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Continuous frames in n-Hilbert spaces and their tensor products

Prasenjit Ghosh and T. K. Samanta

Abstract. We introduce the notion of continuous frame in n-Hilbert space which is a generalization of discrete frame in n-Hilbert space. The tensor product of Hilbert spaces is a very important topic in mathematics. Here we also introduce the concept of continuous frame for the tensor products of n-Hilbert spaces. Further, we study dual continuous frame and continuous Bessel multiplier in n-Hilbert spaces and their tensor products.

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1. Introduction

The notion of frame in Hilbert space was first introduced by Duffin and Schaeffer [4] in connection with some fundamental problem in non-harmonic analysis. Thereafter, it was further developed and popularized by Daubechies et al [3] in 1986. A discrete frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame element. Continuous frames extended the concept of discrete frames when the indices are related to some measurable space. Continuous frame in Hilbert space was studied by A. Rahimi et al [12]. M. H. faroughi and E. Osgooei [6] also studied c-frames and c-Bessel mappings. Continuous frame and discrete frame have been used in image processing, coding theory, wavelet analysis, signal denoising, feature extraction, robust signal processing etc.

In 1970, Diminnie et al [2] introduced the concept of 2-inner product space. A generalization of 2-inner product space for $n \geq 2$ was developed by A. Misiak [11] in 1989. There are several ways to introduced the tensor product of Hilbert spaces. The basic concepts of tensor product of Hilbert spaces were presented by S. Rabinson in [13] and Folland in [5].

In this paper, we give the notions of continuous frames in n-Hilbert spaces and their tensor products. A characterization of continuous frame in n-Hilbert space with the help of its pre-frame operator is discussed. We will see that the image of a continuous frame under a bounded invertible operator in n-Hilbert space is also a continuous frame in n-Hilbert space. Continuous Bessel multipliers and dual continuous frames in n-Hilbert spaces and their tensor product are presented.

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2. Preliminaries

Theorem 2.1. [1] *Let H_1, H_2 be two Hilbert spaces and $U : H_1 \to H_2$ be a bounded linear operator with closed range $\mathcal{R}_U$. Then there exists a bounded linear operator $U^{\dagger} : H_2 \to H_1$ such that $U U^{\dagger} x = x \ \forall\, x \in \mathcal{R}_U$.*

The operator $U^{\dagger}$ is called the pseudo-inverse of U.

Definition 2.1. [12] Let H be a complex Hilbert space and (Ω, μ) be a measure space with positive measure μ. A mapping $F : \Omega \to H$ is called a continuous frame with respect to (Ω, μ) if

- (i) F is weakly-measurable, i. e., for all $f \in H$, $w \to \langle f, F(w) \rangle$ is a measurable function on Ω.
- (ii) there exist constants $0 < A \leq B < \infty$ such that

$$A \, \| f \|^2 \leq \int_{\Omega} | \langle f, F(w) \rangle |^2 \, d\mu(w) \leq B \, \| f \|^2$$

for all $f \in H$. The constants A and B are called continuous frame bounds. If $A = B$, then it is called a tight continuous frame. If the mapping F satisfies only the right inequality, then it is called continuous Bessel mapping with Bessel bound B.

Definition 2.2. [6] Let $L^2(\Omega, \mu)$ be the class of all measurable functions $f : \Omega \to H$ such that $\| f \|_2^2 = \int_{\Omega} \| f(w) \|^2 \, d\mu(w) < \infty$. It can be proved that $L^2(\Omega, \mu)$ is a Hilbert space with respect to the inner product defined by

$$\langle f, g \rangle_{L^2} = \int_{\Omega} \langle f(w), g(w) \rangle \, d\mu(w).$$

Definition 2.3. [6] Let $F : \Omega \to H$ be a Bessel mapping. Then the operator $T_C : L^2(\Omega, \mu) \to H$ is defined by

$$\langle T_C(\varphi), h \rangle = \int_{\Omega} \varphi(w) \, \langle F(w), h \rangle \, d\mu(w)$$

where $\varphi \in L^2(\Omega, \mu)$ and $h \in H$ is well-defined, linear, bounded and its adjoint operator is given by

$$T_C^* : H \to L^2(\Omega, \mu) \ , \ T_C^* f(w) = \langle f, F(w) \rangle \ , \ w \in \Omega.$$

The operator T_C is called a pre-frame operator or synthesis operator and its adjoint operator is called analysis operator of F. The operator $S_C : H \to H$ defined by

$$\langle S_C(f), h \rangle = \int_{\Omega} \langle f, F(w) \rangle \langle F(w), h \rangle \, d\mu(w)$$

is called the frame operator of F.

Definition 2.4. [14] The tensor product of Hilbert spaces H and K is denoted by $H \otimes K$ and it is defined to be an inner product space associated with the inner product

$$\langle f \otimes g , f' \otimes g' \rangle = \langle f, f' \rangle_1 \, \langle g, g' \rangle_2 , \tag{1}$$

for all $f, f' \in H$ and $g, g' \in K$. The norm on $H \otimes K$ is given by

$$\| f \otimes g \| = \| f \|_1 \, \| g \|_2 \ \ \forall\, f \in H \text{ and } g \in K. \tag{2}$$

The space $H \otimes K$ is complete with respect to the above inner product. Therefore the space $H \otimes K$ is a Hilbert space.

For $Q \in \mathcal{B}(H)$ and $T \in \mathcal{B}(K)$, the tensor product of operators Q and T is denoted by $Q \otimes T$ and defined as

$$(Q \otimes T) A = Q A T^{*} \ \forall \ A \in H \otimes K.$$

It can be easily verified that $Q \otimes T \in \mathcal{B}(H \otimes K)$ [5].

Theorem 2.2. [5] *Suppose $Q, Q' \in \mathcal{B}(H)$ and $T, T' \in \mathcal{B}(K)$, then*
(i) $Q \otimes T \in \mathcal{B}(H \otimes K)$ *and* $\|Q \otimes T\| = \|Q\| \|T\|$.
(ii) $(Q \otimes T)(f \otimes g) = Q(f) \otimes T(g)$ *for all* $f \in H, g \in K$.
(iii) $(Q \otimes T)(Q' \otimes T') = (QQ') \otimes (TT')$.
(iv) $Q \otimes T$ *is invertible if and only if* Q *and* T *are invertible, in which case* $(Q \otimes T)^{-1} = \left(Q^{-1} \otimes T^{-1}\right)$.
(v) $(Q \otimes T)^{*} = (Q^{*} \otimes T^{*})$.

Definition 2.5. [7] A real valued function $\|\cdot, \cdots, \cdot\| : H^{n} \to \mathbb{R}$ satisfying the following properties:
(i) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,
(ii) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutations of $x_1, \cdots, x_n$,
(iii) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \|x_1, x_2, \cdots, x_n\|$, $\alpha \in \mathbb{K}$,
(iv) $\|x + y, x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|y, x_2, \cdots, x_n\|$,
for all $x_1, x_2, \cdots, x_n, x, y \in H$, is called n-norm on H. A linear space H, together with a n-norm $\|\cdot, \cdots, \cdot\|$, is called a linear n-normed space.

Definition 2.6. [11] Let $n \in \mathbb{N}$ and H be a linear space of dimension greater than or equal to n over the field $\mathbb{K}$. An n-inner product on H is a map

$$(x, y, x_2, \cdots, x_n) \longmapsto \langle x, y \,|\, x_2, \cdots, x_n \rangle, \ x, y, x_2, \cdots, x_n \in H$$

from H^{n+1} to the set $\mathbb{K}$ such that for every $x, y, x_1, x_2, \cdots, x_n \in H$,
(i) $\langle x_1, x_1 \,|\, x_2, \cdots, x_n \rangle \geq 0$ and $\langle x_1, x_1 \,|\, x_2, \cdots, x_n \rangle = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent,
(ii) $\langle x, y \,|\, x_2, \cdots, x_n \rangle = \langle x, y \,|\, x_{i_2}, \cdots, x_{i_n} \rangle$ for every permutations $(i_2, \cdots, i_n)$ of $(2, \cdots, n)$,
(iii) $\langle x, y \,|\, x_2, \cdots, x_n \rangle = \overline{\langle y, x \,|\, x_2, \cdots, x_n \rangle}$,
(iv) $\langle \alpha x, y \,|\, x_2, \cdots, x_n \rangle = \alpha \langle x, y \,|\, x_2, \cdots, x_n \rangle$, for $\alpha \in \mathbb{K}$,
(v) $\langle x + y, z \,|\, x_2, \cdots, x_n \rangle = \langle x, z \,|\, x_2, \cdots, x_n \rangle + \langle y, z \,|\, x_2, \cdots, x_n \rangle$.
A linear space H together with an n-inner product $\langle \cdot, \cdot \,|\, \cdot, \cdots, \cdot \rangle$ is called an n-inner product space.

Definition 2.7. [7] A sequence $\{x_k\}$ in linear n-normed space H is said to be convergent to $x \in H$ if

$$\lim_{k \to \infty} \|x_k - x, e_2, \cdots, e_n\| = 0$$

for every $e_2, \cdots, e_n \in H$ and it is called a Cauchy sequence if

$$\lim_{l, k \to \infty} \|x_l - x_k, e_2, \cdots, e_n\| = 0$$

for every $e_2, \cdots, e_n \in H$. The space H is said to be complete if every Cauchy sequence in this space is convergent in H. An n-inner product space is called n-Hilbert space if it is complete with respect to its induce norm.

Definition 2.8. [8] Let H be a n-Hilbert space and $a_2, \cdots, a_n$ are fixed elements in H. A sequence $\{f_i\}_{i=1}^{\infty}$ in H is said to be a frame associated to $(a_2, \cdots, a_n)$ if there exists constant $0 < A \leq B < \infty$ such that

$$A \| f, a_2, \cdots, a_n \|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n \rangle|^2 \leq B \| f, a_2, \cdots, a_n \|^2 \quad (3)$$

for all $f \in H$. The constants A, B are called frame bounds. If $\{f_i\}_{i=1}^{\infty}$ satisfies only the right inequality of (3), is called a Bessel sequence associated to $(a_2, \cdots, a_n)$ in H with bound B.

Let $a_2, a_3, \cdots, a_n$ be the fixed elements in H and L_F denote the linear subspace of H spanned by the non-empty finite set $F = \{a_2, a_3, \cdots, a_n\}$. Then the quotient space H / L_F is a normed linear space with respect to the norm, $\| x + L_F \|_F = \| x, a_2, \cdots, a_n \|$ for every $x \in H$. Let M_F be the algebraic complement of L_F, then $H = L_F \oplus M_F$. Define

$$\langle x, y \rangle_F = \langle x, y \,|\, a_2, \cdots, a_n \rangle \text{ on } H.$$

Then $\langle \cdot, \cdot \rangle_F$ is a semi-inner product on H and this semi-inner product induces an inner product on the quotient space H / L_F which is given by

$$\langle x + L_F, y + L_F \rangle_F = \langle x, y \rangle_F = \langle x, y \,|\, a_2, \cdots, a_n \rangle \quad \forall\; x, y \in H.$$

By identifying H / L_F with M_F in an obvious way, we obtain an inner product on M_F. Then M_F is a normed space with respect to the norm $\| \cdot \|_F$ defined by $\| x \|_F = \sqrt{\langle x, x \rangle_F}$ $\forall x \in M_F$. Let H_F be the completion of the inner product space M_F [8].

Theorem 2.3. [8] *Let H be a n-Hilbert space. Then $\{f_i\}_{i=1}^{\infty} \subseteq H$ is a frame associated to $(a_2, \cdots, a_n)$ with bounds A and B if and only if it is a frame for the Hilbert space H_F with bounds A and B.*

For more details on frames in n-Hilbert spaces and their tensor products one can go through the papers [8, 9, 10].

3. Continuous frame in n-Hilbert space

In this section, first we give the definition of a continuous frame in n-Hilbert space and then discuss some of its properties.

Definition 3.1. Let H_1 be a complex n-Hilbert space and $a_2, \cdots, a_n \in H_1$ and (Ω, μ) be a measure space with positive measure μ. A mapping $\mathbb{F} : \Omega \to H_1$ is called a continuous frame or c-frame associated to $(a_2, \cdots, a_n)$ with respect to (Ω, μ) if

- (i) $\mathbb{F}$ is weakly-measurable, i.e., for all $f \in H_1$, the mapping given by $w \to \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle$ is a measurable function on Ω.
- (ii) there exist constants $0 < A \leq B < \infty$ such that

$$A \, \| \, f, \, a_2, \, \cdots, \, a_n \, \|^2 \leq \int_{\Omega} |\, \langle \, f, \, \mathbb{F}\,(\,w\,)\,|\, a_2, \, \cdots, \, a_n \, \rangle \,|^2 \, d\mu(\,w\,)$$
$$\leq B \, \| \, f, \, a_2, \, \cdots, \, a_n \, \|^2 \tag{4}$$

for all $f \in H_1$. The constants A and B are called continuous frame bounds. If $A = B$, then it is called a tight continuous frame associated to $(\,a_2, \, \cdots, \, a_n\,)$. If the mapping $\mathbb{F}$ satisfies only the right inequality of (4), then it is called continuous Bessel mapping associated to $(\,a_2, \, \cdots, \, a_n\,)$ with Bessel bound B.

If μ is a counting measure and $\mu = \mathbb{N}$, $\mathbb{F}$ is called a discrete frame associated to $(\,a_2, \, \cdots, \, a_n\,)$ for H_1.

Remark 3.1. Let $(\,\Omega, \, \mu\,)$ be a measure space with μ is σ-finite. Then the mapping $\mathbb{F} : \Omega \to H_1$ is a continuous frame associated to $(\,a_2, \, \cdots, \, a_n\,)$ with bounds A and B if and only if it is a continuous frame for the Hilbert space H_F with bounds A and B.

Remark 3.2. Define the representation space $L^2_F\,(\,\Omega, \, \mu\,)$

$$= \left\{ \varphi : \Omega \to H_F \,|\, \varphi \text{ is measurable and } \int_{\Omega} \| \, \varphi\,(\,w\,), \, a_2, \, \cdots, \, a_n \, \|^2 \, d\mu(\,w\,) < \infty \right\}.$$

It can be easily proved that $L^2_F\,(\,\Omega, \, \mu\,)$ is a Hilbert space with respect to the inner product defined by

$$\langle \, \varphi, \, \psi \, \rangle_{L^2_F} = \int_{\Omega} \langle \, \varphi\,(\,w\,), \, \psi\,(\,w\,)\,|\, a_2, \, \cdots, \, a_n \, \rangle \, d\mu(\,w\,) \ \text{ for } \ \varphi, \, \psi \in L^2_F\,(\,\Omega, \, \mu\,).$$

Theorem 3.1. *Let $(\,\Omega, \, \mu\,)$ be a measure space and $\mathbb{F} : \Omega \to H_1$ be a continuous Bessel mapping associated to $(\,a_2, \, \cdots, \, a_n\,)$ with bound B. Then the operator $T_C : L^2_F\,(\,\Omega, \, \mu\,) \to H_F$ defined by*

$$\langle \, T_C\,(\,\varphi\,), \, f \,|\, a_2, \, \cdots, \, a_n \, \rangle = \int_{\Omega} \varphi\,(\,w\,) \, \langle \, \mathbb{F}\,(\,w\,), \, f \,|\, a_2, \, \cdots, \, a_n \, \rangle \, d\mu(\,w\,)$$

where $\varphi \in L^2_F\,(\,\Omega, \, \mu\,)$ and $f \in H_F$, is well-defined, bounded and linear. The adjoint operator $T^*_C : H_F \to L^2_F\,(\,\Omega, \, \mu\,)$ given by

$$(\,T^*_C\,)\,(\,w\,) = \langle \, f, \, \mathbb{F}\,(\,w\,)\,|\, a_2, \, \cdots, \, a_n \, \rangle, \ w \in \Omega$$

is also bounded and $\| \, T_C \, \| = \| \, T^*_C \, \| \leq \sqrt{B}$.

Proof. It is easy to verify that T_C is well-defined and linear. Since $\mathbb{F}$ is a continuous Bessel mapping associated to $(\,a_2, \, \cdots, \, a_n\,)$ with bound B, for each $\varphi \in L^2_F\,(\,\Omega, \, \mu\,)$

and $f \in H_F$, we have

$$\begin{aligned}
\| T_C(\varphi), a_2, \cdots, a_n \| &= \sup_{\| f, a_2, \cdots, a_n \| = 1} | \langle T_C(\varphi), f \,|\, a_2, \cdots, a_n \rangle | \\
&\leq \sup_{\| f, a_2, \cdots, a_n \| = 1} \left(\int_\Omega | \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle |^2 \, d\mu(w) \right)^{1/2} \times \\
&\quad \left(\int_\Omega | \varphi(w) |^2 \, d\mu(w) \right)^{1/2} \\
&\leq \sqrt{B} \, \| \varphi \|_2,
\end{aligned}$$

Hence, T_C is bounded. On the other hand, for each $\varphi \in L_F^2(\Omega, \mu)$ and $f \in H_F$,

$$\begin{aligned}
\langle T_C^*(f), \varphi \,|\, a_2, \cdots, a_n \rangle &= \langle f, T_C(\varphi) \,|\, a_2, \cdots, a_n \rangle \\
&= \int_\Omega \overline{\varphi(w)} \, \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle \, d\mu(w) \\
&= \langle \langle f, \mathbb{F} \,|\, a_2, \cdots, a_n \rangle, \varphi \,|\, a_2, \cdots, a_n \rangle .
\end{aligned}$$

This verify that

$$(T_C^* f)(w) = \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle, \; w \in \Omega.$$

Also, for each $f \in H_F$, we have

$$\begin{aligned}
\| T_C^*(f), \varphi \,|\, a_2, \cdots, a_n \|^2 &= \langle T_C^*(f), T_C^*(f) \,|\, a_2, \cdots, a_n \rangle \\
&= \int_\Omega | \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle |^2 \, d\mu(w).
\end{aligned}$$

This implies that

$$\begin{aligned}
\| T_C \| &= \sup_{\| f, a_2, \cdots, a_n \| = 1} \left(\int_\Omega | \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle |^2 \, d\mu(w) \right)^{1/2} \\
&\leq \sqrt{B}.
\end{aligned}$$

□

Remark 3.3. The operator T_C defined in the Theorem 3.1, is called a pre-frame operator or synthesis operator and T_C^* is called an analysis operator of $\mathbb{F}$.

Definition 3.2. The operator $S_C : H_F \to H_F$ defined by

$$\begin{aligned}
S_C(f)(w) = T_C T_C^*(f)(w) &= T_C(\langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle) \\
&= \int_\Omega \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle \, \mathbb{F}(w) \, d\mu(w),
\end{aligned}$$

is called continuous frame operator of $\mathbb{F}$.

Remark 3.4. Let $\mathbb{F} : \Omega \to H_1$ be a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1 with respect to (Ω, μ). For each $f, g \in H_F$, we have

$$\langle S_C f, g \,|\, a_2, \cdots, a_n \rangle = \int_\Omega \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle \langle \mathbb{F}(w), g \,|\, a_2, \cdots, a_n \rangle \, d\mu(w).$$

Thus, for each $f \in H_F$, we get

$$\langle S_C f, f \,|\, a_2, \cdots, a_n \rangle = \int_\Omega |\langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle|^2 \, d\mu(w).$$

Therefore, for each $f \in H_F$, from (4), we get

$$A \langle f, f \,|\, a_2, \cdots, a_n \rangle \leq \langle S_C f, f \,|\, a_2, \cdots, a_n \rangle \leq B \langle f, f \,|\, a_2, \cdots, a_n \rangle.$$

Hence, $A I_F \leq S_C \leq B I_F$.

Theorem 3.2. *Let (Ω, μ) be a measure space, where μ is a σ-finite measure and let $\mathbb{F} : \Omega \to H_1$ be a measurable function. If the operator $T_C : L_F^2(\Omega, \mu) \to H_F$ defined by*

$$\langle T_C(\varphi), f \,|\, a_2, \cdots, a_n \rangle = \int_\Omega \varphi(w) \langle \mathbb{F}(w), f \,|\, a_2, \cdots, a_n \rangle \, d\mu(w)$$

where $\varphi \in L_F^2(\Omega, \mu)$ and $f \in H_F$, is a bounded operator, then $\mathbb{F}$ is a continuous Bessel mapping associated to $(a_2, \cdots, a_n)$.

Proof. By the Theorem 3.1, we have

$$T_C^*(f)(w) = \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle, \; w \in \Omega.$$

Now, for each $f \in H_F$, we have

$$\int_\Omega |\langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle|^2 \, d\mu(w) = \| T_C^*(f), \varphi \,|\, a_2, \cdots, a_n \|^2$$
$$\leq \| T_C \|^2 \, \| f, a_2, \cdots, a_n \|^2.$$

This completes the proof. □

In the next theorem, we give a characterization of a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1 with respect to its pre-frame operator under some sufficient conditions.

Theorem 3.3. *Let (Ω, μ) be a measure space, where μ is a σ-finite measure. Then the mapping $\mathbb{F} : \Omega \to H_1$ is a continuous frame associated to $(a_2, \cdots, a_n)$ with respect to (Ω, μ) if and only if the pre-frame operator T_C is bounded and onto operator.*

Proof. Let $\mathbb{F}$ be a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1. Then by Theorem 3.1, the operator T_C is bounded and it is easy to verify that T_C is one-one, onto.

Conversely, let T_C be bounded and onto operator. Then there exists a bounded operator $T_C^{\dagger} : H_F \to L_F^2(\Omega, \mu)$ such that $T_C T_C^{\dagger} f = f \quad \forall\, f \in H_F$. Since T_C is bounded, by Theorem 3.2, $\mathbb{F}$ is a continuous Bessel mapping associated to $(a_2, \cdots, a_n)$ and

$$\|T_C^*(f), \varphi \,|\, a_2, \cdots, a_n\|^2 = \int_\Omega |\langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n\rangle|^2 \, d\mu(w).$$

Let $f \in H_F$, then

$$\|f, a_2, \cdots, a_n\|^2 \leq \left\|T_C^{\dagger}\right\|^2 \|T_C^*(f), \varphi \,|\, a_2, \cdots, a_n\|^2.$$

Therefore, for each $f \in H_F$, we have

$$\left\|T_C^{\dagger}\right\|^{-2} \|T_C^*(f), \varphi \,|\, a_2, \cdots, a_n\|^2 \leq \int_\Omega |\langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n\rangle|^2 \, d\mu(w).$$

This completes the proof. □

Theorem 3.4. *Let $\mathbb{F} : \Omega \to H_1$ be a continuous frame associated to $(a_2, \cdots, a_n)$ with respect to (Ω, μ) for H_1 with frame operator S_C and let $U : H_F \to H_F$ be a bounded and invertible operator. Then $U\mathbb{F}$ is a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1 with frame operator $U S_C U^*$*

Proof. For each $f \in H_F$, we have

$$w \to \langle U^* f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n\rangle = \langle f, U\mathbb{F}(w) \,|\, a_2, \cdots, a_n\rangle$$

is measurable. Since U is invertible, for each $f \in H_F$, we have

$$\|f, a_2, \cdots, a_n\| \leq \left\|U^{-1}\right\| \|U^* f, a_2, \cdots, a_n\|.$$

Since $\mathbb{F}$ is a continuous frame associated to $(a_2, \cdots, a_n)$ in H_1, for each $f \in H_F$, we have

$$\begin{aligned} A\|U^* f, a_2, \cdots, a_n\|^2 &\leq \int_\Omega |\langle U^* f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n\rangle|^2 \, d\mu(w) \\ &\leq B\|U^* f, a_2, \cdots, a_n\|^2. \end{aligned}$$

Therefore, for each $f \in H_F$, we have

$$\begin{aligned} A\left\|U^{-1}\right\|^{-2} \|f, a_2, \cdots, a_n\|^2 &\leq \int_\Omega |\langle f, U\mathbb{F}(w) \,|\, a_2, \cdots, a_n\rangle|^2 \, d\mu(w) \\ &\leq B\|U\|^2 \|f, a_2, \cdots, a_n\|^2. \end{aligned}$$

Thus, $U\mathbb{F}$ is a continuous frame associated to $(a_2, \cdots, a_n)$ with bounds $A\left\|U^{-1}\right\|^{-2}$ and $B\|U\|^2$.

Furthermore, for each $f, g \in H_F$, we have

$$\int_{\Omega} \langle f, U\mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle \langle U\mathbb{F}(w), g \,|\, a_2, \cdots, a_n \rangle \, d\mu(w)$$
$$= \int_{\Omega} \langle U^* f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle \langle \mathbb{F}(w), U^* g \,|\, a_2, \cdots, a_n \rangle \, d\mu(w)$$
$$= \langle S_C U^* f, U^* g \,|\, a_2, \cdots, a_n \rangle = \langle U S_C U^* f, g \,|\, a_2, \cdots, a_n \rangle.$$

This shows that the corresponding continuous frame operator is $U S_C U^*$. □

Next, we end this section by discussing the continuous Bessel multiplier in H_1.

Definition 3.3. Let $\mathbb{F}$ and $\mathbb{G}$ be continuous Bessel families associated to $(a_2, \cdots, a_n)$ for H_1 with respect to (Ω, μ) having bounds B_1 and B_2 and $m : \Omega \to \mathbb{C}$ be a measurable function. The operator $M_{m, \mathbb{F}, \mathbb{G}} : H_F \to H_F$ defined by

$$\langle M_{m, \mathbb{F}, \mathbb{G}} f, g \,|\, a_2, \cdots, a_n \rangle$$
$$= \int_{\Omega} m(w) \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle \langle \mathbb{G}(w), g \,|\, a_2, \cdots, a_n \rangle \, d\mu(w)$$

is called continuous Bessel multiplier associated to $(a_2, \cdots, a_n)$ of $\mathbb{F}$ and $\mathbb{G}$ with respect to m.

Theorem 3.5. *The continuous Bessel multiplier associated to $(a_2, \cdots, a_n)$ of $\mathbb{F}$ and $\mathbb{G}$ with respect to m is well defined and bounded.*

Proof. For any $f, g \in H_F$, we have

$$|\langle M_{m, \mathbb{F}, \mathbb{G}} f, g \,|\, a_2, \cdots, a_n \rangle|$$
$$= \left| \int_{\Omega} m(w) \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle \langle \mathbb{G}(w), g \,|\, a_2, \cdots, a_n \rangle \, d\mu(w) \right|$$
$$\leq \|m\|_\infty \left(\int_{X_1} |\langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle|^2 \, d\mu(w) \right)^{1/2} \times$$
$$\left(\int_{\Omega} |\langle g, \mathbb{G}(w) \,|\, a_2, \cdots, a_n \rangle|^2 \, d\mu(w) \right)^{1/2} \times$$
$$\leq \|m\|_\infty \sqrt{B_1 B_2} \, \|f, a_2, \cdots, a_n\|^2 \, \|g, a_2, \cdots, a_n\|^2.$$

This shows that $\|M_{m, \mathbb{F}, \mathbb{G}}\| \leq \|m\|_\infty \sqrt{B_1 B_2}$ and so $M_{m, \mathbb{F}, \mathbb{G}}$ is well-defined and bounded. □

Remark 3.5. According to the proof of the Theorem 3.5, for each $f \in H_F$, we have

$$\| M_{m, \mathbb{F}, \mathbb{G}} f, a_2, \cdots, a_n \| = \sup_{\| g, a_2, \cdots, a_n \| = 1} | \langle M_{m, \mathbb{F}, \mathbb{G}} f, g \,|\, a_2, \cdots, a_n \rangle |$$

$$\leq \| m \|_\infty \sqrt{B_2} \left(\int_\Omega | \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle |^2 \, d\mu(w) \right)^{1/2}. \tag{5}$$

and similarly it can be shown that

$$\| M^*_{m, \mathbb{F}, \mathbb{G}} g, a_2, \cdots, a_n \|$$

$$\leq \| m \|_\infty \sqrt{B_1} \left(\int_\Omega | \langle \mathbb{G}(w), g \,|\, a_2, \cdots, a_n \rangle |^2 \, d\mu(w) \right)^{1/2}. \tag{6}$$

Theorem 3.6. *Let $M_{m, \mathbb{F}, \mathbb{G}}$ be the continuous Bessel multiplier associated to $(a_2, \cdots, a_n)$ of $\mathbb{F}$ and $\mathbb{G}$ with respect to m. Then $\mathbb{F}$ is a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1 provided $M_{m, \mathbb{F}, \mathbb{G}}$ is bounded below.*

Proof. Since $M_{m, \mathbb{F}, \mathbb{G}}$ is bounded below, for each $f \in H_F$, there exists $D > 0$ such that

$$\| M_{m, \mathbb{F}, \mathbb{G}} f, a_2, \cdots, a_n \| \geq D \, \| f, a_2, \cdots, a_n \|.$$

Therefore, for each $f \in H_F$, using 5, we get

$$D^2 \, \| f, a_2, \cdots, a_n \|^2 \leq \| m \|^2_\infty B_2 \int_\Omega | \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle |^2 \, d\mu(w)$$

$$\Rightarrow \frac{D^2}{\| m \|^2_\infty B_2} \| f, a_2, \cdots, a_n \|^2 \leq \int_\Omega | \langle f, \mathbb{F}(w) \,|\, a_2, \cdots, a_n \rangle |^2 \, d\mu(w).$$

Thus, $\mathbb{F}$ is a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1 with bounds $\dfrac{D^2}{\| m \|^2_\infty B_2}$ and B_1. This completes the proof. □

Theorem 3.7. *Let $M_{m, \mathbb{F}, \mathbb{G}}$ be the continuous Bessel multiplier associated to $(a_2, \cdots, a_n)$ of $\mathbb{F}$ and $\mathbb{G}$ with respect to m. Suppose $\lambda_1 < 1$, $\lambda_2 > -1$ such that for each $f \in H_F$, we have*

$$\| f - M_{m, \mathbb{F}, \mathbb{G}} f, a_2, \cdots, a_n \| \leq \lambda_1 \, \| f, a_2, \cdots, a_n \| + \lambda_2 \, \| M_{m, \mathbb{F}, \mathbb{G}} f, a_2, \cdots, a_n \|.$$

Then $\mathbb{F}$ is a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1.

Proof. For each $f \in H_F$, we have

$$\| f, a_2, \cdots, a_n \| - \| M_{m, \mathbb{F}, \mathbb{G}} f, a_2, \cdots, a_n \| \leq \| f - M_{m, \mathbb{F}, \mathbb{G}} f, a_2, \cdots, a_n \|$$
$$\leq \lambda_1 \, \| f, a_2, \cdots, a_n \| + \lambda_2 \, \| M_{m, \mathbb{F}, \mathbb{G}} f, a_2, \cdots, a_n \|.$$
$$\Rightarrow (1 - \lambda_1) \, \| f, a_2, \cdots, a_n \| \leq (1 + \lambda_2) \, \| M_{m, \mathbb{F}, \mathbb{G}} f, a_2, \cdots, a_n \|.$$

Now, using (5), we get

$$\frac{(1-\lambda_1)}{(1+\lambda_2)}\,\|f,\,a_2,\,\cdots,\,a_n\|$$
$$\leq \|m\|_\infty \sqrt{B_2}\left(\int_\Omega |\langle f,\,\mathbb{F}(w)\,|\,a_2,\,\cdots,\,a_n\rangle|^2\,d\mu(w)\right)^{1/2}$$
$$\Rightarrow \frac{(1-\lambda_1)^2}{\|m\|_\infty^2\,B_2\,(1+\lambda_2)^2}\,\|f,\,a_2,\,\cdots,\,a_n\|^2$$
$$\leq \int_\Omega |\langle f,\,\mathbb{F}(w)\,|\,a_2,\,\cdots,\,a_n\rangle|^2\,d\mu(w). \tag{7}$$

Thus, $\mathbb{F}$ is a continuous frame associated to $(a_2,\,\cdots,\,a_n)$ for H_1 with bounds $\dfrac{(1-\lambda_1)^2}{\|m\|_\infty^2\,B_2\,(1+\lambda_2)^2}$ and B_1. □

Theorem 3.8. *Let $M_{m,\mathbb{F},\mathbb{G}}$ be the continuous Bessel multiplier associated to $(a_2,\cdots,a_n)$ of $\mathbb{F}$ and $\mathbb{G}$ with respect to m. Suppose $\lambda \in [0,1)$ such that for each $f \in H_F$, we have*

$$\|f - M_{m,\mathbb{F},\mathbb{G}}\,f,\,a_2,\,\cdots,\,a_n\| \leq \lambda\,\|f,\,a_2,\,\cdots,\,a_n\|.$$

Then $\mathbb{F}$ and $\mathbb{G}$ are continuous frames associated to $(a_2,\,\cdots,\,a_n)$ for H_1.

Proof. Putting $\lambda_1 = \lambda$ and $\lambda_2 = 0$ in (7), we get

$$\frac{(1-\lambda)^2}{\|m\|_\infty^2\,B_2}\,\|f,\,a_2,\,\cdots,\,a_n\|^2 \leq \int_\Omega |\langle f,\,\mathbb{F}(w)\,|\,a_2,\,\cdots,\,a_n\rangle|^2\,d\mu(w).$$

Thus, $\mathbb{F}$ is a continuous frame associated to $(a_2,\,\cdots,\,a_n)$ for H_1.
On the other hand, for each $f \in H_F$, we have

$$\left\|f - M^*_{m,\mathbb{F},\mathbb{G}}\,f,\,a_2,\,\cdots,\,a_n\right\| = \left\|(I_F - M_{m,\mathbb{F},\mathbb{G}})^*\,f,\,a_2,\,\cdots,\,a_n\right\|$$
$$\leq \|I_F - M_{m,\mathbb{F},\mathbb{G}}\|\,\|f,\,a_2,\,\cdots,\,a_n\| \leq \lambda\,\|f,\,a_2,\,\cdots,\,a_n\|$$
$$\Rightarrow (1-\lambda)\,\|f,\,a_2,\,\cdots,\,a_n\| \leq \left\|M^*_{m,\mathbb{F},\mathbb{G}}\,f,\,a_2,\,\cdots,\,a_n\right\|.$$

Now, using (6), we get

$$\frac{(1-\lambda)^2}{\|m\|_\infty^2\,B_1}\,\|f,\,a_2,\,\cdots,\,a_n\|^2 \leq \int_\Omega |\langle \mathbb{G}(w),\,f\,|\,a_2,\,\cdots,\,a_n\rangle|^2\,d\mu(w).$$

This shows that $\mathbb{G}$ is a continuous frame associated to $(a_2,\,\cdots,\,a_n)$ for H_1. This completes the proof. □

4. Continuous frame in tensor product of n-Hilbert spaces

In this section, we introduce the concept of continuous frame in tensor product of n-Hilbert spaces and give a characterization. We begin this section with the concept of tensor product of n-Hilbert spaces.

Let H_1 and H_2 be two n-Hilbert spaces associated with the n-inner products $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_1$ and $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_2$, respectively. The tensor product of H_1 and H_2 is denoted by $H_1 \otimes H_2$ and it is defined to be an n-inner product space associated with the n-inner product given by

$$\begin{aligned} &\langle f \otimes g, f_1 \otimes g_1 | f_2 \otimes g_2, \cdots, f_n \otimes g_n \rangle \\ &= \langle f, f_1 | f_2, \cdots, f_n \rangle_1 \langle g, g_1 | g_2, \cdots, g_n \rangle_2 , \end{aligned} \tag{8}$$

for all $f, f_1, f_2, \cdots, f_n \in H_1$ and $g, g_1, g_2, \cdots, g_n \in H_2$.
The n-norm on $H_1 \otimes H_2$ is defined by

$$\begin{aligned} &\| f_1 \otimes g_1, f_2 \otimes g_2, \cdots, f_n \otimes g_n \| \\ &= \| f_1, f_2, \cdots, f_n \|_1 \| g_1, g_2, \cdots, g_n \|_2 , \end{aligned} \tag{9}$$

for all $f_1, f_2, \cdots, f_n \in H_1$ and $g_1, g_2, \cdots, g_n \in H_2$, where the n-norms $\| \cdot, \cdots, \cdot \|_1$ and $\| \cdot, \cdots, \cdot \|_2$ are generated by $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_1$ and $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_2$, respectively. The space $H_1 \otimes H_2$ is completion with respect to the above n-inner product. Therefore the space $H_1 \otimes H_2$ is an n-Hilbert space.

Consider $G = \{ b_2, b_3, \cdots, b_n \}$, where $b_2, b_3, \cdots, b_n$ are fixed elements in H_2 and L_G denote the linear subspace of H_2 spanned by G. Now, we can define the Hilbert space H_G with respect to the inner product is given by

$$\langle f + L_G , g + L_G \rangle_G = \langle f , g \rangle_G = \langle f , g | b_2 , \cdots , b_n \rangle_2 ; \; \forall \; f, g \in H_2.$$

Remark 4.1. According to the definition 2.4, $H_F \otimes H_G$ is the Hilbert space with respect to the inner product:

$$\langle f \otimes g , f' \otimes g' \rangle = \langle f , f' \rangle_F \, \langle g , g' \rangle_G ,$$

for all $f, f' \in H_F$ and $g, g' \in H_G$.

Definition 4.1. Let $(X, \mu) = (X_1 \times X_2, \mu_1 \otimes \mu_2)$ be the product of measure spaces with σ-finite positive measures μ_1, μ_2 and $a_2 \otimes b_2, \cdots, a_n \otimes b_n$ be fixed elements in $H_1 \otimes H_2$. The mapping $\mathcal{F} : X \to H_1 \otimes H_2$ is called a continuous frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ) if

- (i) $\mathcal{F}$ is weakly-measurable, i. e., for all $f \otimes g \in H_1 \otimes H_2$, $x = (x_1, x_2) \to \langle f \otimes g, \mathcal{F}(x) | a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle$ is a measurable function on X.
- (ii) there exist constants $A, B > 0$ such that

$$\begin{aligned} &A \| f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2 \\ &\leq \int_X | \langle f \otimes g, \mathcal{F}(x) | a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle |^2 \, d\mu(x) \\ &\leq B \| f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2 , \end{aligned} \tag{10}$$

for all $f \otimes g \in H_1 \otimes H_2$. The constants A and B are called continuous frame bounds. If $A = B$, then it is called a tight continuous frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$. If the mapping $\mathcal{F}$ satisfies only the right inequality of (10), then it is called Bessel mapping or c-Bessel mapping associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ with Bessel bound B.

In the following theorem, we show that the continuous frame in n-Hilbert space is preserved by the tensor product.

Theorem 4.1. *The mapping* $\mathcal{F} = F_1 \otimes F_2 : X \to H_1 \otimes H_2$ *is a continuous frame associated to* $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ *for* $H_1 \otimes H_2$ *with respect to* (X, μ) *if and only if* F_1 *is a continuous frame associated to* $(a_2, \cdots, a_n)$ *for* H_1 *with respect to* (X_1, μ_1) *and* F_2 *is a continuous frame associated to* $(b_2, \cdots, b_n)$ *for* H_2 *with respect to* (X_2, μ_2)

Proof. Suppose that $\mathcal{F} = F_1 \otimes F_2$ is a continuous frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ). Let $f \in H_1 / \{0\}$ and fix $g \in H_2 / \{0\}$. Then $f \otimes g \in H_1 \otimes H_2$ and by Fubini's theorem we have

$$\int_X |\langle f \otimes g, F_1(x_1) \otimes F_2(x_2) | a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle|^2 \, d\mu(x)$$
$$= \int_{X_1} |\langle f, F_1(x_1) | a_2, \cdots, a_n \rangle_1|^2 \, d\mu_1(x_1) \times \int_{X_2} |\langle g, F_2(x_2) | b_2, \cdots, b_n \rangle_2|^2 \, d\mu_2(x_2).$$

Therefore, for each $f \otimes g \in H_1 \otimes H_2$, (10) can be written as

$$A \, \|f, a_2, \cdots, a_n\|_1^2 \, \|g, b_2, \cdots, b_n\|_2^2$$
$$\leq \int_{X_1} |\langle f, F_1(x_1) | a_2, \cdots, a_n \rangle_1|^2 \, d\mu_1(x_1) \times \int_{X_2} |\langle g, F_2(x_2) | b_2, \cdots, b_n \rangle_2|^2 \, d\mu_2(x_2)$$
$$\leq B \, \|f, a_2, \cdots, a_n\|_1^2 \, \|g, b_2, \cdots, b_n\|_2^2 .$$

Here we may assume that every $F_1(x_1)$ and $a_2, \cdots, a_n$ are linearly independent and also every $F_2(x_2)$ and $b_2, \cdots, b_n$ are linearly independent. Hence

$$\int_{X_1} |\langle f, F_1(x_1) | a_2, \cdots, a_n \rangle_1|^2 \, d\mu_1(x_1),$$

$$\int_{X_2} |\langle g, F_2(x_2) | b_2, \cdots, b_n \rangle_2|^2 \, d\mu_2(x_2)$$

are non-zero. Thus from the above inequality we can write

$$\frac{A \, \|g, b_2, \cdots, b_n\|_2^2}{\int_{X_2} |\langle g, F_2(x_2) | b_2, \cdots, b_n \rangle_2|^2 \, d\mu_2(x_2)} \, \|f, a_2, \cdots, a_n\|_1^2$$
$$\leq \int_{X_1} |\langle f, F_1(x_1) | a_2, \cdots, a_n \rangle_1|^2 \, d\mu_1(x_1)$$
$$\leq \frac{B \, \|g, b_2, \cdots, b_n\|_2^2}{\int_{X_2} |\langle g, F_2(x_2) | b_2, \cdots, b_n \rangle_2|^2 \, d\mu_2(x_2)} \, \|f, a_2, \cdots, a_n\|_1^2 .$$

Thus, for each $f \in H_1 / \{0\}$, we have

$$\begin{aligned} A_1 \| f, a_2, \cdots, a_n \|_1^2 &\leq \int\limits_{X_1} |\langle f, F_1(x_1) | a_2, \cdots, a_n \rangle_1|^2 \, d\mu_1(x_1) \\ &\leq B_1 \| f, a_2, \cdots, a_n \|_1^2, \end{aligned}$$

where

$$A_1 = \inf_{g \in H_2} \left\{ \frac{A \| g, b_2, \cdots, b_n \|_2^2}{\int\limits_{X_2} |\langle g, F_2(x_2) | b_2, \cdots, b_n \rangle_2|^2 \, d\mu_2(x_2)} \right\},$$

and

$$B_1 = \sup_{g \in H_2} \left\{ \frac{B \| g, b_2, \cdots, b_n \|_2^2}{\int\limits_{X_2} |\langle g, F_2(x_2) | b_2, \cdots, b_n \rangle_2|^2 \, d\mu_2(x_2)} \right\}.$$

This shows that F_1 is a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1 with respect to (X_1, μ_1). Similarly, it can be shown that F_2 is a continuous frame associated to $(b_2, \cdots, b_n)$ for H_2 with respect to (X_2, μ_2).

Conversely, suppose that F_1 is a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1 with respect to (X_1, μ_1) having bounds A, B and F_2 is a continuous frame associated to $(b_2, \cdots, b_n)$ for H_2 with respect to (X_2, μ_2) having bounds C, D. By the assumption it is easy to very that $F = F_1 \otimes F_2$ is weakly measurable on $H_1 \otimes H_2$ with respect to (X, μ). Now, for each $f \in H_1 / \{0\}$, $g \in H_2 / \{0\}$, we have

$$\begin{aligned} A \| f, a_2, \cdots, a_n \|_1^2 &\leq \int\limits_{X_1} |\langle f, F_1(x_1) | a_2, \cdots, a_n \rangle_1|^2 \, d\mu_1(x_1) \\ &\leq B \| f, a_2, \cdots, a_n \|_1^2, \end{aligned}$$

$$\begin{aligned} C \| g, b_2, \cdots, b_n \|_2^2 &\leq \int\limits_{X_2} |\langle g, F_2(x_2) | b_2, \cdots, b_n \rangle_2|^2 \, d\mu_2(x_2) \\ &\leq D \| g, b_2, \cdots, b_n \|_2^2. \end{aligned}$$

Multiplying the above two inequalities and using Fubini's theorem we get

$$\begin{aligned} &AC \| f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2 \\ &\leq \int\limits_{X} |\langle f \otimes g, \mathcal{F}(x) | a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle|^2 \, d\mu(x) \\ &\leq BD \| f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2, \end{aligned}$$

for all $f \otimes g \in H_1 \otimes H_2$. This completes the proof. □

Remark 4.2. Let $(X, \mu) = (X_1 \times X_2, \mu_1 \otimes \mu_2)$ be the product of measure spaces with σ-finite positive measures μ_1, μ_2.

Let $L^{2}_{F \otimes G}(X, \mu)$ be the class of all measurable functions $\Psi : X \to H_F \otimes H_G$ such that

$$\int\limits_{X} \| \Psi(x), a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2 \, d\mu(x) < \infty,$$

with the inner product

$$\begin{aligned} \langle \Psi, \Phi \rangle_{L^{2}_{F \otimes G}} &= \int\limits_{X} \langle \Psi(x), \Phi(x) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle \, d\mu(x), \\ &= \int\limits_{X_1} \langle \varphi_1(x_1), \psi_1(x_1) \,|\, a_2, \cdots, a_n \rangle \, d\mu(x_1) \times \\ &\qquad \int\limits_{X_2} \langle \varphi_2(x_2), \psi_2(x_2) \,|\, b_2, \cdots, b_n \rangle \, d\mu(x_2) \\ &= \langle \varphi_1, \psi_1 \rangle_{L^{2}_{F}} \langle \varphi_2, \psi_2 \rangle_{L^{2}_{G}}, \end{aligned}$$

for $\Psi = \varphi_1 \otimes \varphi_2, \Phi = \psi_1 \otimes \psi_2 \in L^{2}_{F \otimes G}(X, \mu)$. The space $L^{2}_{F \otimes G}(X, \mu)$ is completion with respect to the above inner product. Therefore it is an Hilbert space.

Remark 4.3. Let $\mathcal{F}$ be a continuous Bessel family associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ). Then the synthesis operator $T_{\mathcal{F}} : L^{2}_{F \otimes G}(X, \mu) \to H_F \otimes H_G$ defined by

$$\begin{aligned} T_{\mathcal{F}}(\varphi) &= \int\limits_{X} \varphi(x) F(x) \, d\mu(x) \\ &= \int\limits_{X_1} \int\limits_{X_2} \varphi(x_1, x_2) \, F(x_1, x_2) \, d\mu(x_1, x_2) \end{aligned}$$

where $\varphi \in L^{2}_{F \otimes G}(X, \mu)$ is well-defined, bounded and linear. The analysis operator $T^{*}_{\mathcal{F}} : H_F \otimes H_G \to L^{2}_{F \otimes G}(X, \mu)$ given by

$$(T^{*}_{\mathcal{F}}(f \otimes g))(x) = \langle f \otimes g, \mathcal{F}(x) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle,$$

$x \in X, f \otimes g \in H_F \otimes H_G$. The frame operator $S_{\mathcal{F}} : H_F \otimes H_G \to H_F \otimes H_G$ is given by

$$S_{\mathcal{F}}(f \otimes g) = \int\limits_{X} \langle f \otimes g, \mathcal{F}(x) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle \, \mathcal{F}(x) \, d\mu(x)$$

The next theorem demonstrates that the continuous frame operator associated with the tensor product of two continuous frames in n-Hilbert spaces is exactly the tensor product of their respective continuous frame operators.

Theorem 4.2. *Let $\mathcal{F} = F_1 \otimes F_2 : X \to H_1 \otimes H_2$ be a continuous frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ). Then $S_{\mathcal{F}} = S_{F_1} \otimes S_{F_2}$.*

Proof. Suppose that $\mathcal{F} = F_1 \otimes F_2$ is a continuous frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ). Then for each $f \otimes g \in$

$H_F \otimes H_G$, we have

$$S_{\mathcal{F}}(f \otimes g)$$
$$= \int_X \langle f \otimes g, F_1(x_1) \otimes F_2(x_2) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle \, F_1(x_1) \otimes F_2(x_2) d\mu(x)$$
$$= \int_{X_1} \langle f, F_1(x_1) \,|\, a_2, \cdots, a_n \rangle_1 \, F_1(x_1) \, d\mu_1(x_1) \otimes$$
$$\int_{X_2} \langle g, F_2(x_2) \,|\, b_2, \cdots, b_n \rangle_2 \, F_2(x_2) \, d\mu_2(x_2)$$
$$= S_{F_1} f \otimes S_{F_2} g = (S_{F_1} \otimes S_{F_2})(f \otimes g).$$

□

Theorem 4.3. *Let F_1 be a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1 with respect to (X_1, μ_1) having bounds A, B and F_2 be a continuous frame associated to $(b_2, \cdots, b_n)$ for H_2 with respect to (X_2, μ_2) having bounds C, D. Then $A C I_{F \otimes G} \leq S_{F_1 \otimes F_2} \leq B D I_{F \otimes G}$, where $I_{F \otimes G}$ is the identity operator on $H_F \otimes H_G$ and S_{F_1}, S_{F_2} are continuous frame operators of F_1, F_2, respectively.*

Proof. Since S_{F_1} and S_{F_2} are continuous frame operators, we have

$$A I_F \leq S_{F_1} \leq B I_F, \quad C I_G \leq S_{F_2} \leq D I_G,$$

where I_F and I_G are the identity operators on H_F and K_G, respectively. Taking tensor product on the above two inequalities, we get

$$A C (I_F \otimes I_G) \leq (S_{F_1} \otimes S_{F_2}) \leq B D (I_F \otimes I_G)$$
$$\Rightarrow A C I_{F \otimes G} \leq S_{F_1 \otimes F_2} \leq B D I_{F \otimes G}.$$

This completes the proof. □

To each continuous frame in n-Hilbert space one can associate a dual continuous frame which is introduced as follows.

If F_1 is a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1 with respect to (X_1, μ_1) and F_2 is a continuous frame associated to $(b_2, \cdots, b_n)$ for H_2 with respect to (X_2, μ_2), then we may consider the dual continuous frame G_1 associated to $(a_2, \cdots, a_n)$ of F_1 and dual continuous frame G_2 associated to $(b_2, \cdots, b_n)$ of F_2 which satisfies the following:

$$\langle f, g \,|\, a_2, \cdots, a_n \rangle_1$$
$$= \int_{X_1} \langle f, F_1(x_1) \,|\, a_2, \cdots, a_n \rangle_1 \, \langle G_1(x_1), g \,|\, a_2, \cdots, a_n \rangle_1 \, d\mu_1(x_1), \tag{11}$$

$$\langle f_1, g_1 \,|\, b_2, \cdots, b_n \rangle_1$$
$$= \int_{X_2} \langle f_1, F_2(x_2) \,|\, b_2, \cdots, b_n \rangle_2 \, \langle G_2(x_2), g_1 \,|\, b_2, \cdots, b_n \rangle_2 \, d\mu_2(x_2), \tag{12}$$

for all $f, g \in H_1$ and $f_1, g_1 \in H_2$.

Now, we give the definition of dual continuous frame in $H_1 \otimes H_2$.

Definition 4.2. Let $\mathcal{F}$ be a continuous frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ). Then a frame $\mathcal{G}$ associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ satisfying

$$f \otimes g = \int\limits_X \langle f \otimes g, \mathcal{F}(x) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle \, \mathcal{G}(x) \, d\mu(x),$$

for all $f \otimes g \in H_1 \otimes H_2$, is called a dual continuous frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ of $\mathcal{F}$. The pair $(\mathcal{F}, \mathcal{G})$ is called a dual pair of continuous frames associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$.

Next, we give a sufficient condition for two tensor product of continuous frames to form a pair of dual continuous frames in $H_1 \otimes H_2$.

Theorem 4.4. *Let F_1 be a continuous frame associated to $(a_2, \cdots, a_n)$ for H_1 with respect to (X_1, μ_1) and F_2 is a continuous frame associated to $(b_2, \cdots, b_n)$ for H_2 with respect to (X_2, μ_2). Suppose G_1 be the dual continuous frame associated to $(a_2, \cdots, a_n)$ of F_1 and G_2 be the dual continuous frame associated to $(b_2, \cdots, b_n)$ of F_2. Then $\mathcal{G} = G_1 \otimes G_2 : X \to H_2 \otimes H_2$ is a dual continuous frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ) of $\mathcal{F} = F_1 \otimes F_2 : X \to H_1 \otimes H_2$.*

Proof. By theorem 4.1, $\mathcal{F} = F_1 \otimes F_2 : X \to H_1 \otimes H_2$ and $\mathcal{G} = G_1 \otimes G_2 : X \to H_2 \otimes H_2$ are continuous frames associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ). Since G_1 is a dual continuous frame associated to $(a_2, \cdots, a_n)$ of F_1 and G_2 is a dual continuous frame associated to $(b_2, \cdots, b_n)$ of F_2, for $f \in H_1$ and $g \in H_2$, we have

$$f = \int\limits_{X_1} \langle f, F_1(x_1) \,|\, a_2, \cdots, a_n \rangle_1 \, G_1(x_1) \, d\mu_1(x_1),$$

$$g = \int\limits_{X_2} \langle g, F_2(x_2) \,|\, b_2, \cdots, b_n \rangle_2 \, G_2(x_2) \, d\mu_2(x_2).$$

Now, for each $f \otimes g \in H_1 \otimes H_2$, we have

$$f \otimes g$$
$$= \int\limits_{X_1} \int\limits_{X_2} \langle f, F_1(x_1) \,|\, a_2, \cdots, a_n \rangle_1 \, \langle g, F_2(x_2) \,|\, b_2, \cdots, b_n \rangle_2 \mathcal{G}(x) d\mu(x)$$

where $\mathcal{G}(x) = G_1(x_1) \otimes G_2(x_2)$. By Fubini's theorem, we can write

$$f \otimes g = \int\limits_X \langle f \otimes g, \mathcal{F}(x) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle \, \mathcal{G}(x) \, d\mu(x).$$

This completes the proof. □

In the following theorem, we will see that dual pair of continuous Bessel families is a dual pair of continuous frames in $H_1 \otimes H_2$.

Theorem 4.5. *Let $F_1,\, G_1$ be the dual pair of continuous Bessel families associated to $(a_2, \cdots, a_n)$ for H_1 with respect to (X_1, μ_1) having bounds B_1, B_2 and $F_2,\, G_2$ be the dual pair of continuous Bessel families associated to $(b_2, \cdots, b_n)$ for H_2 with respect to (X_2, μ_2) having bounds D_1, D_2. Then $\mathcal{G} = G_1 \otimes G_2 : X \to H_2 \otimes H_2$ is a dual continuous frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ) of $\mathcal{F} = F_1 \otimes F_2 : X \to H_1 \otimes H_2$.*

Proof. First, we show that $\mathcal{F} = F_1 \otimes F_2, \mathcal{G} = G_1 \otimes G_2 : X \to H_1 \otimes H_2$ are continuous frames associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ). Now, for each $f \otimes g \in H_1 \otimes H_2$, using (11) and (12), we have

$$\begin{aligned}
&\| f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2 \\
&= \langle f, f \,|\, a_2, \cdots, a_n \rangle_1 \langle g, g \,|\, a_2, \cdots, a_n \rangle_2 \\
&= \int\limits_{X_1} \langle f, F_1(x_1) \,|\, a_2, \cdots, a_n \rangle_1 \langle G_1(x_1), f \,|\, a_2, \cdots, a_n \rangle_1 \, d\mu_1(x_1) \times \\
&\qquad \int\limits_{X_2} \langle g, F_2(x_2) \,|\, b_2, \cdots, b_n \rangle_2 \langle G_2(x_2), g \,|\, b_2, \cdots, b_n \rangle_2 \, d\mu_2(x_2) \\
&\leq \left(\int\limits_{X_1} |\langle f, F_1(x_1) \,|\, a_2, \cdots, a_n \rangle_1|^2 \, d\mu_1(x_1) \right)^{1/2} \times \\
&\qquad \left(\int\limits_{X_1} |\langle g, G_1(x_1) \,|\, a_2, \cdots, a_n \rangle_1|^2 \, d\mu_1(x_1) \right)^{1/2} \times \\
&\qquad \left(\int\limits_{X_2} |\langle g, F_2(x_2) \,|\, b_2, \cdots, b_n \rangle_2|^2 \, d\mu_2(x_2) \right)^{1/2} \times \\
&\qquad \left(\int\limits_{X_2} |\langle g, G_2(x_2) \,|\, b_2, \cdots, b_n \rangle_2|^2 \, d\mu_2(x_2) \right)^{1/2} \\
&\quad \leq \sqrt{B_2 D_2}\, \| f, a_2, \cdots, a_n \|_1 \, \| g, b_2, \cdots, b_n \|_2 \times \\
&\qquad\quad \left(\int\limits_{X} |\langle f \otimes g, \mathcal{F}(x) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle|^2 \, d\mu(x) \right)^{1/2} \\
&\quad \Rightarrow \frac{1}{B_2 D_2} \| f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2 \\
&\qquad\quad \leq \int\limits_{X} |\langle f \otimes g, \mathcal{F}(x) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle|^2 \, d\mu(x).
\end{aligned}$$

Thus, $\mathcal{F}$ is a continuous frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ) having bounds $\dfrac{1}{B_2 D_2}$ and $B_1 D_1$. Similarly, it can be shown that $\mathcal{G}$ is a continuous frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$

with respect to (X, μ). Now, by theorem 4.4, $(\mathcal{F}, \mathcal{G})$ is a dual pair of continuous frames associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$. This completes the proof. $\square$

Now, we end this section by discussing the idea of continuous Bessel multiplier in $H_1 \otimes H_2$.

Definition 4.3. Let $\mathcal{F}$ and $\mathcal{G}$ be continuous Bessel families associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ) having bounds B_1 and B_2 and $m : X \to \mathbb{C}$ be a measurable function. The operator $\mathcal{M}_{m, \mathcal{F}, \mathcal{G}} : H_F \otimes H_G \to H_F \otimes H_G$ defined by

$$\mathcal{M}_{m, \mathcal{F}, \mathcal{G}}(f \otimes g) =: \int_X m(x) \langle f \otimes g, \mathcal{F}(x) | a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle \mathcal{G}(x) d\mu(x), \tag{13}$$

for all $f \otimes g \in H_F \otimes H_G$, is called continuous Bessel multiplier associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ of $\mathcal{F}$ and $\mathcal{G}$ with respect to m.

Remark 4.4. Let F_1, G_1 be continuous Bessel families associated to $(a_2, \cdots, a_n)$ for H_1 with respect to (X_1, μ_1) and F_2, G_2 be continuous Bessel families associated to $(b_2, \cdots, b_n)$ for H_2 with respect to (X_2, μ_2) and $m_1 : X_1 \to \mathbb{C}$, $m_2 : X_2 \to \mathbb{C}$ be two measurable function. Suppose $M_{m_1, F_1, G_1} : H_F \to H_F$ be a continuous Bessel multiplier associated to $(a_2, \cdots, a_n)$ of F_1 and G_1 with respect to m_1 and $M_{m_2, F_2, G_2} : H_G \to H_G$ be a continuous Bessel multiplier associated to $(b_2, \cdots, b_n)$ of F_2 and G_2 with respect to m_2. Now, by theorem 4.1, $\mathcal{F} = F_1 \otimes F_2, \mathcal{G} = G_1 \otimes G_2 : X \to H_1 \otimes H_2$ are continuous Bessel families associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ with respect to (X, μ). From (13), for each $f \otimes g \in H_F \otimes H_G$, we can write

$$\begin{aligned}
&\mathcal{M}_{m, \mathcal{F}, \mathcal{G}}(f \otimes g) \\
&=: \int_X m(x) \langle f \otimes g, \mathcal{F}(x) | a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle \mathcal{G}(x) d\mu(x) \\
&=: \int_{X_1} m_1(x_1) \langle f, F_1(x_1) | a_2, \cdots, a_n \rangle_1 G_1(x_1) d\mu_1(x_1) \otimes \\
&\qquad \int_{X_2} m_2(x_2) \langle g, F_2(x_2) | b_2, \cdots, b_n \rangle_2 G_2(x_2) d\mu_2(x_2) \\
&=: M_{m_1, F_1, G_1} f \otimes M_{m_2, F_2, G_2} g = (M_{m_1, F_1, G_1} \otimes M_{m_2, F_2, G_2})(f \otimes g).
\end{aligned}$$

Thus, $\mathcal{M}_{m, \mathcal{F}, \mathcal{G}} = M_{m_1, F_1, G_1} \otimes M_{m_2, F_2, G_2}$.

Remark 4.5. According to the theorem 3.5, the continuous Bessel multiplier associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ of $\mathcal{F}$ and $\mathcal{G}$ with respect to m is well defined and bounded.

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(Prasenjit Ghosh) Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata, 700019, West Bengal, India
E-mail address: `prasenjitpuremath@gmail.com`

(T. K. Samanta) Department of Mathematics, Uluberia College, Uluberia, Howrah, 711315, West Bengal, India
E-mail address: `mumpu_tapas5@yahoo.co.in`