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A novel computational method for solving the fractional SIS epidemic model of two different fractional operators

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Abstract. This study proposes a novel computational method for solving the fractional SIS epidemic model involving the Caputo and Caputo-Fabrizio fractional derivatives, that called Elzaki differential transform method (EDTM) which is a coupling of two powerful methods: the Elzaki transform method and the differential transform method. To demonstrate the effectiveness and advantage of the proposed method, a numerical example is presented. The results obtained by the EDTM are compared with well-known exact solutions. This results show that this method is very effective and more accurate for solving this type of problem. Therefore, our proposed method can be employed to study the solutions of a wide range of real problems arising in engineering and natural sciences, which can be modeled by a fractional differential equations.

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1. Introduction

Recently, it has turned out that many phenomena in engineering and natural sciences can be described very successfully by models using mathematical tools from fractional calculus which deals with derivatives and integrals of arbitrary orders. This is what made several mathematicians and physicists give more attention to the modeling problems in which the fractional derivative does arise in applied sciences. Nowadays, there exist many phenomena in several fields which are modeled by nonlinear fractional differential equations such as: fluid mechanics, viscoelasticity, chemistry, nonlinear control theory, nonlinear biology, biomedical, finance, engineering, and physics [4, 9, 12, 13, 14, 18, 19, 22, 23].

As a result of the difficulty in finding an analytical solution for the nonlinear fractional differential equations by using analytical methods, the numerical methods were used. In the literature, there are several numerical methods of solving nonlinear fractional differential equations. Among them: the Adomian decomposition method (ADM) [8], homotopy analysis method (HAM) [20], homotopy perturbation method (HPM) [2], variational iteration method (VIM) [1], modified fractional Taylor series method (MFTSM) [11].

The main objective of this study is to propose a novel computational method for solving the fractional SIS epidemic model with two different fractional derivative operators. This method is a coupling of two powerful methods: the Elzaki transform

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method and the differential transform method, called Elzaki differential transform method (EDTM).

The fractional SIS epidemic model described in the system of ordinary nonlinear first order fractional differential equations in the form

$$\begin{cases} D^{\alpha}s(t) = -rs(t)i(t) - \mu s(t) + \mu, s(0) = s_0, \\ D^{\alpha}i(t) = rs(t)i(t) - \mu i(t), \qquad i(0) = i_0, \end{cases} \tag{1}$$

where D^{α} is the fractional derivative operator in the Caputo sense of order α with $0 < \alpha \leq 1$, $s(t)$ is the susceptible population, $i(t)$ is the infected population, $r > 0$ is the infectivity coefficient, and $\mu > 0$ is the recovery coefficient, while $s_0 > 0$ and $i_0 > 0$ are given constants.

And in the form

$$\begin{cases} \mathcal{D}^{(\alpha)}s(t) = -rs(t)i(t) - \mu s(t) + \mu, s(0) = s_0, \\ \mathcal{D}^{(\alpha)}i(t) = rs(t)i(t) - \mu i(t), \qquad i(0) = i_0, \end{cases} \tag{2}$$

where $\mathcal{D}^{(\alpha)}$ is the fractional derivative operator in the Caputo-Fabrizio sense of order α with $0 < \alpha \leq 1$.

When $\alpha = 1$, the system (1) and (2) reduces to the classical SIS epidemic model. Recently, many methods have been developed to solve the classic SIS epidemic model. Among these methods: Lie group (LG) [17], homotopy analysis method (HAM) [10], differential transformation method (DTM) [3].

The rest of the paper has been organized as follows. In Section 2, we give some basic definitions and preliminaries of the fractional calculus theory and Elzaki transform. In Section 3, we introduce the basic definitions and fundamental theorems of differential transform method. In Section 4, we describe the Elzaki differential transform method (EDTM) to solve the fractional SIS epidemic model (1) and (2). In Section 5, we propose a numerical application which demonstrate the effectiveness of our proposed method. In Section 6, we discuss our obtained results represented by Figures and Tables. Finally, the conclusions are presented in the last section. (Section 7).

2. Basic definitions and preliminaries

This section gives some basic definitions and preliminaries of the fractional calculus theory and Elzaki transform, which are used in this study.

Definition 2.1. [14] The Riemann-Liouville fractional integral of order $\alpha \geq 0$ of a function f in $L^1(\mathbb{R}^+)$, is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi)d\xi, \tag{3}$$

where $\Gamma(.)$ denotes the gamma function.

Definition 2.2. [14] The Caputo fractional derivative of order $\alpha \geq 0$ of a function $f(t)$, is defined as

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi)d\xi, \tag{4}$$

where $n - 1 < \alpha \leq n$ with $n = [\alpha] + 1$ and $[\alpha]$ being the integer part of α.

By changing the kernel $(t-\xi)^{-\alpha}$ by the function $\exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right]$ and $\dfrac{1}{\Gamma(1-\alpha)}$ by $\dfrac{M(\alpha)}{1-\alpha}$ one obtains the new Caputo-Fabrizio fractional derivative of order $0<\alpha\leq 1$, which has been recently introduced by Caputo and Fabrizio in [5].

Definition 2.3. [5] The Caputo–Fabrizio fractional derivative of order $0<\alpha\leq 1$, of a function f in $H^1(\mathbb{R}^+)$ is defined as

$$\mathcal{D}^{(\alpha)}f(t)=\frac{M(\alpha)}{1-\alpha}\int_0^t f'(\xi)\exp\left[-\frac{\alpha\,(t-\xi)}{1-\alpha}\right]d\xi, \tag{5}$$

where $M(\alpha)$ is a normalization function that satisfies $M(0)=M(1)=1$.

If $f\notin H^1(\mathbb{R}^+)$, then its fractional derivative is redefined as [5]

$$\mathcal{D}^{(\alpha)}f(t)=\frac{\alpha M(\alpha)}{1-\alpha}\int_0^t (f(t)-f(\xi))\exp\left[-\frac{\alpha\,(t-\xi)}{1-\alpha}\right]d\xi, t>0.$$

For $n\geq 1$ and $0<\alpha\leq 1$, the fractional derivative of order $(\alpha+n)$ is defined by

$$\mathcal{D}^{(\alpha+n)}f(t)=\mathcal{D}^{(\alpha)}(\mathcal{D}^{(n)}f(t)). \tag{6}$$

The above Caputo–Fabrizio fractional derivative was later modified by Jorge Losada and Juan José Nieto [15] as

$$\mathcal{D}^{(\alpha)}f(t)=\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^t f'(\xi)\exp\left[-\frac{\alpha\,(t-\xi)}{1-\alpha}\right]d\xi, t>0. \tag{7}$$

The fractional integral corresponding to the derivative in equation (7) was defined by Jorge Losada and Juan José Nieto in 2015, as follows.

Definition 2.4. [15] Let $0<\alpha\leq 1$. The fractional integral of order α of f defined by

$$\mathcal{I}^{(\alpha)}f(t)=\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}f(t)+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f(\xi)d\xi, t>0. \tag{8}$$

From the definition in equation (8), the fractional integral of Caputo-Fabrizio type of a function f of order $0<\alpha\leq 1$ is an average between function f and its one order integral, i.e.,

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}+\frac{2\alpha}{(2-\alpha)M(\alpha)}=1.$$

Therefore,

$$M(\alpha)=\frac{2}{2-\alpha}, 0<\alpha\leq 1.$$

Due to this, Losada and Nieto remarked that Caputo-Fabrizio fractional derivative can redefined as

Definition 2.5. [15] Let $0 < \alpha \leq 1$. The fractional Caputo-Fabrizio derivative of order α of a function f is given by

$$\mathcal{D}^{(\alpha)} f(t) = \frac{1}{1-\alpha} \int_0^t f'(\xi) \exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right] d\xi, t > 0, \tag{9}$$

and its fractional integral is defined as

$$\mathcal{I}^{(\alpha)} f(t) = (1-\alpha) f(t) + \alpha \int_0^t f(\xi) d\xi, t > 0.$$

Definition 2.6. [6] The Elzaki transform is defined over the set of functions

$$A = \left\{ f(t) / \exists M, k_1, k_2 > 0, |f(t)| < M \exp\left(\frac{|t|}{k_j}\right), \text{ if } t \in (-1)^j \times [0, \infty) \right\},$$

by the following integral

$$\mathcal{E}\left[f(t)\right] = T(v) = v \int_0^{+\infty} f(t) \exp\left(-\frac{t}{v}\right) dt, t > 0,$$

where v is the factor of variable t.

Property 1: The Elzaki transform is a linear operator. That is, if λ and μ are non-zero constants, then

$$\mathcal{E}\left[\lambda f(t) \pm \mu g(t)\right] = \lambda \mathcal{E}\left[f(t)\right] \pm \mu \mathcal{E}\left[g(t)\right].$$

Property 2: If $f^{(n)}(t)$ is the $n-$th derivative of the function $f(t) \in A$ with respect to "t" then its Elzaki transform is given by

$$\mathcal{E}\left[f^{(n)}(t)\right] = \frac{1}{v^n} T(v) - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0).$$

Property 3: Suppose $T(v)$ and $G(v)$ are the Elzaki transforms of $f(t)$ and $g(t)$, respectively, both defined in the set A. Then

$$\mathcal{E}\left[(f * g)(t)\right] = \frac{1}{v} T(v) G(v),$$

where the convolution of two functions is defined by

$$(f * g)(t) = \int_0^t f(\xi) g(t-\xi) d\xi = \int_0^t f(t-\xi) g(\xi) d\xi.$$

Property 4: The Elzaki transform for some special functions.

$$\begin{aligned}\mathcal{E}(1) &= v^2,\\ \mathcal{E}(t) &= v^3,\\ \mathcal{E}\left(\frac{t^n}{n!}\right) &= v^{n+2}, n=0,1,2,...\\ \mathcal{E}\left(\exp(at)\right) &= \frac{v^2}{1-av},\\ \mathcal{E}\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right) &= v^{\alpha+2}, \alpha>-1.\end{aligned}$$

Theorem 2.1. *If $T(v)$ is the Elzaki transform of $f(t)$, then the Elzaki transform of the Riemann-Liouville fractional integral for the function $f(t)$ of order α, is given by*

$$\mathcal{E}\left[I^\alpha f(t)\right] = v^\alpha T(v). \tag{10}$$

Proof. The Riemann-Liouville fractional integral for the function $f(t)$ as in (3), can be expressed as the convolution

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t). \tag{11}$$

Applying the Elzaki transform on both sides of (11) and using properties 3 and 4, we get

$$\begin{aligned}\mathcal{E}\left[I^\alpha f(t)\right] &= \mathcal{E}\left[\frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t)\right] = v\mathcal{E}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]\mathcal{E}\left[f(t)\right]\\ &= \frac{1}{v} v^{\alpha+1} T(v) = v^\alpha T(v).\end{aligned}$$

The proof is complete. □

Theorem 2.2. *Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $T(v)$ be the Elzaki transform of the function $f(t)$, then the Elzaki transform of the Caputo fractional derivative of $f(t)$ of order α, is given by*

$$\mathcal{E}\left[D^\alpha f(t)\right] = \frac{1}{v^\alpha} T(v) - \sum_{k=0}^{n-1} v^{2-\alpha+k} f^{(k)}(0). \tag{12}$$

Proof. Let

$$g(t) = f^{(n)}(t),$$

then the definition of the Caputo fractional derivative can be expressed as

$$\begin{aligned}D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi\\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} g(\xi) d\xi\\ &= I^{n-\alpha} g(t). \end{aligned} \tag{13}$$

Applying the Elzaki transform on both sides of (13) and using Theorem 2.1, we get

$$\mathcal{E}\left[D^{\alpha} f(t)\right]=\mathcal{E}\left[I^{n-\alpha} g(t)\right]=v^{n-\alpha} G(v), \tag{14}$$

where $G(v)$ denotes the Elzaki transform of function $g(t)$.

Also, from Property 2, we have

$$\begin{aligned} \mathcal{E}[g(t)] &= \mathcal{E}\left[f^{(n)}(t)\right], \\ G(v) &= \frac{1}{v^n} T(v)-\sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0). \end{aligned} \tag{15}$$

Substituting (15) in (14), we get

$$\begin{aligned} \mathcal{E}\left[D^{\alpha} f(t)\right] &= v^{n-\alpha}\left(\frac{1}{v^n} T(v)-\sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0)\right) \\ &= \frac{1}{v^{\alpha}} T(v)-\sum_{k=0}^{n-1} v^{2-\alpha+k} f^{(k)}(0). \end{aligned}$$

The proof is complete. □

Theorem 2.3. *The Elzaki transform of the Caputo-Fabrizio fractional derivative of the function $f(t)$ of order $\alpha+n$, where $0<\alpha \leq 1$ and $n \in \mathbb{N} \cup\{0\}$, is given by*

$$\mathcal{E}\left[\mathcal{D}^{(\alpha+n)} f(t)\right]=\frac{1}{1-\alpha(1-v)}\left[\frac{1}{v^n} \mathcal{E}(f(t))-\sum_{k=0}^{n} v^{2-n+k} f^{(k)}(0)\right]. \tag{16}$$

Proof. According to Definition 2.5 and relation (6), we have

$$\begin{aligned} \mathcal{E}\left[\mathcal{D}^{(\alpha+n)} f(t)\right] &= \mathcal{E}\left[\mathcal{D}^{(\alpha)}(\mathcal{D}^{(n)} f(t))\right] \\ &= \frac{1}{1-\alpha} v \int_0^{+\infty} \exp\left(-\frac{t}{v}\right)\left(\int_0^t f^{(n+1)}(\xi) \exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right] d\xi\right) dt \\ &= \frac{1}{1-\alpha} v \int_0^{+\infty} \exp\left(-\frac{t}{v}\right)\left(f^{(n+1)}(t) * \exp\left[-\frac{\alpha t}{1-\alpha}\right]\right) dt \\ &= \frac{1}{1-\alpha} \mathcal{E}\left(f^{(n+1)}(t) * \exp\left[-\frac{\alpha t}{1-\alpha}\right]\right). \end{aligned}$$

Hence, from properties 2, 3 and 4, we get

$$\begin{aligned} \mathcal{E}\left[\mathcal{D}^{(\alpha+n)} f(t)\right] &= \frac{1}{1-\alpha} \frac{1}{v} \mathcal{E}\left(f^{(n+1)}(t)\right) \mathcal{E}\left(\exp\left[-\frac{\alpha t}{1-\alpha}\right]\right) \\ &= \frac{v}{1-\alpha(1-v)}\left[\frac{1}{v^{n+1}} \mathcal{E}(f(t))-\sum_{k=0}^{n} v^{1-n+k} f^{(k)}(0)\right] \\ &= \frac{1}{1-\alpha(1-v)}\left[\frac{1}{v^n} \mathcal{E}(f(t))-\sum_{k=0}^{n} v^{2-n+k} f^{(k)}(0)\right]. \end{aligned}$$

The proof is complete. □

3. Differential transform method

This section introduces the basic definitions and fundamental theorems of differential transform method are defined and proved in [7, 16].

Definition 3.1. The differential transform of the function $f(t)$ is defined as

$$F(k)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^k}{dt^k}f(t)\right]_{t=t_0}, \tag{17}$$

where $f(t)$ is the original function and $F(k)$ the transformed function

Definition 3.2. The inverse differential transform of $F(k)$ is defined as

$$f(t)=\sum_{k=0}^{\infty}F(k)(t-t_0)^k. \tag{18}$$

Combining equations (17) and (18), we get

$$f(t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^k}{dt^k}f(t)\right]_{t=t_0}(t-t_0)^k. \tag{19}$$

In particular, for $t_0=0$, equation (19) becomes

$$f(t)=\frac{1}{k!}\left[\frac{d^k}{dt^k}f(t)\right]_{t=0}t^k. \tag{20}$$

From the above definitions, the fundamental operations of the differential transform method are given by the following theorems.

Theorem 3.1. *Let $F(k), G(k)$ and $H(k)$ be the differential transforms of the functions $f(t), g(t)$ and $h(t)$ respectively, then*

(1) *if* $h(t)=\lambda f(t)+\mu g(t)$, *then* $H(k)=\lambda F(k)+\mu G(k)$, $\lambda,\ \mu\in\mathbb{R}$.

(2) *if* $h(t)=f(t)g(t)$, *then* $H(k)=\sum_{r=0}^{k}F(r)G(k-r)$.

(3) *if* $h(t)=f_1(t)f_2(t)...f_{n-1}(t)f_n(t)$, *then*

$$\begin{aligned}H(k) &= \sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}...\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}F_1(k_1)F_2(k_2-k_1)\\ &\quad\times...\times F_{n-1}(k_{n-1}-k_{n-2})F_n(k-k_{n-1}).\end{aligned}$$

(4) *if* $h(t)=\frac{d^n}{dt^n}f(t)$, *then*

$$\begin{aligned}H(k) &= (k+1)(k+2)...(k+n)F(k+n)\\ &= \frac{(k+n)!}{k!}F(k+n), n=1,2,3,...\end{aligned}$$

(5) *if* $h(t)=1$, *then*

$$H(k)=\delta(k)=\begin{cases}1, k=0,\\ 0, k\neq 0.\end{cases}$$

4. Analysis of the method

Theorem 4.1. *Let consider the fractional SIS epidemic model involving the Caputo and Caputo-Fabrizio fractional derivatives (1) and (2). The EDTM gives the solution of (1) or (2) in the form of infinite series that rapidly converge to the exact solutions as follows*

$$\begin{cases} s(t) = \sum\limits_{k=0}^{\infty} S(k), \\ i(t) = \sum\limits_{k=0}^{\infty} I(k), \end{cases} \tag{21}$$

where $S(k)$ and $I(k)$ are the fractional differential transformed function of $s(t)$ and $i(t)$, respectively.

Proof. Consider the fractional SIS epidemic model involving the Caputo and Caputo-Fabrizio fractional derivatives (1) and (2).

4.1. In the case of the fractional derivative of the Caputo. First, we apply the Elzaki transform on both sides of (1) and using Theorem 2.2, we obtain

$$\begin{cases} \mathcal{E}\left[s(t)\right] = v^2 s(0) + v^\alpha \mathcal{E}\left[-rs(t)i(t) - \mu s(t) + \mu\right], \\ \mathcal{E}\left[i(t)\right] = v^2 i(0) + v^\alpha \mathcal{E}\left[rs(t)i(t) - \mu i(t)\right]. \end{cases} \tag{22}$$

Then, we take the inverse Elzaki transform on both sides of (22), we have

$$\begin{cases} s(t) = s_0 + \mathcal{E}^{-1}\left(v^\alpha \mathcal{E}\left[-rs(t)i(t) - \mu s(t) + \mu\right]\right), \\ i(t) = i_0 + \mathcal{E}^{-1}\left(v^\alpha \mathcal{E}\left[rs(t)i(t) - \mu i(t)\right]\right). \end{cases} \tag{23}$$

Now, we apply the differential transform method to (23), we get

$$\begin{aligned} S(0) &= s_0, \\ S(k+1) &= \mathcal{E}^{-1}\left(v^\alpha \mathcal{E}\left[-rA(k) - \mu S(k) + \mu\delta(k)\right]\right), k = 0, 1, 2, ... \end{aligned} \tag{24}$$

and

$$\begin{aligned} I(0) &= i_0, \\ I(k+1) &= \mathcal{E}^{-1}\left(v^\alpha \mathcal{E}\left[rA(k) - \mu I(k)\right]\right), k = 0, 1, 2, ... \end{aligned} \tag{25}$$

where

$$\delta(k) = \begin{cases} 1, k = 0, \\ 0, k \neq 0, \end{cases}$$

and $A(k)$ is transformed form of the nonlinear term, $s(t)i(t)$.

According to Theorem 3.1, the first few nonlinear terms are as follows

$$\begin{aligned} A(0) &= S(0)I(0), \\ A(1) &= S(0)I(1) + S(1)I(0), \\ A(2) &= S(0)I(2) + S(1)I(1) + S(2)I(0), \\ A(3) &= S(0)I(3) + S(1)I(2) + S(2)I(1) + S(3)I(0). \end{aligned}$$

From (24) and (25), we have

$$\begin{aligned}
S(0) &= s_0, \\
S(1) &= \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[-rA(0)-\mu S(0)+\mu\right]\right), \\
S(2) &= \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[-rA(1)-\mu S(1)\right]\right), \\
S(3) &= \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[-rA(2)-\mu S(2)\right]\right), \\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
I(0) &= i_0, \\
I(1) &= \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[rA(0)-\mu I(0)\right]\right), \\
I(2) &= \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[rA(1)-\mu I(1)\right]\right), \\
I(3) &= \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[rA(2)-\mu I(2)\right]\right), \\
&\vdots
\end{aligned}$$

Then, the solutions of (1) is given in the form of infinite series as follows

$$\begin{cases} s(t) = \sum\limits_{k=0}^{\infty} S(k), \\ i(t) = \sum\limits_{k=0}^{\infty} I(k). \end{cases}$$

4.2. In the case of the fractional derivative of the Caputo-Fabrizio. First, we apply the Elzaki transform on both sides of (2) and using Theorem 2.3, we obtain

$$\begin{cases} \mathcal{E}\left[s(t)\right] = v^2 s(0) + (1-\alpha(1-v))\,\mathcal{E}\left[-rs(t)i(t)-\mu s(t)+\mu\right], \\ \mathcal{E}\left[i(t)\right] = v^2 i(0) + (1-\alpha(1-v))\,\mathcal{E}\left[rs(t)i(t)-\mu i(t)\right]. \end{cases} \tag{26}$$

Then, we take the inverse Elzaki transform on both sides of (26), we have

$$\begin{cases} s(t) = s_0 + \mathcal{E}^{-1}\left((1-\alpha(1-v))\,\mathcal{E}\left[-rs(t)i(t)-\mu s(t)+\mu\right]\right), \\ i(t) = i_0 + \mathcal{E}^{-1}\left((1-\alpha(1-v))\,\mathcal{E}\left[rs(t)i(t)-\mu i(t)\right]\right). \end{cases} \tag{27}$$

Now, we apply the differential transform method to (27), we get

$$\begin{aligned}
S(0) &= s_0, \\
S(k+1) &= \mathcal{E}^{-1}\left((1-\alpha(1-v))\,\mathcal{E}\left[-rA(k)-\mu S(k)+\mu\delta(k)\right]\right), k=0,1,2,...
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
I(0) &= i_0, \\
I(k+1) &= \mathcal{E}^{-1}\left((1-\alpha(1-v))\,\mathcal{E}\left[rA(k)-\mu I(k)\right]\right), k=0,1,2,...
\end{aligned} \tag{29}$$

where

$$\delta(k) = \begin{cases} 1, k=0, \\ 0, k\neq 0, \end{cases}$$

and $A(k)$ is transformed form of the nonlinear term, $s(t)i(t)$.

From (28) and (29), we have

$$\begin{array}{rcl} S(0) & = & s_0, \\ S(1) & = & \mathcal{E}^{-1}\left((1-\alpha(1-v))\,\mathcal{E}\left[-rA(0)-\mu S(0)+\mu\right]\right), \\ S(2) & = & \mathcal{E}^{-1}\left((1-\alpha(1-v))\,\mathcal{E}\left[-rA(1)-\mu S(1)\right]\right), \\ S(3) & = & \mathcal{E}^{-1}\left((1-\alpha(1-v))\,\mathcal{E}\left[-rA(2)-\mu S(2)\right]\right), \\ & & \vdots \end{array}$$

and

$$\begin{array}{rcl} I(0) & = & i_0, \\ I(1) & = & \mathcal{E}^{-1}\left((1-\alpha(1-v))\,\mathcal{E}\left[rA(0)-\mu I(0)\right]\right), \\ I(2) & = & \mathcal{E}^{-1}\left((1-\alpha(1-v))\,\mathcal{E}\left[rA(1)-\mu I(1)\right]\right), \\ I(3) & = & \mathcal{E}^{-1}\left((1-\alpha(1-v))\,\mathcal{E}\left[rA(2)-\mu I(2)\right]\right), \\ & & \vdots \end{array}$$

Then, the solutions of (2) is given in the form of infinite series as follows

$$\begin{cases} s(t) = \sum\limits_{k=0}^{\infty} S(k), \\ i(t) = \sum\limits_{k=0}^{\infty} I(k). \end{cases}$$

The proof is complete. □

Remark 4.1. The $n-$term approximate solution of (1) or (2) is given by

$$\begin{cases} s(t) = \sum\limits_{k=0}^{n-1} S(k) = S(0)+S(1)+S(2)+...+S(n-1), \\ i(t) = \sum\limits_{k=0}^{n-1} I(k) = I(0)+I(1)+I(2)+...+I(n-1). \end{cases} \tag{30}$$

5. Numerical application

This section presents a numerical application to illustrate the applicability, simplicity and high accuracy of the proposed method. Numerical results are very encouraging.

Example 5.1. Consider the fractional SIS epidemic model involving the Caputo fractional derivative

$$\begin{cases} D^{\alpha}s(t) = -2s(t)i(t) - s(t) + 1, s(0) = 0.3, \\ D^{\alpha}i(t) = 2s(t)i(t) - i(t), \qquad i(0) = 0.7. \end{cases} \tag{31}$$

For $\alpha = 1$, the exact solutions of (31) is (See. [21])

$$\begin{cases} S(t) = 1 - \dfrac{1}{2-\frac{4}{7}e^{-t}}, \\ I(t) = \dfrac{1}{2-\frac{4}{7}e^{-t}}. \end{cases}$$

Following the analysis presented in Section 4 gives

$$\begin{cases} s(t) = \sum_{k=0}^{\infty} S(k), \\ i(t) = \sum_{k=0}^{\infty} I(k), \end{cases}$$

and

$$\begin{aligned} S(0) &= 0.3, \\ S(1) &= \frac{0.28}{\Gamma(\alpha+1)} t^{\alpha}, \\ S(2) &= -\frac{0.504}{\Gamma(2\alpha+1)} t^{2\alpha}, \\ S(3) &= \left(\frac{0.9072\Gamma^2(\alpha+1) + 0.1568\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right) t^{3\alpha}, \\ &\vdots \end{aligned}$$

$$\begin{aligned} I(0) &= 0.7, \\ I(1) &= -\frac{0.28}{\Gamma(\alpha+1)} t^{\alpha}, \\ I(2) &= \frac{0.504}{\Gamma(2\alpha+1)} t^{2\alpha}, \\ I(3) &= -\left(\frac{0.9072\Gamma^2(\alpha+1) + 0.1568\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right) t^{3\alpha}, \\ &\vdots \end{aligned}$$

Hence, the solutions of (31), is given by

$$\begin{aligned} S(t) &= 0.3 + \frac{0.28}{\Gamma(\alpha+1)} t^{\alpha} - \frac{0.504}{\Gamma(2\alpha+1)} t^{2\alpha} \\ &\quad + \left(\frac{0.9072\Gamma^2(\alpha+1) + 0.1568\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right) t^{3\alpha} + ... \\ I(t) &= 0.7 - \frac{0.28}{\Gamma(\alpha+1)} t^{\alpha} + \frac{0.504}{\Gamma(2\alpha+1)} t^{2\alpha} \\ &\quad - \left(\frac{0.9072\Gamma^2(\alpha+1) + 0.1568\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right) t^{3\alpha} + ... \end{aligned}$$

When $\alpha = 1$, we obtained the following results

$$\begin{aligned} S(t) &= 0.3 + 0.28t - 0.252t^2 + 0.20347t^3 + ... \\ I(t) &= 0.7 - 0.28t + 0.252t^2 - 0.20347t^3 + ... \end{aligned}$$

which is the numerical solutions of the classical SIS epidemic model [3].

Now, we consider the fractional SIS epidemic model involving the Caputo-Fabrizio fractional derivative

$$\begin{cases} \mathcal{D}^{(\alpha)}s(t) = -2s(t)i(t) - s(t) + 1, s(0) = 0.3, \\ \mathcal{D}^{(\alpha)}i(t) = 2s(t)i(t) - i(t), \qquad i(0) = 0.7. \end{cases} \tag{32}$$

For $\alpha = 1$, the exact solutions of (32) is (See. [21])

$$\begin{cases} S(t) = 1 - \dfrac{1}{2 - \frac{4}{7}e^{-t}}, \\ I(t) = \dfrac{1}{2 - \frac{4}{7}e^{-t}}. \end{cases}$$

Following the analysis presented in Section 4 gives

$$\begin{cases} s(t) = \sum\limits_{k=0}^{\infty} S(k), \\ i(t) = \sum\limits_{k=0}^{\infty} I(k), \end{cases}$$

and

$$\begin{aligned} S(0) &= 0.3, \\ S(1) &= 0.28\left((1-\alpha) + \alpha t\right), \\ S(2) &= -0.504\left((1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2}{2}t^2\right), \\ S(3) &= 1.064\left(1-\alpha\right)^3 + 3.192\alpha(1-\alpha)^2 t + 1.6744\alpha^2(1-\alpha)t^2 + 0.20347\alpha^3 t^3, \\ &\vdots \end{aligned}$$

$$\begin{aligned} I(0) &= 0.7, \\ I(1) &= -0.28\left((1-\alpha) + \alpha t\right), \\ I(2) &= 0.504\left((1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2}{2}t^2\right), \\ I(3) &= -1.064\left(1-\alpha\right)^3 - 3.192\alpha(1-\alpha)^2 t - 1.6744\alpha^2(1-\alpha)t^2 - 0.20347\alpha^3 t^3, \\ &\vdots \end{aligned}$$

Hence, the solutions of (32), is given by

$$\begin{aligned} S(t) &= S(0) + S(1) + S(2) + S(3) + ... \\ I(t) &= I(0) + I(1) + I(2) + I(3) + ... \end{aligned}$$

When $\alpha = 1$, we obtained the following results

$$\begin{aligned} S(t) &= 0.3 + 0.28t - 0.252t^2 + 0.20347t^3 ... \\ I(t) &= 0.7 - 0.28t + 0.252t^2 - 0.20347t^3 ... \end{aligned}$$

which is the numerical solutions of the classical SIS epidemic model [3].

6. Numerical results and discussions

In Figures 1 and 2, we plotted the graphs of the 4−term approximate solutions obtained by the EDTM and exact solutions among different values of t at $\alpha = 0.7, 0.8, 0.95, 1$ respectively for (31) and (32). It can be seen from Figures 1 and 2 that, in the limit as $\alpha \longrightarrow 1$, the EDTM-solutions approaches the exact solutions of the classical SIS epidemic model. In Tables 1–4, the numerical values are of the 4−approximate solutions obtained by the EDTM and exact solutions through different values of t at $\alpha = 0.7, 0.8, 0.95, 1$ respectively for (31) and (32). From the Tables 1–4, we observe that absolute error is very small which means that the EDTM is very effective to obtain the analytical solutions for the fractional SIS epidemic model easily without any assumption. Higher accuracy can be obtained by using more terms.

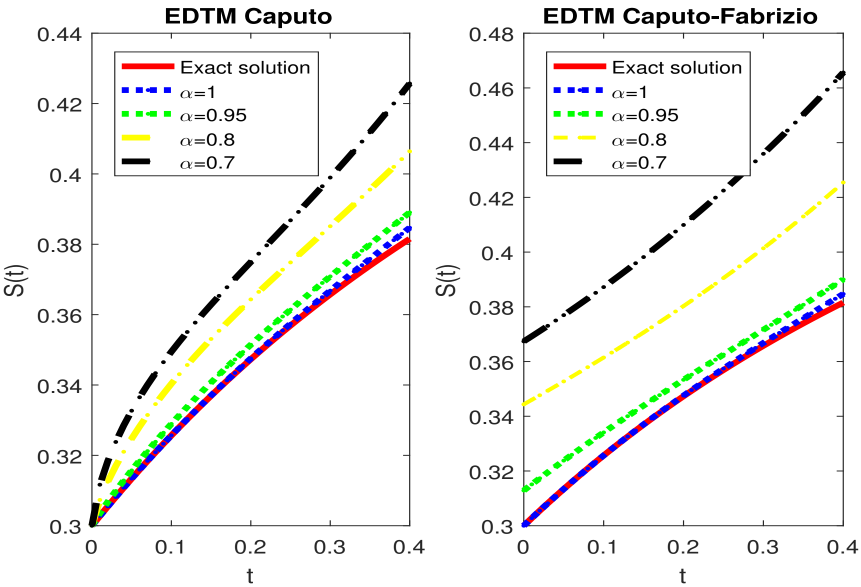

FIGURE 1. 2D plots graphs of the 4−term approximate solutions and exact solution $s(t)$

TABLE 1. The numerical values of the exact solution $s(t)$ and 4−term approximate solutions by Caputo-EDTM for different values of fractional order α

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.95$	$\alpha = 1$	s_{exact}	$\lvert s_{exact} - s_{C-EDTM} \rvert$
0.01	0.31166	0.30734	0.30355	0.30278	0.30278	1.6117×10^{-9}
0.02	0.31837	0.31251	0.30679	0.30550	0.30550	2.5558×10^{-8}
0.03	0.32381	0.31698	0.30987	0.30818	0.30818	1.2834×10^{-7}
0.04	0.32850	0.32102	0.31284	0.31081	0.31081	4.0241×10^{-7}
0.05	0.33269	0.32474	0.31571	0.31340	0.31339	9.7480×10^{-7}

TABLE 2. The numerical values of the exact solution $s(t)$ and 4−term approximate solutions by Caputo-Fabrizio-EDTM for different values of fractional order α

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.95$	$\alpha = 1$	s_{exact}	$\lvert s_{exact} - s_{CF-EDTM}\rvert$
0.01	0.36923	0.34601	0.31511	0.30278	0.30278	1.6117×10^{-9}
0.02	0.37113	0.34767	0.31733	0.30550	0.30550	2.5558×10^{-8}
0.03	0.37304	0.34935	0.31951	0.30818	0.30818	1.2834×10^{-7}
0.04	0.37499	0.35104	0.32167	0.31081	0.31081	4.0241×10^{-7}
0.05	0.37695	0.35274	0.32380	0.31340	0.31339	9.7480×10^{-7}

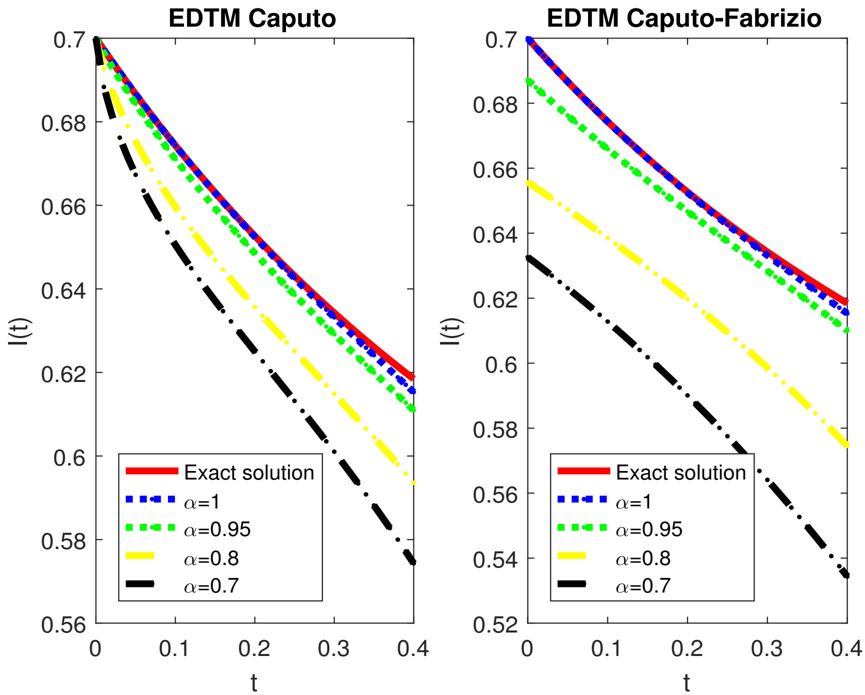

FIGURE 2. 2D plots graphs of the 4−term approximate solutions and exact solutions $i(t)$

7. Conclusion

In this study, a novel computational method has been proposed for solving the fractional SIS epidemic model involving the Caputo and Caputo-Fabrizio fractional derivatives. This method called, the Elzaki differential transform method (EDTM). The numerical example presented in this paper illustrates the simplicity, precision and effciency of the proposed method. In addition, this method gives the solutions in the form of infinite series that rapidly converge to the exact solutions. Finally, based on

TABLE 3. The numerical values of the exact solution $i(t)$ and 4−term approximate solutions by Caputo-EDTM for different values of fractional order α

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.95$	$\alpha = 1$	i_{exact}	$\lvert i_{exact} - i_{C-EDTM} \rvert$
0.01	0.68834	0.69266	0.69645	0.69722	0.69722	1.6117×10^{-9}
0.02	0.68163	0.68749	0.69321	0.69450	0.69450	2.5558×10^{-8}
0.03	0.67619	0.68302	0.69013	0.69182	0.69182	1.2834×10^{-7}
0.04	0.67150	0.67898	0.68716	0.68919	0.68919	4.0241×10^{-7}
0.05	0.66731	0.67526	0.68429	0.68661	0.68660	9.7480×10^{-7}

TABLE 4. The numerical values of the exact solution $i(t)$ and 4−term approximate solutions by Caputo-Fabrizio-EDTM for different values of fractional order α

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.95$	$\alpha = 1$	i_{exact}	$\lvert i_{exact} - i_{CF-EDTM} \rvert$
0.01	0.63077	0.65399	0.68489	0.69722	0.69722	1.6117×10^{-9}
0.02	0.62887	0.65233	0.68267	0.69450	0.69450	2.5558×10^{-8}
0.03	0.62696	0.65065	0.68049	0.69182	0.69182	1.2834×10^{-7}
0.04	0.62501	0.64896	0.67833	0.68919	0.68919	4.0241×10^{-7}
0.05	0.62305	0.64726	0.67620	0.68661	0.68660	9.7480×10^{-7}

this study, we conclude that the proposed method is a powerful and effective mathematical tool for studing a wide range of real problems arising in engineering and natural sciences, which can be modeled by a fractional differential equations.

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