Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 31, 2004, Pages 85–93
ISSN: 1223-6934

Existence of parabolic orbits for the restricted three-body problem

Chouhaïd Souissi

Abstract. In this paper, we show, using a variational formulation, the existence of Parabolic or homoclinic orbits at infinity of the restricted three-body problem.

$$\ddot{z}(t) + \alpha \frac{z}{(z(t)^2 + r^2)^{\frac{\alpha}{2}+1}} = 0.$$

For this, we prove the existence of a minimax critical level of functionals defined on the spaces of periodic functions H^1_{2mT}, we get a sequence $(z_{\alpha,m})_{m\in\mathbb{N}}$. By the Ascoli Theorem, we prove that $(z_{\alpha,m})$ converges to a parabolic orbit.

2000 Mathematics Subject Classification. Primary 60J05; Secondary 60J20.
Key words and phrases. periodic, parabolic, variational formulation.

1. Introduction

The restricted three-body problem belongs to a large category of problems called the Hamiltonian systems. An orbit of this problem is said to be parabolic if two of the three bodies remain bounded while the third goes to infinity with vanishing velocity. The existence of infinitely many periodic and parabolic solutions of such systems has already been largely studied in the litterature ([8]-[11], [14]-[16], [18], [22]-[24], [29]-[32], [38]). In [26], Moser treated this problem using the geometry of the Bernoulli-Shift and the symbolic dynamics. In his method, one has to prove the existence of homoclinic points, which needs the verification of the transversality of stable and unstable manifolds near the hyperbolic fixed points of the systems.

Sitnikov [35] proved the existence of solutions said to be oscillatory, i.e. which correspond to motions in which two bodies remain bounded while the third limits to infinity with zero velocity. He was based on symmetry conditions. In [1], Alekseev treated the problem in a more general point of view using the theory of Poincaré [27]. Than came R.Mc Gehee [20] who studied the application of Poincaré associated to a periodic orbit of Sitnikov's exemple. He showed that the intersection of stable and unstable manifolds to a degenerated fixed point, is an homoclinic point. And nearby we can remark the behaviour of the oscillatory solutions. This result is a generalization of the theory of Slotnik [36] dealing with stable manifolds near degenerated fixed points.

Many authors have been interested in the research of periodic orbits for N-body type problems ([14], [15], [22]-[24], [29], [38] ...) and especially those of the restricted three-body problem ([3], [5], [6], [21], [33] -[35] ...). In the last few years, and precisely from the work of Rabinowitz [28], some of them focused their works on the new and powerful tool consisting on variational methods. In fact, we can find many references based on this tool, one can cite for example [2], [4], [10], [14]- [17], [20], [21], [37] ...

Received: November 11, 2004.

However, there have been less interest in the looking for homoclinic orbits, and lesser for the research of parabolic ones. We can cite, as examples the works of A. Bahri and P.H. Rabinowitz [5] which was revisited by H. Riahi ([29] -[31]), and those of E. Séré [32], V. Coti-Zelati and I. Ekeland [17]...

In these methods, the existence of homoclinic orbits and hyperbolic fixed points is no longer studied. We avoid by the way to look for the intersection or to prove the transversality of manifolds. Thus, we transform the geometric approach to solve this problem to a purely analytic one.

This paper is organized as follows: in section 2, we set the problem in a general frame, then we introduce the particular case studied in this work. Section 3 treats the case $0 < \alpha < 1$ and contains two steps: in the first one, we prove the existence of a series of minimax critical levels of functionals defined on the spaces of periodic functions H^1_{2mT}: we get a sequence $(z_{\alpha,m})_{m\in\mathbb{N}}$. In the second step, using the Ascoli-Arzelà Theorem, we prove that $(z_{\alpha,m})$ converges to a parabolic orbit. Finally, in section 4, taking a sequence $(\alpha_n)_{n\in\mathbb{N}} \subset]0,1[$, and making it converge to 1, we obtain the response for $\alpha = 1$.

2. Problem setting

In this paper, we study the existence of parabolic solutions of the restricted three-body problem. We are interested in the configuration studied by Moser [26], Sitnikov [35] and Mathlouthi [21]. We consider two mass points $m_1 = m_2 > 0$, moving in the plane under Newton's attraction low in the elliptic orbits such that their center of mass O is at rest. We consider a third mass point m moving on the line perpendicular to the plane containing m_1 and m_2, and going through O. We also suppose that m does not influence the motion of m_1 and m_2. Let z be the coordinate describing the motion of m, so that $z = 0$ corresponds to O. The restricted three-body problem consists on determining z such that

$$\ddot{z} + V'(z) = 0. \tag{1}$$

where

$$V(z) = -\frac{1}{(z^2+r^2)^{\alpha/2}}. \qquad 0 < \alpha \leq 1,$$

and $r(t), t \in \mathbb{R}$ is the distance from the center of mass to anyone of the first two mass points. In the papers [21], [26] and [35], it was considered that

$$r(t) = \frac{1}{2}(1 - \varepsilon cost) + O(\varepsilon^2),$$

$r : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, T-periodic, $T > 0$, and $r(t) > 0$, for all $t \in \mathbb{R}$, and the existence of a periodic solution was proved. In this work, we consider the autonomous case which corresponds to $\varepsilon = 0$, and we take $r(t) = r > 0$. So that m_1 and m_2 will have circular trajectories of constant radius r.

Definition 2.1. (Parabolic orbit)
A solution z of problem 1 is said to be a parabolic orbit (or homoclinic to infinity) if it is of class C^2 on $\mathbb{R}$ and satisfies

$$\begin{aligned} &(i) \lim_{|t|\to+\infty} \|z(t)\| = \infty, \\ &(ii) \lim_{|t|\to+\infty} \|\dot{z}(t)\| = 0. \end{aligned} \tag{2}$$

The main result of this paper is the following

Theorem 2.1. *For $0 < \alpha \leq 1$, problem 1 has at least one parabolic orbit.*

3. Variational formulation

3.1. Step1. In this paragraph, we treat the case $0 < \alpha < 1$. For this, we consider the initial conditions

$$z(-mT) - z(mT) = \dot{z}(-mT) - \dot{z}(mT) = 0.$$

where $T > 0$ and $r > 0$. We denote by

$$H^1_{2mT} = \left\{ z \in H^1_{loc}(\mathbb{R}, \mathbb{R}) \quad ; z(-mT) = z(mT) \right\},$$

the space equipped with norm

$$\|z\|^2 = \int_{-mT}^{mT} \left(|z(t)|^2 + |\dot{z}(t)|^2 \right) dt,$$

and we define on H^1_{2mT} the functional

$$f_m(z) = \int_{-mT}^{mT} \left(\frac{1}{2} |\dot{z}(t)|^2 + \frac{1}{(z(t)^2 + r^2)^{\frac{\alpha}{2}}} \right) dt.$$

It's easy to prove that $f_m \in C^2 \left(H^1_{2mT}, \mathbb{R} \right)$ and to see that the solutions of the system are the critical points of the functional f_m. On the other hand, Palais-Smale condition $(PS)_c$ holds at $c > 0$. We recall the following result given in [21]

Lemma 3.1. *a) The functional f_m is even and satisfies the Palais-Smale condition at every level $c > 0$.*

b) If z is a solution of (1), then $z(-t)$, $-z(t)$ and $-z(-t)$ are too.

Let Σ denote the class of sets $A \subset H^1_{2mT} \setminus \{0\}$ such that A is closed in H^1_{2mT} and symmetric with respect to 0. In what follows we define:

Definition 3.1. (Cogenus and Genus)
For $A \in \Sigma$ we define the cogenus of A to be

$$\gamma^-(A) = \inf \left\{ n \in \mathbb{N} / \exists \phi : A \to S^{n-1} \quad \text{odd and continuous} \right\},$$

and the genus to be

$$\gamma^+(A) = \sup \left\{ n \in \mathbb{N} / \exists \phi : S^{n-1} \to A \ \text{ odd and continuous} \right\}.$$

When such a ϕ doesn't exist, we set $\gamma^\pm(A) = \infty$

We recall in this proposition some of the properties of the cogenus. For more details, the reader can be referred, for example, to [4], [7] and [12].

Proposition 3.1. *(i) If $z \neq 0$, $\gamma^\pm(\{z\} \cup \{-z\}) = 1$.*

(ii) If Z is a k-dimensional subspace of H^1_{2mT} and $S = \{z \in Z / \|z\| = r\}$, $r > 0$, then, $\gamma^\pm(S) = k$.

(iii) If Z is a subspace of H^1_{2mT} of codimension k and $\gamma^\pm(A) > k$, then $A \cap Z \neq \emptyset$.

For all $k \in \mathbb{N}$, we set

$$\gamma_k = \left\{ A \in \Sigma, A \text{ is compact}, \gamma^-(A) \geq k \right\},$$

and

$$c_k(m) = \inf_{A \in \gamma_{k+1}} \sup_{z \in A} f_m(z).$$

Then we have

$$c_k(m) \leq c_{k+1}(m).$$

Remark 3.1. *(i) If f_m satisfies the Palais-Smale condition at the level $c_k(m)$, then, using the deformation lemma, we prove that $c_k(m)$ is a critical level of f_m.*

(ii) If $c_k(m) = c_{k+1}(m) < f_m(0)$ and f_m satisfies the Palais-Smale condition at the level $c_k(m)$, then $f_m^{-1}(\{c_m\})$ contains infinitely many distinct critical points.

(iii) $0 = c_0(m) < c_1(m)$.

(iv) $c_{2k_1}(m) < f_m(0)$, *where* $k_1 = \max \left\{ k \in \mathbb{N}/k < \dfrac{mT}{\pi r} \right\}$.

For more details about these remarks, the reader can be referred, for example, to [4], [7], [12] and [21] and the references therein.

Next, we set

$$c_m = c_1(m) = \inf_{A \in \gamma_2} \sup_{z \in A} f_m(z).$$

We obtain the following result

Lemma 3.2. *There exist two constants $c > 0$ and $0 < \theta < 1$, independent of m such that*

$$c_m \leq c m^\theta.$$

Proof. Let $\frac{1}{2} < \beta < 1$, we consider the subspace of H^1_{2mT} generated by the $2mT-$periodic functions

$$\varphi_\beta(t) = |t|^\beta, \qquad \forall t \in [-mT, mT]$$
$$\psi_\beta(t) = (mT - t)^\beta, \qquad \forall t \in [0, mT], \qquad \text{and even}$$

and we set

$$K = \left\{ z \in H^1_{2mT} \ ; \ z = \varepsilon_1 \tau \varphi_\beta + \varepsilon_2 (1-\tau) \psi_\beta, \ \tau \in [0,1] \text{ and } \varepsilon_1, \varepsilon_2 \in \{-1, 1\} \right\}.$$

The set K is compact, symmetric not containing zero, homomorphic to the subset S^1 of H^1_{2mT}. This implies that the cogenus of K is equal to 2. Then,

$$c_m \leq \max_{z \in K} f_m(z).$$

Indeed, we have

$$\dot{z}_{\tau,\varepsilon,\varepsilon_1} = \begin{cases} \beta \varepsilon \tau t^{\beta-1} - \beta(1-\tau)\varepsilon_1 (mT-t)^{\beta-1} & \text{if} \quad t \in]0, mT[\,, \\ \\ -\beta \varepsilon \tau t^{\beta-1} + \beta(1-\tau)\varepsilon_1 (mT-t)^{\beta-1} & \text{if} \quad t \in]-mT, 0[\,, \end{cases}$$

Then

$$\begin{aligned} \int_{-mT}^{mT} |\dot{z}_{\tau,\varepsilon,\varepsilon_1}|^2 dt &\leq (mT)^{2\beta-1} \left[\frac{2\beta^2}{2\beta-1} (\tau^2 + (1-\tau)^2) + 4\beta(1-\tau)\tau \frac{\Gamma(2\beta)}{\Gamma(\beta)^2} \right] \\ &\leq (mT)^{2\beta-1} \left[\frac{2\beta^2}{2\beta-1} + \beta \frac{\Gamma(2\beta)}{\Gamma(\beta)^2} \right], \end{aligned}$$

where $\Gamma(t) = \int_0^{+\infty} s^{t-1}e^{-s}ds$. Moreover, we have

$$\int_0^{mT} \frac{dt}{\left(z^2_{\tau,\varepsilon,\varepsilon_1}(t) + r^2\right)^{\frac{\alpha}{2}}} \leq (mT)^{1-\alpha\beta} \int_0^1 \frac{dt}{|\tau t^\beta - (1-\tau)(1-t)^\beta|^\alpha}.$$

For $\tau \in]0,1[$, we have

$$\int_0^{\frac{1}{2}} \frac{1}{|\tau t^\beta - (1-\tau)(1-t)^\beta|^\alpha} dt \leq$$

$$\leq \begin{cases} \dfrac{2^{\alpha\beta}}{\beta}\left[\dfrac{(1-\tau)^{-\alpha+1} - (1-2\tau)^{-\alpha+1}}{\tau(1-\alpha)}\right], & \text{if} \quad 0 < \tau \leq \dfrac{1}{2}. \\ \dfrac{2^{\alpha\beta}}{\beta}\left[\dfrac{(1-\tau)^{-\alpha+1} + (-1+2\tau)^{-\alpha+1}}{\tau(1-\alpha)}\right], & \text{if} \quad \dfrac{1}{2} \leq \tau < 1. \end{cases}$$

In the other hand,

$$\int_{\frac{1}{2}}^1 \frac{dt}{|\tau t^\beta - (1-\tau)(1-t)^\beta|^\alpha} \leq$$

$$\leq \begin{cases} \dfrac{2^{\alpha\beta}}{\beta}\left[\dfrac{\tau^{-\alpha+1} + (1-2\tau)^{-\alpha+1}}{(1-\tau)(1-\alpha)}\right], & \text{if} \quad 0 < \tau \leq \dfrac{1}{2}. \\ \dfrac{2^{\alpha\beta}}{\beta}\left[\dfrac{\tau^{-\alpha+1} - (-1+2\tau)^{-\alpha+1}}{(1-\tau)(1-\alpha)}\right], & \text{if} \quad \dfrac{1}{2} \leq \tau < 1. \end{cases}$$

Then, by Taylor expansion in 0 and 1, we deduce that

$$\int_0^{mT} \frac{dt}{\left(z^2_{\tau,\varepsilon,\varepsilon_1}(t) + r^2\right)^{\frac{\alpha}{2}}} \leq c(mT)^{1-\alpha\beta},$$

where c is a constant depending on β and α. Then, taking

$$\theta = max(2\beta - 1, 1 - \alpha\beta),$$

we deduce that for all $\tau \in]0,1[$, we have $c_m \leq cm^\theta$. Hence the lemma follows. □

If $z_{\alpha,m}$ is a critical point corresponding to the critical level c_m, then we have

Lemma 3.3. $\|z_{\alpha,m}\|_\infty \longrightarrow +\infty$ *when* $m \longrightarrow +\infty$.

Proof. We have,

$$\frac{2mT}{(\|z_{\alpha,m}\|^2_\infty + r^2)^{\frac{\alpha}{2}}} = \int_{-mT}^{mT} \frac{dt}{(\|z_{\alpha,m}\|^2_\infty + r^2)^{\frac{\alpha}{2}}} \leq f_m(z_{\alpha,m}) \leq cm^\theta.$$

This implies

$$\|z_{\alpha,m}\|^2_\infty \geq \frac{1}{c} m^{\frac{2(1-\theta)}{\alpha}} - r^2 \longrightarrow +\infty \quad \text{when} \quad m \longrightarrow +\infty.$$

□

Remark 3.2. *If $z_{\alpha,m}$ is a solution of* (1)*, then $z_{\alpha,m}$ has at least one zero t_{0m} in $[-mT, mT]$. Then, there exists $n(m) \in \mathbb{Z}$ such that $n(m)T \leq t_{0m} < (n(m)+1)T$. We set*

$$\bar{z}_{\alpha,m}(t) = z_{\alpha,m}(t + n(m)T),$$

then $\bar{z}_{\alpha,m}$ is a solution of 1*, having a zero in $[0, T]$. In the following, for simplicity, we denote $\bar{z}_{\alpha,m}$ by $z_{\alpha,m}$.*

Lemma 3.4. *$(\dot{z}_{\alpha,m})_{m\in\mathbb{N}}$ is uniformly bounded.*

Proof. Let $m \in \mathbb{N}$ and $z_{\alpha,m}$ be a solution of 1. We take the energy function

$$E_m(t) = \frac{1}{2}|\dot{z}_{\alpha,m}(t)|^2 - \frac{1}{\left(z_{\alpha,m}^2 + r^2\right)^{\frac{\alpha}{2}}}. \tag{3}$$

The derivative of E_m with respect to t is such that $E_m'(t) = 0$ for all $t \in \mathbb{R}$ and for each solution $z_{\alpha,m}$ of (1). This implies that $E_m(t) = C(m)$.

We consider an extremum t_m of $z_{\alpha,m}$. Then we have

$$\dot{z}_{\alpha,m}(t_m) = 0.$$

So that we have

$$|C(m)| = \frac{1}{(z_{\alpha,m}^2(t_m) + r^2)^{\frac{\alpha}{2}}} \leq \frac{1}{r^\alpha}$$

From 3 we deduce

$$|\dot{z}_{\alpha,m}(t)|^2 \leq \frac{2}{\left(z_{\alpha,m}^2(t) + r^2\right)^{\frac{\alpha}{2}}} + 2|C(m)| \leq \frac{4}{r^\alpha}$$

□

PROOF OF THE THEOREM FOR $0 < \alpha < 1$

As $z_{\alpha,m}$ vanishes on $[0, T]$, for all $m \in \mathbb{N}$, Lemma 3 implies that $(z_{\alpha,m})_{m\in\mathbb{N}}$ is uniformly bounded on any compact set of $\mathbb{R}$. From 3.4, we deduce that $(z_{\alpha,m})_{m\in\mathbb{N}}$ is uniformly equicontinues on $\mathbb{R}$. Then applying the Ascoli-Arzelà theorem, we can extract a subspace of $(z_{\alpha,m})_{m\in\mathbb{N}}$ converging uniformly on any compact set of $\mathbb{R}$ to a limit z_α, which is a solution of (1). z_α is also of class C^2 on $\mathbb{R}$. On the other hand, it satisfies the energy equation

$$E = \frac{1}{2}|\dot{z}_\alpha|^2 - \frac{1}{(z_\alpha^2 + r^2)^{\frac{\alpha}{2}}} = 0$$

Which implies that

$$\frac{1}{2}|\dot{z}_\alpha|^2 = \frac{1}{(z_\alpha^2 + r^2)^{\frac{\alpha}{2}}} > 0. \tag{4}$$

So we can deduce that z_α is strictly increasing or strictly decreasing on $\mathbb{R}$. We suppose that z_α is bounded. So that there exists a constant $B > 0$ such that $|z_\alpha(t)| < B$ for all $t \in \mathbb{R}$. By 4, there exists $\gamma > 0$ such that $|\dot{z}_\alpha(t)| > \gamma$, for all $t \in \mathbb{R}$, which implies:

$$|z_\alpha(t) - z_\alpha(0)| = |\int_0^t \dot{z}_\alpha(t)dt| > \gamma|t| \to \infty \ \ when \ \ t \to \infty$$

This result contradicts the hypothesis "z_α is bounded". So that: $z_\alpha(t) \to \infty$ when $t \to \infty$.

And again from 4 we have: $|\dot{z}_\alpha(t)| \to 0$ when $|t| \to +\infty$.
It follows that z_α is a parabolic orbit of problem 1.

3.2. Step2.

Lemma 3.5. *For $0 < \alpha < 1$, the solution z_α vanishes on $[0, T]$.*

Proof. We suppose that z_α doesn't vanish on $[0, T]$. Than z_α doesn't change sign on $[0, T]$. Assume, for example, that $z_\alpha(t) > 0$, for all $t \in [0, T]$ (the case "$z_\alpha(t) < 0$, for all $t \in [0, T]$" is similar). As z_α is continues on the compact set $[0, T]$, there exists a constant $\beta > 0$ such that $z_\alpha(t) > \beta$ for all $t \in [0, T]$. But $z_{\alpha,m}$ converges uniformly to z_α on $[0, T]$, so we deduce the existence of an integer N such that

$$m \geq N \Longrightarrow |z_{\alpha,m}(t) - z_\alpha(t)| < \frac{\beta}{2}$$

so that

$$z_{\alpha,m}(t) > \frac{\beta}{2} \qquad \forall t \in [0, T] \quad \text{and} \quad \forall m \geq N$$

It follows that $z_{\alpha,m}(t) \neq 0 \;\; \forall t \in [0, T]$. A contradiction. $\square$

PROOF OF THE THEOREM FOR $\alpha = 1$

Let $(\alpha_n)_{n\in\mathbb{N}}$ be an increasing sequence of $]0, 1[$ which converges to 1. For each $n \in \mathbb{N}$, we denote by $z_n = z_{\alpha_n}$ the solution, constructed below, of the problem

$$\begin{cases} \ddot{z_n}(t) + \alpha_n \dfrac{z_n(t)}{(z_n(t)^2 + r^2)^{\frac{\alpha_n}{2}+1}} & = & 0, \\ \lim\limits_{|t|\to\infty} |z_n(t)| & = & +\infty, \\ \lim\limits_{|t|\to\infty} |\dot{z}_n(t)| & = & 0. \end{cases} \tag{5}$$

Applying 4 for z_n, we obtain

$$|\dot{z}_n(t)|^2 = \frac{2}{(z_n^2(t) + r^2)^{\frac{\alpha}{2}}} \leq \frac{2}{r^\alpha} \qquad \forall t \in \mathbb{R}.$$

which implies that $(\dot{z}_n)_n$ is uniformly bounded on $\mathbb{R}$.

As z_n vanishes on $[0, T]$, for all $n \in \mathbb{N}$, we can deduce that $(\dot{z}_n)_n$ is uniformly bounded on every compact set of $\mathbb{R}$ and uniformly equicontinues on $\mathbb{R}$. It follows that we can again apply the Ascoli-Arzelà Theorem (as we did with the sequence $(z_{\alpha,m})_{m\in\mathbb{N}}$ in Step 1) and extract from z_n a subsequence that is uniformly converging on any compact set of $\mathbb{R}$ to a function z_1 which is a solution of problem 1 for $\alpha = 1$.

Arguing as in the previous step, we deduce by 4, as $|t| \longrightarrow \infty$, that we have

$$\begin{cases} |z_1(t)| \longrightarrow \infty \\ \dot{z}_1(t) \longrightarrow 0 \end{cases}$$

So that z_1 is a parabolic orbit of problem 1.

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(Chouhaïd Souissi) Department de Mathematiques, Faculté des Sciences de Monastir, Route de Kairouan, Monastir 5000, Tunisie, Tel. 0021697238557 & Fax: 0021673500278
E-mail address: chsouissi@yahoo.fr