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Insensitizing controls for linear ODE's

Marcos López-García, Alberto Peña-García, and Luz de Teresa

Abstract. In this paper we present some results regarding insensitizing controls for finite dimensional systems. The concept was introduced by J. L. Lions in [7] in the context of partial differential equations and, as far as we know, is a problem that has not been treated in literature for ordinary differential equations. The concept in this situation arises in a natural way when treating the semidiscrete one for the heat equation. We present some results in the linear framework.

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1. Introduction

In the late 1980's, J.L. Lions introduced several notions, inspired probably by the sensitivity theory for ODE's, to the control theory of Partial Differential Equations in order to handle uncertainties. Sentinels, least regret control and insensitizing controls were proposed in [5], [6] and [7]. In this paper we propose the study of *insensitizing controls* in the framework of finite dimensional systems. The idea behind this important concept is to act on the system in such a way that a functional defined on the solutions of the equation (a "performance index" or "cost" see e.g. [8]) is not sensible to some *pollution* or *noise* in some datum of the system. There are of course several uncertainties and functionals that arise when modeling a system. One of the possible functionals is precisely the solution to the system. In this situation there are some results in the literature that study the *evolution of sensitivities* [4].

Let us recall the insensitizing control problem for the semilinear heat equation. Let $\Omega \subset \mathbb{R}^n, n \geq 1$, be a bounded and open set with boundary $\partial\Omega \in \mathcal{C}^2$. Let $T > 0$ and ω be an open and non empty subset of Ω. Consider the parabolic system

$$\begin{cases} \partial_t y - \Delta y + f(y) = 1_\omega v + \xi & \text{in } Q = \Omega \times (0,T), \\ y = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,T) \\ y(0) = y_0 + \tau w_0 \quad \text{in } \Omega \end{cases} \tag{1}$$

where f is a globally Lipschitz-continuous function, $\xi \in L^2(Q)$ and $y_0 \in L^2(\Omega)$ are given. In system (1), $y = y(x,t)$ is the state and $v = v(x,t)$ is a control function

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supported in ω. Notice that y depends on the data y_0, ξ, v, w_0, τ.

The data in system (1) are incomplete in the following sense:

- $w_0 \in L^2(\Omega)$ is unknown and $|w_0|_{L^2(\Omega)} = 1$,
- $\tau \in \mathbb{R}$ is unknown and small enough.

Let Ψ be a differentiable functional defined on the set of solutions to (1). It is said that the control v insensitizes $\Psi(y)$ for the initial data y_0 and the source term ξ if

$$\left.\frac{\partial \Psi\left(y\left[y_0, \xi, v, w_0, \tau\right]\right)}{\partial \tau}\right|_{\tau=0} = 0, \quad \text{for all } w_0 \in L^2(\Omega). \tag{2}$$

When (2) holds the functional Ψ is locally insensitive to the perturbations of the initial data. In [9] the author analyzes the case when Ψ is the square of the L^2-norm of the state y in some observation subset $\mathcal{O} \subset \Omega$, namely,

$$\Psi(y) := \frac{1}{2} \int_0^T \int_{\mathcal{O}} y^2 dx dt. \tag{3}$$

With respect to this particular functional, the author shows that the insensitivity condition (2) is equivalent to a null-control problem for a coupled system of parabolic PDEs: consider the cascade system of semilinear parabolic equations

$$\begin{cases} \partial_t y - \Delta y + f(y) = 1_\omega v + \xi \text{ in } Q, \\ y = 0 \quad \text{ on } \Sigma, \\ y(0) = y_0 \quad \text{ in } \Omega, \end{cases} \\ \begin{cases} -\partial_t q - \Delta q + f'(y) q = 1_{\mathcal{O}} y \quad \text{ in } Q, \\ q = 0 \quad \text{ on } \Sigma, \\ q(T) = 0 \text{ in } \Omega. \end{cases} \tag{4}$$

Then, a control v satisfies the insensitivity condition (2) for the functional (3) and the system (1), if and only if the component q of the associated solution of (4) fulfills

$$q(0) = 0. \tag{5}$$

Notice that (5) is a null controllability property for the cascade system (4), but we emphasize that the control v acts indirectly on the equation satisfied by q by means of the localized coupling term $1_{\mathcal{O}} y$, and that the first equation in (4) is forward in time while the second one is backward in time. Under suitable conditions on the data f, y_0, ξ and the sets ω and $\mathcal{O}$, the author proves the existence of a control v insensitizing the functional (3). We refer to [9] for the details.

In [1] the authors address the insensitizing control problem from the point of view of numerical methods. They build a semi discrete approximation of the system (1) and by means of semi discrete Carleman estimates they deduce a "relaxed" observability inequality for the linearized equation, which is uniform with respect to the discretization parameter. This yields the existence of suitable insensitizing semi discrete controls within this framework for the initial nonlinear problem.

For writing convenience the authors analyze the case $\Omega = (0, L)$ and consider the elliptic operator $\mathcal{A} = -\partial_x^2$ with homogeneous Dirichlet boundary conditions.

Let $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = L$, they refer to this discretization as to the primal mesh $\mathfrak{M} := \{x_i : i = 1, \ldots, N\}$, and the boundary points are denoted by $\partial\mathfrak{M} = \{x_0, x_{N+1}\} = \{0, L\}$.

Set $h_{i+\frac{1}{2}} = x_{i+1} - x_i$ and $x_{i+\frac{1}{2}} = (x_{i+1} + x_i)/2, i = 0, \ldots, N$. The step size is denoted by $h^{\mathfrak{M}} = \max_i h_{i+\frac{1}{2}}$.

$\mathbb{R}^{\mathfrak{M}}$ stands for the set of discrete functions defined on $\mathfrak{M}$. If $u \in \mathbb{R}^{\mathfrak{M}}$, u_i denotes the value of u at x_i. For $u \in \mathbb{R}^{\mathfrak{M}}$ define

$$u^{\mathfrak{M}} = \sum_{i=1}^{N} \mathbf{1}_{\left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]} u_i, \quad \text{and } u^{\partial\mathfrak{M}} = \{u(0), u(L)\}.$$

Thus, the authors consider the 1-D semi discrete system

$$\begin{cases} \partial_t y^{\mathfrak{M}} + \mathcal{A}^{\mathfrak{M}} y^{\mathfrak{M}} + f\left(y^{\mathfrak{M}}\right) = \mathbf{1}_\omega v^{\mathfrak{M}} + \xi^{\mathfrak{M}} \text{ in } \mathbb{R}^{\mathfrak{M}},\ t \in (0, T), \\ y^{\partial\mathfrak{M}} = 0 \quad \text{in } (0, T), \\ y^{\mathfrak{M}}(0) = y_0^{\mathfrak{M}} + \tau w_0^{\mathfrak{M}}, \end{cases} \tag{6}$$

where f is a C^1 globally Lipschitz-continuous function, with $f(0) = 0$. Here $\mathcal{A}^{\mathfrak{M}}$ is the discrete approximation of $\mathcal{A} := -\partial_x^2$ on the mesh $\mathfrak{M}$, i.e $\mathcal{A}^{\mathfrak{M}}$ is the symmetric tridiagonal matrix $h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{N \times N}$.

They study the existence of uniformly bounded semi discrete controls that insensitize the functional

$$\Psi\left(y^{\mathfrak{M}}\right) := \frac{1}{2} \int_0^T \int_{\mathcal{O}} \left|y^{\mathfrak{M}}\right|^2 dxdt \tag{7}$$

where $y^{\mathfrak{M}}$ is the solution to (6). As in the continuous case, it is proved that the insensitizing control problem for (7) is equivalent to find bounded families of semi discrete controls $\left(v^{\mathfrak{M}}\right)_{\mathfrak{M}}$ such that the solution $\left(y^{\mathfrak{M}}, q^{\mathfrak{M}}\right)$ of the coupled problem

$$\begin{cases} \partial_t y^{\mathfrak{M}} + \mathcal{A}^{\mathfrak{M}} y^{\mathfrak{M}} + f\left(y^{\mathfrak{M}}\right) = \mathbf{1}_\omega v^{\mathfrak{M}} + \xi^{\mathfrak{M}} \quad \text{in } \mathbb{R}^{\mathfrak{M}} \times (0, T), \\ -\partial_t q^{\mathfrak{M}} + \mathcal{A}^{\mathfrak{M}} q^{\mathfrak{M}} + f'\left(y^{\mathfrak{M}}\right) q^{\mathfrak{M}} = \mathbf{1}_{\mathcal{O}} y^{\mathfrak{M}} \text{ in } \mathbb{R}^{\mathfrak{M}} \times (0, T), \\ y^{\partial\mathfrak{M}} = q^{\partial\mathfrak{M}} = 0 \quad \text{in } (0, T), \\ y^{\mathfrak{M}}(0) = y_0^{\mathfrak{M}}, \quad q^{\mathfrak{M}}(T) = 0, \end{cases} \tag{8}$$

satisfies the condition

$$q^{\mathfrak{M}}(0) = 0. \tag{9}$$

To get this, they adapt the approach introduced in [9], but taking into account the semi discrete nature of the problem. Firstly, they analyze controllability properties of the linearized version of (8). Then, a fixed point argument helps to obtain the controllability result for the nonlinear system (8). We refer to [1] for the details.

It is easy to see that the linear version of system (8) (i.e taking $f(s) = as$) can be written as

$$\begin{aligned} \dot{\mathbf{y}} &= A\mathbf{y} + B\mathbf{u} + \widehat{\xi}, \quad t \in (0, T) \\ \dot{\mathbf{q}} &= -A\mathbf{q} + Q\mathbf{y}, \quad t \in (0, T) \\ \mathbf{y}(0) &= \mathbf{y}_0, \quad \mathbf{q}(T) = 0, \end{aligned}$$

where

$A = \dfrac{1}{h^2}\text{tridiag}(1,-2,1)+aI,\ B = \text{diag}(\mathbf{1}_\omega(x_1),\dots,\mathbf{1}_\omega(x_N)),\ Q = \text{diag}(\mathbf{1}_\mathcal{O}(x_1),\dots,\mathbf{1}_\mathcal{O}(x_N)),$

$\mathbf{y}(t) = (y(x_1,t),\dots,y(x_N,t)),\ \mathbf{q}(t) = (q(x_1,t),\dots,q(x_N,t)),\ \mathbf{u}(t) = (v(x_1,t),\dots,v(x_N,t)),$

and

$$\mathbf{y}_0 = (y_0(x_1),\dots,y_0(x_N)), \quad \widehat{\xi}(t) = (\xi(x_1,t),\dots,\xi(x_N,t)).$$

The insensitizing condition (9) is equivalent to drive to zero $\mathbf{q}(0)$.

The last system motivate us to study the partial null controllability of cascade systems in the setting of ordinary differential equations, where the equations are not in the same direction of time, i.e we have a forward equation coupled with a backward equation.

On the other hand, we can mimic the insensitizing control problem in the context of ordinary differential equations and study it in its own right. This is done in the next section.

2. Statement of the problem

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n\backslash\{0\}$, and $T > 0$ we consider the Cauchy problem

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + Bu, \quad t \in (0,T) \\ \mathbf{x}(0) &= \mathbf{x}_0 + \tau\mathbf{x}_1, \end{aligned} \tag{10}$$

where $\mathbf{x} \in \mathbb{R}^n$, $u \in L^2([0,T];\mathbb{R})$ and $\mathbf{x}_0 \in \mathbb{R}^n$ is known. The data of the system are incomplete in the following sense

- $\mathbf{x}_1 \in \mathbb{R}^n$ is unknown but $\|\mathbf{x}_1\| = 1$.
- τ is a small unknown parameter.

Given a functional $\mathcal{J}$ defined on the set of solutions to (10), we say that a control u insensitizes the functional $\mathcal{J}(\mathbf{x})$ with $\mathbf{x}$ solution to (10) (or insensitizes $\mathcal{J}(\mathbf{x})$ to abridge) if

$$\left.\frac{\partial \mathcal{J}}{\partial \tau}\right|_{\tau=0} = 0 \quad \text{for all } \mathbf{x}_1 \in \mathbb{R}^n,\ \|\mathbf{x}_1\| = 1. \tag{11}$$

There is a large set of functionals that may be interesting for the purpose of a insensitizing control. Here we consider the functional defined as

$$\mathcal{J}_T(\mathbf{x}(u)) = \int_0^T \mathbf{x}^*(t)\, G\mathbf{x}(t)\, dt, \tag{12}$$

where $G \in \mathbb{R}^{n\times n}$ is an arbitrary matrix. Observe that in (7) $G = I$, however we will generalize to a more interesting problem with G a general matrix.

Also it can be seen that when G is an antisymmetric matrix the problem is trivial since $\frac{\partial \mathcal{J}}{\partial \tau} \equiv 0$. So all along the paper we will consider G a non antisymmetric matrix.

The following result shows that, for $\mathcal{J}_T$ defined in (12), the insensitizing condition (11) is equivalent to the partial null controllability of a coupled system.

Proposition 2.1. *Let $T > 0$ be given, define $\mathcal{J}_T$ be as in (12). Then a control $u \in L^2([0,T];\mathbb{R})$ insensitizes $\mathcal{J}_T(\mathbf{x})$ with $\mathbf{x}$ solution to (10) if and only if $\mathbf{p}(0) = 0$, where $\mathbf{p}$ satisfies*

$$\begin{aligned} &\dot{\mathbf{x}} = A\mathbf{x} + Bu, \quad t \in (0,T) \\ &\dot{\mathbf{p}} = -A^*\mathbf{p} + Q\mathbf{x}, \quad t \in (0,T) \\ &\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{p}(T) = 0, \end{aligned} \tag{13}$$

with $Q = G + G^* \neq 0$ and $(^*)$ denotes the transpose of a matrix.

Remark 2.1. Observe that the insensitizing condition is equivalent to a *partial null controllability* of system (13) where we want to drive only the p component of the system to zero in time T. Observe also that the equations verified by $\mathbf{x}$ and $\mathbf{p}$ are not in the same direction of time, so we have a forward equation ($\mathbf{x}$ component) coupled with a backward equation ($\mathbf{p}$ component). This fact introduces new technical difficulties and the results differ from the classical ones of controlling only partially a system where all the components go in the same time direction. See e.g. [2] for results on partial controllability for ODE and see [10] chapter 11 for results regarding partial stabilization of a ODE.

Remark 2.2. Observe that it is also interesting to study some optimal control problem associated with the quadratic functional $\mathcal{J}_T$ that relates the optimal control u to the adjoint equation given by $\mathbf{p}$. See e.g. [11], part III.

Throughout this paper $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$, and $\|\cdot\|$ stands for the corresponding norm.

Proof of Proposition 2.1. Let $\mathbf{x}_0 \in \mathbb{R}^n$ fixed. For $\mathbf{x}_1 \in \mathbb{R}^n$ such that $\|\mathbf{x}_1\| = 1$ we consider the solution $\mathbf{x}(\cdot,\cdot;\mathbf{x}_1) : [0,T] \times (-\delta,\delta) \to \mathbb{R}^n$ of (10) that is given as follows

$$\mathbf{x}(t,\tau) := \mathbf{x}(t,\tau;\mathbf{x}_1) = e^{At}(\mathbf{x}_0 + \tau\mathbf{x}_1) + e^{At}\int_0^t e^{-As}Bu(s)ds.$$

We observe that the functional $\mathcal{J}_T$ is differentiable with respect to τ. Hence the insensitizing condition can be rewritten in this way

$$\int_0^T \left\langle \frac{\partial \mathbf{x}}{\partial \tau}(t,\tau), Q\mathbf{x}(t,\tau) \right\rangle dt \bigg|_{\tau=0} = \int_0^T \left\langle \frac{\partial \mathbf{x}}{\partial \tau}(t,0), Q\mathbf{x}(t,0) \right\rangle dt = 0. \tag{14}$$

We have that $\frac{\partial \mathbf{x}}{\partial \tau}(t,0) = \mathbf{w}$ with $\mathbf{w}$ the corresponding solution to

$$\begin{aligned} &\dot{\mathbf{w}} = A\mathbf{w}, \quad t \in (0,T) \\ &\mathbf{w}(0) = \mathbf{x}_1. \end{aligned} \tag{15}$$

Then, taking the inner product of (15) with $\mathbf{p}$ given in (13) we obtain:

$$\begin{aligned} \int_0^T \langle \mathbf{w}, \dot{\mathbf{p}}(t) \rangle \, dt &= -\int_0^T \langle \dot{\mathbf{w}}, \mathbf{p}(t) \rangle \, dt - \langle \mathbf{x}_1, \mathbf{p}(0) \rangle \\ &= -\int_0^T \langle A\mathbf{w}, \mathbf{p}(t) \rangle \, dt - \langle \mathbf{x}_1, \mathbf{p}(0) \rangle . \end{aligned}$$

From (14) we get that

$$0 = \int_0^T \left\langle \frac{\partial \mathbf{x}}{\partial \tau}(t,0), \dot{\mathbf{p}}(t) + A^*\mathbf{p}(t) \right\rangle dt = -\langle \mathbf{x}_1, \mathbf{p}(0) \rangle ,$$

for all $\mathbf{x}_1 \in \mathbb{R}^n$ with $\|\mathbf{x}_1\| = 1$. That means that for every $\widetilde{\mathbf{x}}_1 \neq 0$, we get $-\left\langle \frac{\widetilde{\mathbf{x}}_1}{\|\widetilde{\mathbf{x}}_1\|}, \mathbf{p}(0) \right\rangle = 0$, so $\langle \widetilde{\mathbf{x}}_1, \mathbf{p}(0) \rangle = 0$ for every $\widetilde{\mathbf{x}}_1 \neq 0$ and the conclusion is immediate. □

The rest of the paper is organized as follows. In section 3 we reformulate Proposition 2.1 in a more general situation, that is, we will consider a partial null controllability problem for a forward-backward cascade system. In section 4, we consider the partial null controllability of a non-autonomous forward-backward coupled system.

3. Insensitizing controls for linear ODEs

In what follows we will treat a general linear forward-backward model with the component $\mathbf{p}$ of the system described by a general matrix C (that is not necessarily $-A^*$). That is, we will consider the equation verified by $\mathbf{x}$ as forward in time and the system verified by $\mathbf{p}$ as backward in time with $\mathbf{p}(T) = 0$.

For $T > 0$, $A, C, Q \in \mathbb{R}^{n\times n}$, $Q \neq 0$, and $B \in \mathbb{R}^n \setminus \{0\}$ consider the following cascade system

$$\begin{aligned} &\dot{\mathbf{x}} = A\mathbf{x} + Bu, \quad t \in (0,T) \\ &\dot{\mathbf{p}} = C\mathbf{p} + Q\mathbf{x}, \quad t \in (0,T) \\ &\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{p}(T) = 0. \end{aligned} \tag{16}$$

Definition 3.1. We will say that (16) is *partially null controllable* if for every $\mathbf{x}_0 \in \mathbb{R}^n$ it exists $u \in L^2([0,T];\mathbb{R})$ such that $\mathbf{p}(0) = 0$.

We introduce the matrix-valued analytic function

$$g(t) := \int_0^t e^{sC} Q e^{-sA} ds\, e^{tA}, \quad t \in \mathbb{R}.$$

By using the variation of constants formula we get the solution $\mathbf{p}(t)$ of the system (16), then we evaluate at $t = T$ to get

$$\mathbf{p}(0) = -e^{-TC} g(T)\mathbf{x}_0 - e^{-TC} \int_0^T g(T-s)Bu(s)ds. \tag{17}$$

The next result gives an explicit representation of g as a power series in terms of A, C and Q.

Proposition 3.1. *The function g has the following representation in power series*

$$g(t) = \sum_{j=0}^{\infty} \frac{t^{j+1}}{(j+1)!} D_j, \quad t \in \mathbb{R}, \tag{18}$$

where

$$D_j := \sum_{\ell=0}^{j} C^\ell Q A^{j-\ell}, \quad j \geq 0. \tag{19}$$

Proof. Since $g(t)$ is an analytic function at $t = 0$ with infinite convergence radius and $g(0) = 0$ we write

$$g(t) = \sum_{j=0}^{\infty} \frac{g^{(j+1)}(0)}{(j+1)!} t^{j+1}.$$

Straightforward computations show

$$g^{(j+1)}(0) = (-1)^j \sum_{\ell=0}^{j} \left(\sum_{k=\ell}^{j} (-1)^k \binom{j+1}{k+1} \binom{k}{\ell} \right) (-C)^\ell Q A^{j-\ell}, \quad j \geq 0. \tag{20}$$

Observe that the proof of (20) is straightforward when A, Q and C commute. For $x \in \mathbb{R}$ fixed, $x \neq -1$, consider the function

$$\tilde{g}_x(t) := \int_0^t e^{-sx} e^{-s} ds\, e^t, \quad t \in \mathbb{R}.$$

Therefore

$$\begin{aligned} \tilde{g}_x(t) &= \frac{1}{1+x}(e^t - e^{-tx}) = \sum_{j=0}^{\infty} \frac{1-(-x)^{j+1}}{1+x} \frac{t^{j+1}}{(j+1)!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{\ell=0}^{j} (-1)^\ell x^\ell \right) \frac{t^{j+1}}{(j+1)!} \\ &= \sum_{j=0}^{\infty} \left((-1)^j \sum_{\ell=0}^{j} \left(\sum_{k=\ell}^{j} (-1)^k \binom{j+1}{k+1} \binom{k}{\ell} \right) x^\ell \right) \frac{t^{j+1}}{(j+1)!}. \end{aligned}$$

Hence for all $x \neq -1$ we have

$$\sum_{\ell=0}^{j} (-1)^j \left(\sum_{k=\ell}^{j} (-1)^k \binom{j+1}{k+1} \binom{k}{\ell} \right) x^\ell = \sum_{\ell=0}^{j} (-1)^\ell x^\ell, \quad j \geq 0,$$

it follows that

$$(-1)^j \sum_{k=\ell}^{j} (-1)^k \binom{j+1}{k+1} \binom{k}{\ell} = (-1)^\ell$$

for all $\ell = 0, \ldots, j$ and $j \geq 0$. The result follows from (20). □

From Proposition 3.1, we get a characterization of the partial null controllability property of system (16). In this aim, for a linear map $f : V \to W$ we put $\mathcal{R}(f) := \{f(v) : v \in V\}$ to denote the image of f.

Theorem 3.2. *Let $T > 0$ be given, $A, C, Q \in \mathbb{R}^{n\times n}$, $Q \neq 0$, and $B \in \mathbb{R}^n \backslash \{0\}$. The following statements are equivalent.*

(1) The cascade system (16) is partially null controllable at time $T > 0$.

(2) There exists $c > 0$ such that the observability inequality

$$\|g(T)^* \mathbf{w}_0\|^2 \leq c \int_0^T |\langle g(t)B, \mathbf{w}_0\rangle|^2 dt \tag{21}$$

holds for all $\mathbf{w}_0 \in \mathbb{R}^n$.

(3) $\mathcal{R}(h(T)) \subset span\{D_j B\}_{j\geq 0}$, where

$$h(T) = \int_0^T e^{sC} Q e^{-sA} ds. \tag{22}$$

Proof. *1)* $\Leftrightarrow$ *2)* Consider the operator $\mathcal{C} : L^2(0,T) \to \mathbb{R}^n$ defined as

$$\mathcal{C}u := -\int_0^T g(T-s)Bu(s)ds.$$

A straighforward computation shows that $\mathcal{C}^* : \mathbb{R}^n \to L^2(0,T)$ is given as

$$\mathcal{C}^*\mathbf{w}_0 = -\langle g(T-\cdot)B, \mathbf{w}_0\rangle.$$

The system (16) is partially null controllable at time $T > 0$ iff $\mathcal{R}(g(T)) \subset \mathcal{R}(\mathcal{C})$ iff there exists a constant $c > 0$ such that

$$\|g(T)^*\mathbf{w}_0\| \le c\|\mathcal{C}^*\mathbf{w}_0\|$$

for all $\mathbf{w}_0 \in \mathbb{R}^n$, see [3, Lemma 2.48, page 58].

2) $\Rightarrow$ *3)* Let $\mathbf{w}_0 \in \{span\{D_jB\}_{j\ge 0}\}^\perp$. Then $\langle D_jB, \mathbf{w}_0\rangle = 0$ for all $j \ge 0$, from (18) we get that $\langle g(t)B, \mathbf{w}_0\rangle = 0$ for all $t > 0$ and the hypothesis implies that $g(T)^*\mathbf{w}_0 = 0$, i.e. $\mathbf{w}_0 \in \mathcal{R}(g(T))^\perp$. Thus, $\{span\{D_jB\}_{j\ge 0}\}^\perp \subset \mathcal{R}(g(T))^\perp$.

3) $\Rightarrow$ *1)* If $span\{D_jB\}_{j\ge 0} = \{0\}$ then $h(T) = 0$, thus $g(T) = 0$. Hence we choose the trivial control $u \equiv 0$ on $[0,T]$ for all $\mathbf{x}_0 \in \mathbb{R}^n$, and the result follows from (17).

Now we assume that $span\{D_jB\}_{j\ge 0} \ne \{0\}$. We introduce the following positive semi-definite matrix

$$\mathcal{M} := \int_0^T g(t)BB^*g(t)^*dt.$$

We have $\mathbf{w}_0 \in \ker\mathcal{M}$ if and only if

$$0 = \langle \mathcal{M}\mathbf{w}_0, \mathbf{w}_0\rangle = \int_0^T |B^*g(t)^*\mathbf{w}_0|^2 dt$$

iff $\langle \mathbf{w}_0, g(t)B\rangle = 0$ for all $t \in [0,T]$ iff $\mathbf{w}_0 \in \{span\{D_jB\}_{j\ge 0}\}^\perp$ (by (18)). It follows that $(\ker\mathcal{M})^\perp = span\{D_jB\}_{j\ge 0} \ne 0$.

Since $\mathcal{M}$ is symmetric it can be diagonalized and $\mathcal{M} : (\ker\mathcal{M})^\perp \to (\ker\mathcal{M})^\perp$ is bijective. The hypothesis implies that $\mathcal{R}(g(T)) \subset span\{D_jB\}_{j\ge 0}$. Thus, for $\mathbf{x}_0 \in \mathbb{R}^n$ we set $u(t) := -B^*g(T-t)^*\mathcal{M}^{-1}g(T)\mathbf{x}_0$, $t \in [0,T]$, and the corresponding solution $\mathbf{p}(t)$ of the system (16) satisfies

$$e^{TC}\mathbf{p}(0) = -g(T)\mathbf{x}_0 + \mathcal{M}\mathcal{M}^{-1}g(T)\mathbf{x}_0 = 0.$$

□

The following remark shows that the control u obtained in the proof of Theorem 3.2 is the one of minimal norm in $L^2([0,T];\mathbb{R})$.

Remark 3.1. 1.-Let $\widetilde{u} \in L^2([0,T];\mathbb{R})$ be another control in system (16) such that the corresponding solution $\widetilde{\mathbf{p}}(t)$ satisfies $\widetilde{\mathbf{p}}(0) = 0$ then

$$\int_0^T g(T-t)B\widetilde{u}(t)dt = \int_0^T g(T-t)Bu(t)dt.$$

We set $v(t) := \widetilde{u}(t) - u(t)$, thus

$$\|\widetilde{u}\|^2_{L^2(0,T)} = \|u\|^2_{L^2(0,T)} + \|v\|^2_{L^2(0,T)} + 2\int_0^T u(t)v(t)dt,$$

and

$$\int_0^T u(t)v(t)dt = -\int_0^T \langle \mathcal{M}^{-1} g(T)\mathbf{x}_0, g(T-t)B\rangle v(t)dt$$
$$= -\langle \mathcal{M}^{-1} g(T)\mathbf{x}_0, \int_0^T g(T-t)Bv(t)dt\rangle = 0$$

imply that $\|u\|_{L^2(0,T)} \leq \|\widetilde{u}\|_{L^2(0,T)}$.
2.- The control $u(t)$ is the unique solution of a (Hausdorff) moment problem on $[0,T]$: Since

$$\int_0^T t^n (T-t)^{j+1} dt = T^{n+j+2}\mathcal{B}(n+1, j+2) = T^{n+j+2}\frac{n!(j+1)!}{(n+j+2)!},$$

where $\mathcal{B}(\cdot,\cdot)$ stands for the Beta function, we get

$$\int_0^T t^n u(t)dt = -n!\sum_{j=0}^{\infty}\frac{T^{n+j+2}}{(n+j+2)!}B^* D_j^* \mathcal{M}^{-1} g(T)x_0 \text{ for all } n \geq 0.$$

3.- If $span\{D_j B\}_{j\geq 0} = \mathbb{R}^n$ then the system (16) is partially null controllable at any time $T > 0$.

From Proposition 2.1 and Theorem 3.2 we have the following result.

Corollary 3.3. *Let $T > 0$ and $\mathcal{J}_T$ be given by (12). Consider $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n \backslash \{0\}$. Then there exists a control $u \in L^2([0,T];\mathbb{R})$ that insensitizes $\mathcal{J}_T(\mathbf{x})$ with $\mathbf{x}$ the corresponding solution to*

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu, \quad t \in (0,T)$$
$$\mathbf{x}(0) = \mathbf{x}_0 + \tau\mathbf{x}_1,$$

if and only if

$$\mathcal{R}(\widetilde{h}(T)) \subset span\{\widetilde{D}_j B\}_{j\geq 0}, \tag{23}$$

where $Q = G + G^* \neq 0$,

$$\widetilde{h}(T) = \int_0^T e^{-sA^*} Q e^{-sA} ds, \quad and \quad \widetilde{D}_j := \sum_{\ell=0}^{j} (-A^*)^\ell Q A^{j-\ell}, \quad j \geq 0 \tag{24}$$

Notice that $\int_0^T e^{-sA^*} e^{-sA} ds$ is an invertible operator on $\mathbb{R}^n$ for all $T > 0$.

Remark 3.2. 1.- If Q commutes with A^*, then $\mathcal{R}(\widetilde{h}(T)) = \mathcal{R}(Q)$ for all $T > 0$. In this case,

$$span\left\{\sum_{\ell=0}^{j}(-A^*)^\ell A^{j-\ell}B\right\}_{j\geq 0} = \mathbb{R}^n \tag{25}$$

is a sufficient condition for which (23) holds for all $T > 0$. For instance, if we also assume that A is (anti)symmetric and (A^2, B) is a controllable pair then (25) holds. Conversely, if Q is invertible, A is (anti)symmetric and (23) holds, then Cayley-Hamilton Theorem implies that (A^2, B) is a controllable pair.

2.- If Q commutes with A, then

$$span\left\{\sum_{\ell=0}^{j}(-A^*)^\ell A^{j-\ell}QB\right\}_{j\geq 0} = \mathbb{R}^n \tag{26}$$

is a sufficient condition for which (23) holds for all $T > 0$. For instance, if we also assume that A is (anti)symmetric and (A^2, QB) is a controllable pair then (26) holds. Conversely, if Q is invertible, A is (anti)symmetric and (23) holds, then the Cayley-Hamilton theorem implies that (A^2, QB) is a controllable pair.

4. Partial null controllability: the linear non-autonomous case

In this section we analyze the partial null controllability of the non-autonomous version of system (16).

Let $T > 0$ be given, for $A(t), C(t), Q(t) \in C^0([0,T]; \mathbb{R}^{n\times n})$, $B(t) \in C^0(([0,T]; \mathbb{R}^n)$ and $u \in L^2([0,T];\mathbb{R})$, we consider the following non-autonomous system

$$\begin{aligned} \dot{\mathbf{x}} &= A(t)\,\mathbf{x} + B(t)\,u, \quad t \in (0,T) \\ \dot{\mathbf{p}} &= C(t)\,\mathbf{p} + Q(t)\mathbf{x}, \quad t \in (0,T) \\ \mathbf{x}(0) &= \mathbf{x}_0, \quad \mathbf{p}(T) = 0. \end{aligned} \tag{27}$$

Let $R_A(\cdot,\cdot), R_C(\cdot,\cdot) : [0,T]^2 \to \mathbb{R}^{n\times n}$ be the resolvents corresponding to the time-varying linear systems $\dot{\mathbf{x}} = A(t)\,\mathbf{x}$ and $\dot{\mathbf{x}} = C(t)\,\mathbf{x}$ respectively (see [3, Proposition 1.5, page 5]), therefore

$$\begin{aligned} \mathbf{x}(t) &= R_A(t,0)\mathbf{x}_0 + \int_0^t R_A(t,\tau)B(\tau)u(\tau)d\tau, \quad t \in [0,T], \\ \mathbf{p}(t) &= -\int_t^T R_C(t,\tau)Q(\tau)\mathbf{x}(\tau)d\tau, \quad t \in [0,T], \end{aligned}$$

thus

$$\begin{aligned} -\mathbf{p}(0) &= \int_0^T R_C(0,\tau)Q(\tau)R_A(\tau,0)d\tau\,\mathbf{x}_0 \\ &\qquad + \int_0^T R_C(0,\tau)Q(\tau)\int_0^\tau R_A(\tau,s)B(s)u(s)ds d\tau \\ &= \int_0^T R_C(0,\tau)Q(\tau)R_A(\tau,0)d\tau\,\mathbf{x}_0 \\ &\qquad + \int_0^T \int_s^T R_C(0,\tau)Q(\tau)R_A(\tau,s)d\tau B(s)u(s)ds. \end{aligned} \tag{28}$$

We introduce the operator $\mathcal{C} : L^2(0,T) \to \mathbb{R}^n$ as follows

$$\mathcal{C}u := -\int_0^T \int_s^T R_C(0,\tau)Q(\tau)R_A(\tau,s)d\tau B(s)u(s)ds.$$

An easy computation shows that $\mathcal{C}^* : \mathbb{R}^n \to L^2(0,T)$ is given by

$$(\mathcal{C}^*\mathbf{w}_0)(s) = -\left\langle \int_s^T R_C(0,\tau)Q(\tau)R_A(\tau,s)d\tau\, B(s), \mathbf{w}_0 \right\rangle.$$

We also consider the following operators

$$\mathcal{G}_T := \int_0^T R_C(0,\tau)Q(\tau)R_A(\tau,0)d\tau,$$

$$\mathcal{G}(s) := \int_s^T R_C(0,\tau)Q(\tau)R_A(\tau,s)d\tau,$$

$$\mathcal{M} := \int_0^T \mathcal{G}(s)B(s)B(s)^*\mathcal{G}(s)^* ds \geq 0.$$

Notice that $\mathcal{M}$ is a positive semi-definite matrix.

Similar to Theorem 3.2, we have the following result.

Theorem 4.1. *Let $T > 0$ be given and $A(t), C(t), Q(t) \in C^0([0,T];\mathbb{R}^{n\times n})$, $B(t) \in C^0([0,T];\mathbb{R}^n)$ with $Q \not\equiv 0$, $B \not\equiv 0$. The following statements are equivalent.*

(1) The cascade system (27) is partially null controllable at time $T > 0$.

(2) There exists $c > 0$ such that the observabillity inequality

$$\|\mathcal{G}_T^* \mathbf{w}_0\|^2 \leq c \int_0^T |(\mathcal{C}^*\mathbf{w}_0)(s)|^2 ds \tag{29}$$

holds for all $\mathbf{w}_0 \in \mathbb{R}^n$.

(3) $\mathcal{R}(\mathcal{G}_T) \subset (\ker \mathcal{M})^\perp$.

Proof. 1) ⇔ 2) We just notice that (28) is equivalent to

$$-\mathbf{p}(0) = \mathcal{G}_T \mathbf{x}_0 - \mathcal{C}u, \tag{30}$$

and we proceed as in the proof of *i) ⇔ ii)* in Theorem 3.2.

2) ⇒ 3) Let $\mathbf{w}_0 \in \ker \mathcal{M}$. Since $\langle \mathcal{M}\mathbf{w}_0, \mathbf{w}_0\rangle = 0$ it follows that $(\mathcal{C}^*\mathbf{w}_0)(s) = 0$ for all $s \in [0,T]$, thus $\mathcal{G}_T^*\mathbf{w}_0 = 0$ and the result follows.

3) ⇒ 1) If $(\ker \mathcal{M})^\perp = \{0\}$ then $\mathcal{G}_T = 0$. Hence we choose the trivial control $u \equiv 0$ on $[0,T]$ for all $\mathbf{x}_0 \in \mathbb{R}^n$, and the result follows from (30).

Now we assume $(\ker \mathcal{M})^\perp \neq \{0\}$. We have that $\mathcal{M}$ is diagonalisable and $\mathcal{M} : (\ker \mathcal{M})^\perp \to (\ker \mathcal{M})^\perp$ is bijective. By hypothesis $\mathcal{R}(\mathcal{G}_T) \subset (\ker \mathcal{M})^\perp$. Given $\mathbf{x}_0 \in \mathbb{R}^n$ we set $u(t) := -B(t)^*\mathcal{G}(t)^*\mathcal{M}^{-1}\mathcal{G}_T\mathbf{x}_0$, so the corresponding solution $\mathbf{p}(t)$ of the system (27) satisfies

$$-\mathbf{p}(0) = \mathcal{G}_T\mathbf{x}_0 - \mathcal{M}\mathcal{M}^{-1}\mathcal{G}_T\mathbf{x}_0 = 0.$$

□

In order to get an algebraic test to verify the issue *iii)* in the last result, from now on we assume $A(t), C(t), Q(t) \in C^\infty([0,T];\mathbb{R}^{n\times n})$, $B(t) \in C^\infty([0,T];\mathbb{R}^n)$. For $i, j \geq 0$ we introduce the following sequences of vector-valued functions defined on $[0,T]$,

$$\begin{aligned} B_0(t) &:= B(t), & H_0^j(t) &:= Q(t)B_j(t), \\ B_{i+1}(t) &:= \dot{B}_i(t) - A(t)B_i(t), & H_{i+1}^j(t) &:= \dot{H}_i^j(t) - C(t)H_i^j(t), \end{aligned} \tag{31}$$

we also set for $s \in [0,T], m \geq 1$,

$$Y_{s,0} := \mathcal{G}(s)B(s), \quad Y_{s,m} := -R_C(0,s)\sum_{j=0}^{m-1} H_{m-1-j}^j(s) + \mathcal{G}(s)B_m(s), \tag{32}$$

and $E_s := span\{Y_{s,m} : m \geq 0\}$, $s \in [0,T]$.

Lemma 4.2. *We have $E_s \subset (\ker \mathcal{M})^\perp$ for all $s \in [0,T]$. If $A(t), C(t), Q(t) \in C^\omega([0,T];\mathbb{R}^{n\times n})$ and $B(t) \in C^\omega([0,T];\mathbb{R}^n)$ then $(\ker \mathcal{M})^\perp = \bigcap_{s\in[0,T]} E_s$.*

Proof. We claim that

$$\frac{d^m}{ds^m}\left(\mathcal{G}B(s)\right) = Y_{s,m}, \quad \text{for all } m \geq 1, s \in [0,T]. \tag{33}$$

First, we use that $\frac{\partial R_A}{\partial s}(\tau, s) = -R_A(\tau, s)A(s)$ to get

$$\begin{aligned}
\frac{d}{ds}\left(\mathcal{G}B_j(s)\right) &= -R_C(0,s)Q(s)R_A(s,s)B_j(s) \\
&\quad + \int_s^T R_C(0,\tau)Q(\tau)R_A(\tau,s)(\dot{B}_j(s) - A(s)B_j(s))d\tau \\
&= -R_C(0,s)Q(s)B_j(s) + \int_s^T R_C(0,\tau)Q(\tau)R_A(\tau,s)B_{j+1}(s)d\tau \\
&= -R_C(0,s)H_0^j(s) + \mathcal{G}(s)B_{j+1}(s), \quad \text{for all } j \geq 0.
\end{aligned}$$

The last equality (with $j=0$) implies (33) for $m=1$.

Now we assume that (33) holds for some $m \geq 1$. Thus,

$$\begin{aligned}
\frac{d^{m+1}}{ds^{m+1}}\left(\mathcal{G}B(s)\right) &= -R_C(0,s)\sum_{j=0}^{m-1}\left[\dot{H}_{m-1-j}^j(s) - C(s)\,H_{m-1-j}^j(s)\right] + \frac{d}{ds}\left(\mathcal{G}B_m(s)\right) \\
&= -R_C(0,s)\sum_{j=0}^{m-1} H_{m-j}^j(s) - R_C(0,s)H_0^m(s) + \mathcal{G}(s)B_{m+1}(s) = Y_{s,m+1}.
\end{aligned}$$

If $\mathbf{w}_0 \in \ker \mathcal{M}$ then $\mathbf{w}_0^*\mathcal{G}(s)B(s) = -(\mathcal{C}^*\mathbf{w}_0)(s) = 0$ for all $s \in [0,T]$, therefore (33) implies that

$$\langle \mathbf{w}_0, Y_{s,m}\rangle = 0 \quad \text{for all } m \geq 1, s \in [0,T]$$

and the first part of the result follows.

To prove the second part we use that for each $s \in [0,T]$ there exists $\delta_s > 0$ such that

$$\mathcal{G}(t)B(t) = \sum_{m=0}^{\infty}\frac{d^m(\mathcal{G}B)}{ds^m}(s)\frac{(t-s)^m}{m!} = \sum_{m=0}^{\infty} Y_{s,m}\frac{(t-s)^m}{m!}$$

for $t \in (s-\delta_s, s+\delta_s)$.
If $\mathbf{w}_0 \in E_s^\perp$ for all $s \in [0,T]$, then $\mathbf{w}_0^*\mathcal{G}(t)B(t) = -(\mathcal{C}^*\mathbf{w}_0)(t) = 0$ for all $t \in (s-\delta_s, s+\delta_s), s \in [0,T]$. It follows that $\langle \mathcal{M}\mathbf{w}_0, \mathbf{w}_0\rangle = 0$, so $\mathbf{w}_0 \in \ker \mathcal{M}$ and the result has been proved. □

As a consequence we have the following result.

Corollary 4.3. *Let $T > 0$ be given, if there exists $\bar{t} \in [0,T]$ such that $\mathcal{R}(\mathcal{G}_T) \subset E_{\bar{t}}$, then (27) is null controllable at time T. Conversely, if A, B, C, Q are analytic functions on $[0,T]$ and (27) is partially null controllable at time T, then $\mathcal{R}(\mathcal{G}_T) \subset E_s$ for all $s \in [0,T]$.*

Example 4.1. Suppose that $A(t), B(t), C(t), Q(t)$ are constant functions on $[0,T]$. In this case, (31) yields

$$B_i = (-1)^i A^i B, \quad H_i^j = (-1)^{i+j}C^iQA^jB \quad \text{for all } i,j \geq 0.$$

Since $R_A(t,s) = e^{(t-s)A}, R_C(t,s) = e^{(t-s)C}$, it follows by (22) that

$$\mathcal{G}_T = \int_0^T R_C(0, T-s) Q R_A(T-s, 0) ds = e^{-TC} h(T) e^{TA},$$

so $\mathcal{R}(\mathcal{G}_T) = \mathcal{R}(h(T))$, and

$$E_T = span \left\{ e^{-TC} \sum_{j=0}^{m-1} H_{m-1-j}^j : m \geq 1 \right\} = span \{D_n B : n \geq 0\}.$$

where D_n is given in (19). The last corollary implies that system (16) is partially null controllable at time T iff $\mathcal{R}(h(T)) \subset span \{D_n B : n \geq 1\}$. This is another proof of Theorem 3.2.

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(Marcos López-García) Unidad Cuernavaca, Instituto de Matemáticas, Universidad Nacional Autónoma de México, México. ORCID ID 0000-0001-5966-3878
E-mail address: marcos.lopez@im.unam.mx

(Alberto Peña-García) Instituto de Matematicas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., C. P. 04510 Ciudad de México, México. ORCID ID
E-mail address: jof10z@hotmail.com

(Luz de Teresa) Instituto de Matematicas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., C. P. 04510 Ciudad de México, México. ORCID ID 0000-0002-4051-1214
E-mail address: ldeteresa@im.unam.mx