i}{2}\right]} \frac{(2k-1)!!}{(2k-2)!!}, \ \textit{if} \ i \ \textit{is odd}, \ i \geq 3 \end{cases} \tag{8}$$

and

$$b_i = \begin{cases} 1, \quad \textit{if} \quad i = 0 \\ 0, \quad \textit{if} \quad i = 1 \\ (i-1)!!, \quad \textit{if} \ i \ \textit{is even}, \ i \geq 2 \\ \dfrac{1}{2}(i-1)!! \displaystyle\sum_{k=1}^{\left[\frac{i}{2}\right]} \frac{(2k-1)!!}{(2k-2)!!}, \quad \textit{if} \ i \ \textit{is odd}, \ i \geq 3. \end{cases} \tag{9}$$

Theorem 1.5. *Let $f : [0,1] \to \mathbb{R}$ be a s times derivable function in the point $x \in [0,1]$, s is a natural number, s even.*
If $s = 0$, then

$$\lim_{m\to\infty} (B_m f)(x) = f(x), \tag{10}$$

and if $s \geq 2$, then

$$\lim_{m\to\infty} m^{\frac{s}{2}} \left[(B_m f)(x) - \sum_{i=0}^{s-1} \frac{1}{m^i i!} T_{m,i}(x) f^{(i)}(x) \right] = \tag{11}$$

$$= \frac{(s-1)!!}{s!} [x(1-x)]^{\frac{s}{2}} f^{(s)}(x).$$

2. Preliminaries

Remind some known notions and results.

We immediately have

$$T_{m,0}(x) = \sum_{k=0}^{m} p_{m,k}(x) = 1, \ T_{m,1}(x) = \sum_{k=0}^{m} (k - mx) p_{m,k}(x) = 0,$$

$$T_{m,2}\left(x\right)=\sum_{k=0}^{m}(k-mx)^2 p_{m,k}(x)=mx(1-x)\,,$$

for any x$\in[0,1]$, for any m a non zero natural number. Let the functions $e_k:[0,1]\to\mathbb{R}, e_k(x)=x^k$, for any $x\in[0,1]$, where k is a natural number.

Lemma 2.1. *Let m and i be non zero natural number. Then*

$$T_{m,i}(x)=m^i\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}x^{i-j}\big(B_m e_j\big)(x)\,, \tag{12}$$

for any $x\in[0,1]$.

Proof. We know that

$$\begin{aligned}T_{m,i}\left(x\right)&=\sum_{k=0}^{m}(k-mx)^i p_{m,k}(x)=m^i\sum_{k=0}^{m}\left(\frac{k}{m}-x\right)^i p_{m,k}(x)=\\ =\ & m^i\sum_{k=0}^{m}\left[(-x)+\frac{k}{m}\right]^i p_{m,k}(x)=m^i\sum_{k=0}^{m}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}x^{i-j}\left(\frac{k}{m}\right)^j p_{m,k}(x)=\\ =\ & m^i\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}x^{i-j}\left[\sum_{k=0}^{m}\left(\frac{k}{m}\right)^j p_{m,k}(x)\right],\quad \text{and hence, (12) results.}\end{aligned}$$

□

Lemma 2.2. *If m and p are natural numbers, $m\neq 0$, we have*

$$\big(B_m e_p\big)(x)=\sum_{i=0}^{p}\frac{1}{m^i i!}\,T_{m,i}(x)e_p^{(i)}(x)\,, \tag{13}$$

for any $x\in[0,1]$.

Proof. We successively have

$$\begin{aligned}\big(B_m e_p\big)(x)\ &=\ \sum_{k=0}^{m}p_{m,k}(x)\left(\frac{k}{m}\right)^p=\frac{1}{m^p}\sum_{k=0}^{m}p_{m,k}(x)k^p=\\ &=\ \frac{1}{m^p}\sum_{k=0}^{m}p_{m,k}(x)\big[mx+(k-mx)\big]^p=\\ &=\ \frac{1}{m^p}\sum_{k=0}^{m}p_{m,k}(x)\sum_{i=0}^{p}\binom{p}{i}(mx)^{p-i}(k-mx)^i=\\ &=\ \frac{1}{m^p}\sum_{i=0}^{p}\left[\sum_{k=0}^{m}(k-mx)^i p_{m,k}(x)\right]\binom{p}{i}m^{p-i}x^{p-i}=\\ &=\ \sum_{i=0}^{p}\frac{1}{m^i}T_{m,i}(x)\binom{p}{i}x^{p-i}=\sum_{i=0}^{p}\frac{1}{m^i}\,T_{m,i}(x)\frac{p!}{i!(p-i)!}\,x^{p-i}=\\ &=\ \sum_{i=0}^{p}\frac{1}{m^i i!}T_{m,i}(x)(x^p)^{(i)}\,,\quad\text{and so (13) yields.}\end{aligned}$$

□

Corollary 2.1. *If m is a non zero natural number and P is a polynomial with real coefficients of degree $P = p$, then*

$$\left(B_m P\right)(x) = \sum_{i=0}^{p} \frac{1}{m^i i!} T_{m,i}(x) P^{(i)}(x)\,, \tag{14}$$

for any $x \in [0,1]$.

Proof. The proof is a result of Lemma 2.2. □

Corollary 2.2. *If m, p, s are natural numbers,$m \neq 0$ and $p \leq s$, then*

$$\big(B_m e_p\big)(x) = \sum_{i=0}^{s} \frac{1}{m^i i!} T_{m,i}(x) e_p^{(i)}(x)\,, \tag{15}$$

for any $x \in [0,1]$.

Corollary 2.3. *If m, p, s are natural numbers, $m \neq 0$ and $p \leq s$, P is a polynomial with real coefficients of degree $P = p$, then*

$$\big(B_m P\big)(x) = \sum_{i=0}^{s} \frac{1}{m^i i!} T_{m,i}(x) P^{(i)}(x)\,, \tag{16}$$

for any $x \in [0,1]$.

3. Main results

Lemma 3.1. *Let $s \geq 2$ be an even natural number. Then*

$$\frac{s}{2} - i + \left[\frac{i}{2}\right] \geq 0 \tag{17}$$

for all i natural numbers, $i \leq s$ and the equality in (3.1) takes place if and only if $i \in \{s-1, s\}$.

Proof. If i is an even number, $i \leq s$, then $\frac{s}{2} - i + \left[\frac{i}{2}\right] = \frac{s}{2} - 2\left[\frac{i}{2}\right] + + \left[\frac{i}{2}\right] = \frac{s}{2} - \left[\frac{i}{2}\right] \geq 0$, with equality if and only if i=s. If i is an odd number, because s is an even number it results that $i < s$, then $i + 1 \leq s$. We have that $\frac{s}{2} - i + \left[\frac{i}{2}\right] = \frac{s}{2} - \big(2\left[\frac{i}{2}\right] + 1\big) + \left[\frac{i}{2}\right] = \frac{s}{2} - \left[\frac{i}{2}\right] - 1 \geq 0$, with equality if and only if $i + 1 = s$, then $i = s - 1$. □

Theorem 3.1. *Let $f : [0,1] \to \mathbb{R}$ be s times derivable function in the point $x \in [0,1]$, s even natural number.*

If $s = 0$, then

$$\lim_{m\to\infty} \left(B_m f\right)(x) = f(x), \tag{18}$$

if $s = 2$, then

$$\lim_{m\to\infty} m\left[\left(B_m f\right)(x) - f\left(x\right)\right] = \frac{x\left(1-x\right)}{2} f''\left(x\right) \tag{19}$$

and if $s \geq 4$, then

$$\lim_{m\to\infty} m^{\frac{s}{2}} \left[\big(B_m f\big)(x) - \sum_{i=0}^{s-2} \frac{1}{m^i i!} T_{m,i}(x) f^{(i)}(x)\right] = \frac{1}{(s-1)!} \left[x\left(x-1\right)\right]^{\frac{s}{2}-1} \cdot \tag{20}$$

$$\cdot\left\{-\frac{(s-1)!!}{s}f^{(s)}(x)\,x^2+\left[\frac{(s-1)!!}{s}f^{(s)}(x)-(s-2)!!f^{(s-1)}(x)\sum_{k=1}^{\frac{s}{2}-1}\frac{(2k-1)!!}{(2k-2)!!}\right]x+\right.$$
$$\left.+\frac{1}{2}(s-2)!!f^{(s-1)}(x)\sum_{k=1}^{\frac{s}{2}-1}\frac{(2k-1)!!}{(2k-2)!!}\right\}.$$

Proof. From the Theorem 1.3 for $s=0$ we have that $\lim_{m\to\infty}\left[(B_mf)(x)-T_{m,0}(x)f^{(0)}(x)\right]=0$ and because $T_{m,0}(x)=1$ and $f^{(0)}(x)=f(x)$, obtain the relation (18). For $s=2$, from the Theorem 1.3 we have that

$$\lim_{m\to\infty}m\left[(B_mf)(x)-T_{m,0}(x)f^{(0)}(x)-\frac{1}{m}T_{m,1}(x)f'(x)-\right.$$
$$\left.-\frac{1}{m^2 2!}T_{m,2}(x)f''(x)\right]=0.$$

But $T_{m,1}(x)=0$, $T_{m,2}(x)=mx(1-x)$ and replacing above, we obtain the relation (19).

Let $s\in\mathbb{N}$, s even number, $s\geq 4$. For $i\in\{0,1,...,s\}$, we study $\lim_{m\to\infty}m^{\frac{s}{2}}\frac{1}{m^i i!}T_{m,i}(x)f^{(i)}(x)=\frac{1}{i!}f^{(i)}(x)\lim_{m\to\infty}\frac{T_{m,i}(x)}{m^{\left[\frac{i}{2}\right]}}m^{\frac{s}{2}-i+\left[\frac{i}{2}\right]}$ and keeping in mind of Theorem 1.4, we have that

$$\lim_{m\to\infty}m^{\frac{s}{2}}\frac{1}{m^i i!}T_{m,i}(x)f^{(i)}(x)= \tag{21}$$
$$=\frac{1}{i!}f^{(i)}(x)\left[x(1-x)\right]^{\left[\frac{i}{2}\right]}(\alpha_i x+\beta_i)\lim_{m\to\infty}m^{\frac{s}{2}-i+\left[\frac{i}{2}\right]}.$$

Taking account of Lemma 3.1, it results that its limit $\lim_{m\to\infty}m^{\frac{s}{2}-i+\left[\frac{i}{2}\right]}$ is finite if and only if $i\in\{s-1,s\}$. With this observation, from relation (6), we have that

$$\lim_{m\to\infty}m^{\frac{s}{2}}\left[(B_mf)(x)-\sum_{i-0}^{s-2}\frac{1}{m^i i!}T_{m,i}(x)f^{(i)}(x)\right]=$$
$$=\lim_{m\to\infty}\left[m^{\frac{s}{2}}\frac{1}{m^{s-1}(s-1)!}T_{m,s-1}(x)f^{(s-1)}(x)+m^{\frac{s}{2}}\frac{1}{m^s s!}T_{m,s}(x)f^{(s)}(x)\right]$$

and keeping in mind of (21), Lemma 3.1 and Theorem 1.4, we obtain relation (20). □

Observation 3.1. *Relation* (19) *from the Theorem* 3.1 *is the result obtained by E. Voronovskaja in Theorem* 1.1.

Observation 3.2. *For* $s=4$ *in Theorem* 3.1*, we obtain Theorem* 1.2.

Corollary 3.1. *If* $f:[0,1]\to\mathbb{R}$ *is a* s *times derivable function in the point* $x\in[0,1]$, s *is a non zero natural number,* s *even, then*

$$\lim_{m\to\infty}m^{\frac{s}{2}}\left[(B_mf)(x)-\sum_{i=0}^{s-1}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}x^{i-j}\cdot\right. \tag{22}$$
$$\left.\cdot(B_me_j)(x)\frac{1}{i!}f^{(i)}(x)\right]=\frac{(s-1)!!}{s!}\left[x(1-x)\right]^{\frac{s}{2}}f^{(s)}(x).$$

Proof. It results from Theorem 1.5 and from Lemma 2.1. □

Observation 3.3. *We ask our selves if a relation of the same type as the one in Theorem* 1.3 *takes place for odd number s. The answer is negative. Considering in the following an* $f \in C([0,1])$ *and an* $x \in [0,1]$ *arbitrary. Assuming the contrary, then for* $s = 3$ *there exists* $\alpha_3 \in \mathbb{R}$ *such that*

$$\lim_{m\to\infty} m^{\alpha_3}\left[(B_m f)(x) - \sum_{i=0}^{3}\frac{1}{m^i i!}T_{m,i}(x)f^{(i)}(x)\right] = 0. \tag{23}$$

Taking into acount the relation (4) *and because* $T_{m,3} = m(1-2x)x(1-x)$ *we have*

$$\lim_{m\to\infty} m^{2}\left[(B_m f)(x) - \sum_{i=0}^{3}\frac{1}{m^i i!}T_{m,i}(x)f^{(i)}(x)\right] = \tag{24}$$
$$= \frac{x^2(1-x)^2}{8}f^{(IV)}(x)$$

From (23) *and* (24) *we obtain a contradiction.*

References

[1] O. Agratini, *Aproximare prin operatori liniari*, Cluj-Napoca, Presa Universitară Clujeană, 2000 (Romanian).
[2] S. Bernstein, Complément à l'article de E. Voronovskaja, *C. R. Acad. Sci. URSS*, 86-92 (1932).
[3] G.G. Lorentz, *Bernstein polynomials*, Toronto, University of Toronto Press, 1953.
[4] G.G. Lorentz, *Aproximation of functions*, New York, Holt, Rinehart and Winston, 1966.
[5] O.T. Pop, About Bernstein's polynomial, to appear in *Octogon Mathematical Magazine*.
[6] D.D. Stancu, Gh. Coman, O. Agratini, R. Trîmbiţaş, *Analiza numerică şi teoria aproximării*, **I**, Cluj-Napoca, Presa Universitara Clujeana, 2001 (Romanian).
[7] D.D. Stancu, *Curs şi culegere de probleme de analiză numerică*, **I**, Cluj-Napoca, Universitatea "Babeş Bolyai" Cluj-Napoca, Facultatea de Matematică, 1977 (Romanian).
[8] E. Voronovskaja, Détérmination de la forme asymtotique d'aproximation des functions par les polynomes de M. Bernstein, *C. R. Acad. Sci. URSS*, 79-85 (1932).

(Ovidiu T. Pop) National College "Mihai Eminescu"
Mihai Eminescu Street, No. 13, Satu Mare 440014, Romania
E-mail address: ovidiutiberiu@yahoo.com