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On convex functions and related inequalities

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Abstract. The main result of this paper is to give refinement and reverse the celebrated Jensen inequality. We directly apply our results to establish several weighted arithmetic-geometric mean inequality. We also present a stronger estimate for the first inequality in the Hermite-Hadamard inequality.

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1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the C^*–algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. As customary, we reserve m, M for scalars and $\mathbf{1}_{\mathcal{H}}$ for the identity operator on $\mathcal{H}$. A self-adjoint operator A is said to be positive (written $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ holds for all $x \in \mathcal{H}$ also an operator A is said to be strictly positive (denoted by $A > 0$) if A is positive and invertible. If A and B are self-adjoint, we write $B \geq A$ in case $B - A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$–isomorphism between the C^*–algebra $C(\sigma(A))$ of continuous functions on the spectrum $\sigma(A)$ of a selfadjoint operator A and the C^*–algebra generated by A and the identity operator $\mathbf{1}_{\mathcal{H}}$. If $f, g \in C(\sigma(A))$, then $f(t) \geq g(t)$ $(t \in \sigma(A))$ implies that $f(A) \geq g(A)$.

A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It's said to be unital if $\Phi(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$. A continuous function f defined on the interval J is called an operator convex function if $f((1-v)A + vB) \leq (1-v)f(A) + vf(B)$ for every $0 < v < 1$ and for every pair of bounded self-adjoint operators A and B whose spectra are both in J.

The well-known Jensen inequality for the convex functions states that if f is a convex function on the interval $[m, M]$, then

$$f\left(\sum_{i=1}^{n} w_i a_i\right) \leq \sum_{i=1}^{n} w_i f(a_i) \tag{1}$$

for all $a_i \in [m, M]$ and $w_i \in [0, 1]$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^{n} w_i = 1$.
There is an extensive amount of literature devoted to Jensen's inequality concerning different generalizations, refinements, and converse results, see, for example [6, 10]. Mond and Pečarić [5] gave an operator extension of the Jensen inequality as follows: Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with $\sigma(A) \subseteq [m, M]$, and let $f(t)$ be a convex function on $[m, M]$, then for any unit vector $x \in \mathcal{H}$,

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle .$$

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Choi [1] showed if $f : J \to \mathbb{R}$ is an operator convex function, A is a self-adjoint operator with the spectra in J, and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is unital positive linear mapping, then

$$f(\Phi(A)) \le \Phi(f(A)). \tag{2}$$

Though in the case of convex function the inequality (2) does not hold in general, we have the following estimate [3, Lemma 2.1]:

$$f(\langle \Phi(A)x, x\rangle) \le \langle \Phi(f(A))x, x\rangle \tag{3}$$

for any unit vector $x \in \mathcal{K}$.
We here cite [4] and [11] as pertinent references to inequalities of types (2) and (3). For other recent results treating the Jensen operator inequality, we refer the reader to [3, 7, 8].

The current paper gives extensions of Jensen-type inequalities.

2. Main results

For our purpose, we need the following well-known result. See, for example, [7].

Lemma 2.1. *Let $f : J \to \mathbb{R}$ be a convex function and let $a, b \in J$. Then*

$$f((1-t)a + tb) \le (1-t)f(a) + tf(b) - 2r\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right)$$

and

$$(1-t)f(a) + tf(b) \le f((1-t)a + tb) + 2R\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right)$$

where $r = \min\{1-t, t\}$, $R = \max\{1-t, t\}$, and $0 \le t \le 1$.

In our first result, we present a refinement of the Jensen inequality.

Theorem 2.2. *Let $f : J \to \mathbb{R}$ be a convex function, $x_1, x_2, \ldots, x_n \in J$, and let $w_1, w_2, \ldots, w_n$ be positive numbers with $\sum_{i=1}^n w_i = 1$. Then*

$$\begin{aligned} & f\left(\sum_{i=1}^n w_i x_i\right) \\ & \le \sum_{i=1}^n w_i f(x_i) - 2\left(\frac{f\left(\sum_{i=1}^n w_i x_i\right) + \sum_{i=1}^n w_i f(x_i)}{2} - \sum_{i=1}^n w_i f\left(\frac{\sum_{j=1}^n w_j x_j + x_i}{2}\right)\right) \\ & \le \sum_{i=1}^n w_i f(x_i). \end{aligned}$$

Proof. From Lemma 2.1, we infer that

$$\frac{f(a + t(b-a)) - f(a)}{t} \le f(b) - f(a) - \frac{2\min\{t, 1-t\}}{t}\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right),$$

or

$$\frac{f(a + t(b-a)) - f(a)}{t} \le f(b) - f(a) - \frac{1 - |2t-1|}{t}\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right).$$

Now by letting $t \to 0$, we get

$$f(a) + f'(a)(b-a) \le f(b) - 2\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right). \tag{4}$$

Since for any convex function

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2},$$

we have

$$f(a) + f'(a)(b-a) \le f(b) - 2\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right) \le f(b).$$

Put $a = \sum_{i=1}^n w_i x_i$ in , we get

$$\begin{aligned}
& f\left(\sum_{i=1}^n w_i x_i\right) + bf'\left(\sum_{i=1}^n w_i x_i\right) - f'\left(\sum_{i=1}^n w_i x_i\right)\sum_{i=1}^n w_i x_i \\
& \le f(b) - 2\left(\frac{f\left(\sum_{i=1}^n w_i x_i\right) + f(b)}{2} - f\left(\frac{\sum_{i=1}^n w_i x_i + b}{2}\right)\right) \\
& \le f(b).
\end{aligned}$$

By replacing $b = x_i$, and then multiply by w_i and summing from 1 to n, we get

$$\begin{aligned}
& f\left(\sum_{i=1}^n w_i x_i\right) \\
& \le \sum_{i=1}^n w_i f(x_i) - 2\left(\frac{f\left(\sum_{i=1}^n w_i x_i\right) + \sum_{i=1}^n w_i f(x_i)}{2} - \sum_{i=1}^n w_i f\left(\frac{\sum_{j=1}^n w_j x_j + x_i}{2}\right)\right) \\
& \le \sum_{i=1}^n w_i f(x_i).
\end{aligned}$$

□

As a direct consequence of Theorem 2.2, we can obtain:

Corollary 2.3. *Let $f : J \to \mathbb{R}$ be a convex function, $x_1, x_2, \ldots, x_n \in J$, and let $w_1, w_2, \ldots, w_n$ be positive numbers with $\sum_{i=1}^n w_i = 1$. Then*

$$f\left(\sum_{i=1}^n w_i x_i\right) \le \sum_{i=1}^n w_i f\left(\frac{\sum_{j=1}^n w_j x_j + x_i}{2}\right) \le \sum_{i=1}^n w_i f(x_i).$$

Proof. It follows from Theorem 2.2 (see also [9]),

$$\sum_{i=1}^n w_i f\left(\frac{\sum_{j=1}^n w_j x_j + x_i}{2}\right) \le \frac{f\left(\sum\limits_{i=1}^n w_i x_i\right) + \sum\limits_{i=1}^n w_i f(x_i)}{2}.$$

Now, by the Jensen inequality, we have

$$
\begin{aligned}
f\left(\sum_{i=1}^{n} w_i x_i\right) &\le \sum_{i=1}^{n} w_i f\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right) \\
&\le \frac{f\left(\sum\limits_{i=1}^{n} w_i x_i\right) + \sum\limits_{i=1}^{n} w_i f(x_i)}{2} \\
&\le \sum_{i=1}^{n} w_i f(x_i).
\end{aligned}
$$

□

Corollary 2.3 provides a multiplicative refinement of the weighted arithmetic-geometric mean inequality.

Corollary 2.4. *Let $x_1, x_2, \ldots, x_n$ be positive numbers, and $w_1, w_2, \ldots, w_n$ be positive numbers with $\sum_{i=1}^{n} w_i = 1$. Then*

$$
\prod_{i=1}^{n} x_i^{w_i} \le \left(\frac{\prod\limits_{i=1}^{n}\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)^{2w_i}}{\prod_{i=1}^{n} x_i^{w_i} \sum_{i=1}^{n} w_i x_i}\right) \prod_{i=1}^{n} x_i^{w_i} \le \sum_{i=1}^{n} w_i x_i.
$$

Proof. Since $f(t) = -\log t$ is a convex function, we infer from Theorem 2.2,

$$
\begin{aligned}
\log\left(\prod_{i=1}^{n} x_i^{w_i}\right) &\le \log\left(\left(\frac{\prod\limits_{i=1}^{n}\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)^{2w_i}}{\prod_{i=1}^{n} x_i^{w_i} \sum_{i=1}^{n} w_i x_i}\right) \prod_{i=1}^{n} x_i^{w_i}\right) \\
&\le \log\left(\sum_{i=1}^{n} w_i x_i\right).
\end{aligned}
$$

We get the desired inequality by applying exp from both sides of the above inequality. □

The following theorem gives a refinement of an inequality proved by Dragomir and Ionescu in 1994 [2].

Theorem 2.5. *Let $f : J \to \mathbb{R}$ be a convex and differentiable function, $x_1, x_2, \ldots, x_n \in J$, and let $w_1, w_2, \ldots, w_n$ be positive numbers with $\sum_{i=1}^{n} w_i = 1$. Then*

$$
\begin{aligned}
&\sum_{i=1}^{n} w_i f(x_i) + \sum_{i=1}^{n} w_i x_i \sum_{i=1}^{n} w_i f'(x_i) - \sum_{i=1}^{n} w_i x_i f'(x_i) \\
&\le f\left(\sum_{i=1}^{n} w_i x_i\right) - 2\left(\frac{\sum_{i=1}^{n} w_i f(x_i) + f\left(\sum_{i=1}^{n} w_i x_i\right)}{2} - \sum_{i=1}^{n} w_i f\left(\frac{x_i + \sum_{j=1}^{n} w_j x_j}{2}\right)\right) \\
&\le f\left(\sum_{i=1}^{n} w_i x_i\right).
\end{aligned}
$$

Proof. As we have showed in the proof of Theorem 2.2,

$$f(a)+f'(a)(b-a)\le f(b)-2\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right).$$

By replacing $b=\sum_{i=1}^{n}w_ix_i$, we get

$$\begin{aligned}&f(a)+\sum_{i=1}^{n}w_ix_if'(a)-af'(a)\\&\le f\left(\sum_{i=1}^{n}w_ix_i\right)-2\left(\frac{f(a)+f\left(\sum_{i=1}^{n}w_ix_i\right)}{2}-f\left(\frac{a+\sum_{i=1}^{n}w_ix_i}{2}\right)\right)\\&\le f\left(\sum_{i=1}^{n}w_ix_i\right)\end{aligned}$$

By setting $a=x_i$, and then multiply by w_i and summing from 1 to n, we get

$$\begin{aligned}&\sum_{i=1}^{n}w_if(x_i)+\sum_{i=1}^{n}w_ix_i\sum_{i=1}^{n}w_if'(x_i)-\sum_{i=1}^{n}w_ix_if'(x_i)\\&\le f\left(\sum_{i=1}^{n}w_ix_i\right)-2\left(\frac{\sum_{i=1}^{n}w_if(x_i)+f\left(\sum_{i=1}^{n}w_ix_i\right)}{2}-\sum_{i=1}^{n}w_if\left(\frac{x_i+\sum_{j=1}^{n}w_jx_j}{2}\right)\right)\\&\le f\left(\sum_{i=1}^{n}w_ix_i\right).\end{aligned}$$

□

In the following result, we obtain a multiplicative reverse of the weighted arithmetic-geometric mean inequality by Theorem 2.5.

Corollary 2.6. *Let $x_1,x_2,\ldots,x_n$ be positive numbers, $w_1,w_2,\ldots,w_n$ be positive numbers with $\sum_{i=1}^{n}w_i=1$. Then*

$$\frac{\prod_{i=1}^{n}\left(\frac{x_i+\sum_{j=1}^{n}w_jx_j}{2}\right)^{2w_i}}{\exp\left(\sum_{i=1}^{n}w_ix_i\sum_{i=1}^{n}\frac{w_i}{x_i}-1\right)\sum_{i=1}^{n}w_ix_i\prod_{i=1}^{n}x_i^{w_i}}\sum_{i=1}^{n}w_ix_i\le\prod_{i=1}^{n}x_i^{w_i}.$$

Theorem 2.7. *Let $f:J\to\mathbb{R}$ be a convex function, $x_1,x_2,\ldots,x_n\in J$, and let $w_1,w_2,\ldots,w_n$ be positive numbers. Then*

$$\begin{aligned}&2\left(\frac{\sum_{i=1}^{n}w_if(x_i)+\sum_{i=1}^{n}w_if(y_i)}{2}-\sum_{i=1}^{n}w_if\left(\frac{x_i+y_i}{2}\right)\right)+\sum_{i=1}^{n}w_if'(y_i)x_i\\&-\sum_{i=1}^{n}w_if'(y_i)y_i\le\sum_{i=1}^{n}w_if(x_i)-\sum_{i=1}^{n}w_if(y_i).\end{aligned}$$

Proof. If we apply (4) for the selection $a=y_i$, $b=x_i$ $(i=1,2,\ldots,n)$, we may write

$$2\left(\frac{f(x_i)+f(y_i)}{2}-f\left(\frac{x_i+y_i}{2}\right)\right)+f'(y_i)(x_i-y_i)\le f(x_i)-f(y_i),\qquad(5)$$

for any $i = 1, 2, \ldots, n$. Multiplying (5) by $w_i \geq 0$ $(i = 1, 2, \ldots, n)$ and summing over i from 1 to n we deduce

$$2\left(\frac{\sum_{i=1}^{n} w_i f(x_i) + \sum_{i=1}^{n} w_i f(y_i)}{2} - \sum_{i=1}^{n} w_i f\left(\frac{x_i + y_i}{2}\right)\right) + \sum_{i=1}^{n} w_i f'(y_i)\, x_i$$
$$- \sum_{i=1}^{n} w_i f'(y_i)\, y_i \quad \leq \sum_{i=1}^{n} w_i f(x_i) - \sum_{i=1}^{n} w_i f(y_i).$$

□

If we choose $y_1 = y_2 = \cdots = y_n = 1/W_n \ \sum_{i=1}^{n} w_i x_i$ in Theorem 2.7, we get:

Corollary 2.8. *Let $f : J \to \mathbb{R}$ be a convex function, $x_1, x_2, \ldots, x_n \in J$, $w_1, w_2, \ldots, w_n$ be positive numbers, and $W_n = \sum_{i=1}^{n} w_i$. Then*

$$f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i f(x_i)$$
$$- 2\left(\frac{1/W_n \ \sum_{i=1}^{n} w_i f(x_i) + f\left(1/W_n \ \sum_{i=1}^{n} w_i x_i\right)}{2} - \frac{1}{W_n} f\left(\frac{x_i + 1/W_n \ \sum_{i=1}^{n} w_i x_i}{2}\right)\right).$$

If one chooses in Theorem 2.7, $x_1 = x_2 = \cdots = x_2 = 1/W_n \ \sum_{i=1}^{n} w_i y_i$, then we may deduce the following counterpart of Jensen's inequality.

Corollary 2.9. *Let $f : J \to \mathbb{R}$ be a convex function, $x_1, x_2, \ldots, x_n \in J$, $w_1, w_2, \ldots, w_n$ be positive numbers, and $W_n = \sum_{i=1}^{n} w_i$. Then*

$$\frac{1}{W_n} \sum_{i=1}^{n} w_i f(y_i) + \frac{1}{W_n} \sum_{i=1}^{n} w_i f'(y_i) \frac{1}{W_n} \sum_{i=1}^{n} w_i y_i - \frac{1}{W_n} \sum_{i=1}^{n} w_i f'(y_i)\, y_i$$
$$+ 2\left(\frac{f\left(1/W_n \ \sum_{i=1}^{n} w_i y_i\right) + 1/W_n \ \sum_{i=1}^{n} w_i f(y_i)}{2} - \frac{1}{W_n} f\left(\frac{1/W_n \ \sum_{i=1}^{n} w_i y_i + y_i}{2}\right)\right)$$
$$\leq f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i y_i\right).$$

We close this section by providing a new refinement and a reverse for the first inequality in the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f((1-t)\, a + tb)\, dt \leq \frac{f(a) + f(b)}{2}.$$

Theorem 2.10. *Let $f : J \to \mathbb{R}$ be a convex function and let $a, b \in J$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f((1-t)\, a + tb)\, dt - \int_0^1 \frac{1-t}{1+|2t-1|} f((1-t)\, a + tb)\, dt$$
$$- \int_0^1 \frac{t}{1+|2t-1|} f((1-t)\, b + ta)\, dt + \int_0^1 \frac{1}{1+|2t-1|} f\left(2(a-b)\left(t^2 - t\right) + a\right) dt$$

and

$$\int_0^1 f((1-t)a+tb)\,dt \le f\left(\frac{a+b}{2}\right) + \int_0^1 \frac{1-t}{1-|1-2t|} f((1-t)a+tb)\,dt$$
$$+ \int_0^1 \frac{t}{1-|1-2t|} f((1-t)b+ta) - \int_0^1 \frac{1}{1-|1-2t|} f\left(2(a-b)\left(t^2-t\right)+a\right) dt.$$

Proof. It follows from Lemma 2.1,

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{2R}\left((1-t)f(a)+tf(b)-f((1-t)a+tb)\right).$$

Now, by replacing $a=(1-t)a+tb$ and $b=(1-t)b+ta$, we get

$$f\left(\frac{a+b}{2}\right) \le \frac{f((1-t)a+tb)+f((1-t)b+ta)}{2}$$
$$-\frac{1}{2R}\left((1-t)f((1-t)a+tb)+tf((1-t)b+ta)-f\left(2(a-b)\left(t^2-t\right)+a\right)\right).$$

Thus,

$$f\left(\frac{a+b}{2}\right) \le \frac{f((1-t)a+tb)+f((1-t)b+ta)}{2} - \frac{1-t}{1+|2t-1|} f((1-t)a+tb)$$
$$-\frac{t}{1+|2t-1|} f((1-t)b+ta) + \frac{1}{1+|2t-1|} f\left(2(a-b)\left(t^2-t\right)+a\right).$$

Now, by taking integral over $0 \le t \le 1$, we get

$$f\left(\frac{a+b}{2}\right) \le \int_0^1 f((1-t)a+tb)\,dt - \int_0^1 \frac{1-t}{1+|2t-1|} f((1-t)a+tb)\,dt$$
$$-\int_0^1 \frac{t}{1+|2t-1|} f((1-t)b+ta)\,dt + \int_0^1 \frac{1}{1+|2t-1|} f\left(2(a-b)\left(t^2-t\right)+a\right) dt.$$

On the other hand, by Lemma 2.1

$$\frac{f(a)+f(b)}{2} \le f\left(\frac{a+b}{2}\right) + \frac{1}{2r}\left((1-t)f(a)+tf(b)-f((1-t)a+tb)\right).$$

Therefore,

$$\int_0^1 f((1-t)a+tb)\,dt \le f\left(\frac{a+b}{2}\right) + \int_0^1 \frac{1-t}{1-|1-2t|} f((1-t)a+tb)\,dt$$
$$+ \int_0^1 \frac{t}{1-|1-2t|} f((1-t)b+ta) - \int_0^1 \frac{1}{1-|1-2t|} f\left(2(a-b)\left(t^2-t\right)+a\right) dt.$$

□

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