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Edge resolving number of pentagonal circular ladder

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Abstract. Let $G = G(V, E)$ be a non-trivial simple connected graph. The length of the shortest path between two vertices p and q, represented by $d(p, q)$, is called the distance between the vertices p and q. The distance between an edge $\varepsilon = pq$ and a vertex r in G is defined as $d(\varepsilon, r) = \min\{d(p, r), d(q, r)\}$. If $d(r, p) \neq d(r, q)$, then the vertex r is said to distinguish (resolve or recognize) two elements (edges or vertices) $p, q \in V \cup E$. The minimum cardinality of a subset R (R_e) of vertices such that all other vertices (edges) of the graph G are uniquely determined by their distances to the vertices in R (R_e) is the metric dimension (edge metric dimension) of a graph G. In this article, we consider a family of pentagonal circular ladder (P_m) and investigate its edge metric dimension. We show that, for P_m the edge metric dimension is strictly greater than its metric dimension. Additionally, we answer a problem raised in the recent past, regarding the edge metric dimension of a family of a planar graph R_m (exists in the literature).

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1. Introduction

The problem of metric dimension was initiated in the seventies by Slater [16] and Harary & Melter [4] independently. Suppose $G = (V, E)$ to be a simple non-trivial connected graph. The length of the shortest path between any two vertices p and q, denoted by $d(p, q)$, is called the distance between vertices p and q. The totality of edges that are incident to a vertex of G is known as its degree (valency).

An ordered subset $R \subseteq V$ of distinct vertices is said to be a *resolving set* if every pair of different vertices of G are resolved by at least one vertex of R. In other words, for a subset of vertices, $R = \{x_1, x_2, x_3, ..., x_k\}$ of G, any vertex $\beta \in V$ can be represented uniquely in the form of a k-vector $\gamma(\beta|R) = (d(x_1, \beta), d(x_2, \beta), d(x_3, \beta), ..., d(x_k, \beta))$. Then, the set R is a resolving set for G, if $\gamma(p|R) = \gamma(q|R)$ implies that $p = q$ for all $p, q \in V$. Next, the resolving set R is said to be the *metric basis* for G, if the set R has the least possible cardinality in G, and this least cardinality is known as the *metric dimension* (*location number*) of G, represented by $dim(G)$. A subset R of distinguishable vertices in G is said to be an independent resolving set for G, if R is independent as well as resolving set.

For a subset $R = \{x_1, x_2, x_3, ..., x_k\}$ of distinct ordered vertices in G, the q^{th} component (distance coordinate) of $\gamma(x|R)$ is zero if and only if $x = x_l$. Therefore, in order to check that the set R is a resolving set in G, it is sufficient to prove that $\gamma(p|R) \neq \gamma(q|R)$ for each pair of distinct vertices $p, q \in V(G) \setminus R$.

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The first paper consisting of the concepts of resolving set and that of minimum resolving set were introduced by Slater [16], in association with the problem of recognizing the location of a thief or an intruder in a given network. He used the terms *location number* and *locating set*, to describe the cardinality of a minimum resolving set and a resolving set of a given network, respectively. Harary and Melter [4] independently introduced the same concept, but used the terms resolving set and metric dimension, rather than locating set and location number as used by Slater, respectively. After these introductory papers, varieties of distinct resolving sets with different properties have also been presented, such as strong metric dimension, fault-tolerant metric dimension, local metric dimension, independent resolving sets, resolving dominating sets, and many others.

The resolving set and metric dimension for a graph provide some useful information regarding the vertices of the graph, it is quite natural to ask if there is any other graph invariant or parameter, which deals with the edges of the graphs in a similar way as the resolving set for the graph. Answer to that important question was put forward by Kelenc et al. [7], where the authors introduced the concept of edge metric dimension of graphs. Firstly, they defined the distance between a vertex $p \in V$ and an edge $\varepsilon = ab$ in the following manner:

$$d(\varepsilon, p) = min\{d(a,p), d(b,p)\}$$

For a subset of vertices, $R_e = \{x_1, x_2, x_3, ..., x_k\}$ of G, any edge $e \in E$ can be represented uniquely in the form of a k-vector

$$\gamma'(e|R_e) = (d(x_1,e), d(x_2,e), d(x_3,e), ..., d(x_k,e)).$$

Then, the set R_e is an edge resolving set for G, if $\gamma'(e_1|R_e) = \gamma'(e_2|R_e)$ implies that $e_1 = e_2$ for all $e_1, e_2 \in E$. Next, the edge resolving set R_e is said to be the *edge metric basis* for G, if the set R_e has the least possible cardinality in G, and this least cardinality is known as the *edge metric dimension* of G, represented by $edim(G)$. A subset R_e of distinguishable vertices in G is said to be an independent edge resolving set for G, if R is independent as well as edge resolving set. Some results regarding the metric dimension and the edge metric dimension are as follows:

Proposition 1. [7] For every positive integer $m \geq 3$, $edim(P_m) = dim(P_m) = 1$ (P_m is a path on m vertices), $dim(K_m) = edim(K_m) = m - 1$ (K_m is a complete graph on m vertices), and $dim(C_m) = edim(C_m) = 2$ (C_m is a cycle on m vertices).

Afterward, these concepts were studied in-depth, Melter and Tomescu [9] employed the concept of metric dimension in image processing and pattern recognition, Sebo and Tannier [12] discussed the notion of metric dimension in terms of combinatorial optimization, Cáceres et al. [2] employed these concepts on coin weighing problems and mastermind games, Khuller et al. [8] found an application of metric dimension in the navigation of robots, Beerloiva et al. [1] discussed these ideas to network discovery and verification, Chartrand et al. [3] studied applications to chemical science, Slater [16] discussed problems related to SONAR (Sound Navigation and Ranging), coastguard LORAN (Long-Range Navigation), facility location problems, etc.

In elementary geometry, a polytope is a geometric object with flat sides. Convex polytopes are defined as polytopes with the extra property of being convex sets and being enclosed in the m-dimensional space $\mathbb{R}^m$ (Euclidean space). Convex polytopes play an important role in a variety of disciplines of mathematics as well as in applied areas, most notably in linear programming. For several distinct classes of convex polytopes, the concept of metric and edge metric dimension have been discussed in [5, 6, 10, 13, 14, 15, 18].

In this paper, we determine the edge metric dimension for a rotationally symmetrical family of planar graphs P_m ([5], pentagonal circular ladder), known from the literature. We show that for the pentagonal circular ladder P_m, the edge metric dimension is strictly greater than its metric dimension i.e., $edim(P_m) > dim(P_m)$, for every $m \geq 3$. We also give an answer to a problem raised by Raza et al. [11] regarding the edge metric dimension of the family of planar graph R_m (exists in the literature), and we prove for this family R_m, that $edim(R_m) = 5$. Moreover, for both of these families of planar graphs, we show that the edge metric basis sets R_e are independent.

Next, we give some known outcomes regarding the metric dimension of the two aforementioned families of the planar graphs (viz., P_m and R_m), which are as follows:

Theorem 1. [5] Let P_m be the graph of pentagonal circular ladder. Then, $dim(P_m) = 2$ for every $n \geq 6$.

Theorem 2. [11] Let R_m be the rotationally symmetric planar graph. Then, $dim(R_m) = 3$ for every $n \geq 6$.

Throughout this article, all vertex indices are taken to be modulo m. The present paper is organized as follows: In section 2, we study the edge metric dimension of the pentagonal circular ladder P_m (see Fig. 1 and 2). In section 3, we study the edge metric dimension of the rotationally symmetrical planar graph R_m (see Fig. 3), and in our last section, we conclude our results and findings regarding the aforementioned families of the planar graphs.

2. Edge metric dimension of pentagonal circular ladder

The planar graph P_m [5], comprises of $3m$ number of vertices and $4m$ number of edges. It has n faces each having five sides (pentagonal faces) and two n-sided faces, as shown in Fig. 2. We represent the set of vertices and edges for the pentagonal circular ladder P_m as $V(P_m)$ and $E(P_m)$, respectively. The sets $V(P_m)$ and $E(P_m)$ are as follows:

$$V(P_m) = \{p_l, q_l, r_l : 1 \leq l \leq m\}$$

and

$$E(P_m) = \{p_l q_l, q_l r_l, p_l p_{l+1}, r_l q_{l+1} : 1 \leq l \leq m\}$$

We name the set $A = \{p_l : 1 \leq l \leq m\}$ of vertices in P_m, as the p-vertices, the set $B = \{q_l : 1 \leq l \leq m\}$ of vertices in P_m, as the q-vertices, and the set $C = \{r_l : 1 \leq l \leq m\}$ of vertices in P_m, as the r-vertices. For our purpose, we take $p_{m+1} = p_1$, $r_{m+1} = r_1$, and $q_{m+1} = q_1$ (whenever necessary). In this section, we study the notion of edge metric dimension for the pentagonal circular ladder P_m. We prove that $edim(P_m) = 3$ for $3 \leq m \leq 14$ and $edim(P_m) = \lceil \frac{m}{6} \rceil$ for $m \geq 15$. Additionally, we prove that an edge resolving set R_e for P_m is independent. Next, we have the following result regarding the edge metric dimension of P_m.

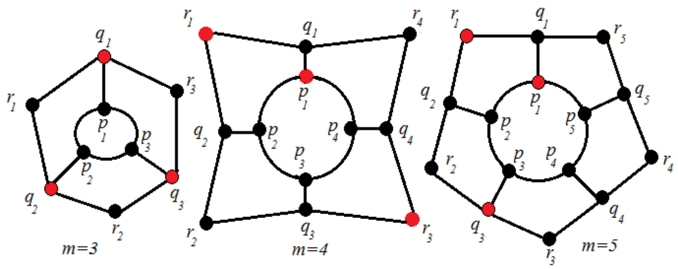

FIGURE 1. Pentagonal circular ladder P_m, for $m = 3, 4, 5$.

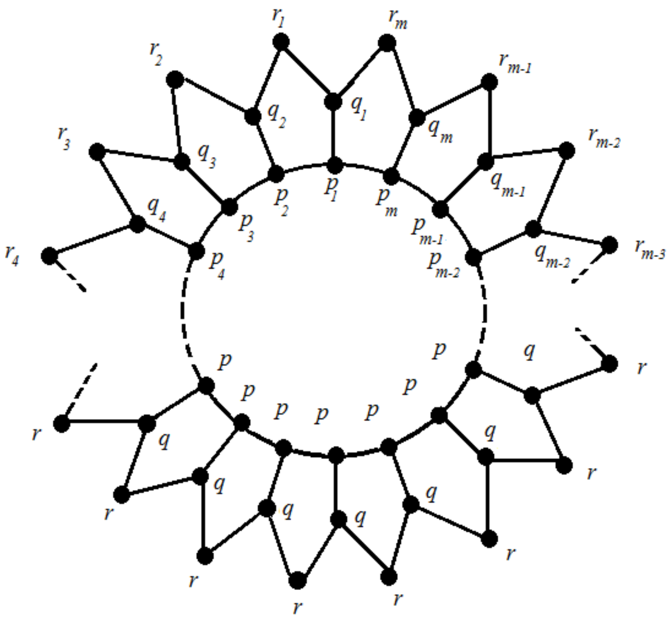

FIGURE 2. Pentagonal circular ladder P_m, for $m \geq 6$.

Lemma 1. The set $A \subset V(P_m)$ of p-vertices is not an edge resolving set for P_m.

Proof. Suppose on the contrary, that the set A of p-vertices is an edge resolving set for the planar graph P_m. Then, from Fig. 2, we find that $\gamma'(q_l r_l|A) = \gamma'(q_l r_{l-1}|A)$, for all $1 \leq l \leq m$, a contradiction. $\square$

Lemma 2. Suppose $R_e \subset V(P_m)$ has vertices only from the set $B = \{q_l : 1 \leq l \leq m\}$ $(C = \{r_l : 1 \leq l \leq m\})$. Let $d(a, b) \geq 10$, for every $a, b \in R_e$. Then, R_e is not an edge resolving set for P_m for $m \geq 15$.

Proof. Suppose on the contrary, that the set R_e with the above said properties is an edge resolving set for P_m. Then, without loss of generality, let $a = r_2$ and $b = r_9$, then $\gamma'(q_6 r_6|R_e) = \gamma'(q_5 r_6|R_e)$, a contradiction. $\square$

Lemma 3. The cardinality of any edge resolving set R_e for P_m is $\geq \lceil \frac{m}{6} \rceil$, for every $m \geq 15$.

Proof. It is obvious from Lemma 2. □

Next, we obtain the edge metric dimension for P_m; $m \geq 3$. We also investigate the property of independence in an edge resolving set for P_m. For this, we have the following result:

Theorem 3. Let P_m be the graph of pentagonal circular ladder. Then,

$$edim(P_m) = \begin{cases} 3, & \text{if } 3 \leq m \leq 14; \\ \lceil \frac{m}{6} \rceil & \text{if } m \geq 15. \end{cases}$$

Proof. For $3 \leq n \leq 14$, it is easy to check that the edge metric dimension of P_m is 3. For $m = 3, 4$, and 5 one can find that the position of the edge basis vertices (color in red) in Fig. 1, and for $6 \leq m \leq 14$ (where $m = 2s$ or $m = 2s+1$; $s \in \mathbb{N}$) the set of vertices $R_e = \{r_2, r_{s+1}, r_m\}$, is an edge resolving set for P_m. Now, for $m \geq 15$, we have following cases to be considered.

Case (I) $m \equiv\ 0\ (mod\ 6)$.
Then, we write $m = 6k = 2s$, where $k, s \in \mathbb{N}$ and $k \geq 3$. First, we prove that $edim(P_m) \leq \lceil \frac{m}{6} \rceil$. For this, suppose $R_e = \{r_2, r_8, r_{14}, ..., r_{6k-10}, r_{6k-4}\} \subset V(P_m)$. We will prove that R_e is an edge resolving set with minimum cardinality for P_m. By total enumeration, one can verify easily that the set R_e is an edge resolving set with minimum cardinality for P_m, whenever k =3, 4, and 5. Next, for $k \geq 6$, we have to prove that the cardinality of minimum edge resolving set R_e for P_m is $\leq \lceil \frac{m}{6} \rceil$. For this, we show that the edge metric codes with respect to the set R_e, are distinct for every two distinct members of $E(P_m)$.

Suppose $R_e^* = \{r_2, r_8, r_{6k-4}\}$. Next, we give edge metric codes for every edge of P_m; $m \geq 15$, corresponding to the set R_e^*. The edge metric codes for the set of edges $E_1 = \{e = p_l p_{l+1} | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 1:

TABLE 1. Edge metric codes for the edges present in E_1

Edges	$\gamma'(e)$
$p_l p_{l+1}$; $l = 1$	$(2, 8, 6)$
$p_l p_{l+1}$; $l = 2$	$(2, 7, 7)$
$p_l p_{l+1}$; $3 \leq l \leq 7$	$(l-1, 9-l, l+5)$
$p_l p_{l+1}$; $l = 8$	$(7, 2, 13)$
$p_l p_{l+1}$; $9 \leq l \leq s-4$	$(l-1, l-7, l+5)$
$p_l p_{l+1}$; $s-3 \leq l \leq s+2$	$(l-1, l-7, 6k-l-3)$
$p_l p_{l+1}$; $s+3 \leq l \leq s+8$	$(6k-l+3, l-7, 6k-l-3)$
$p_l p_{l+1}$; $s+9 \leq l \leq 6k-5$	$(6k-l+3, 6k-l+9, 6k-l-3)$
$p_l p_{l+1}$; $l = 6k-4$	$(6k-l+3, 6k-l+9, 2)$
$p_l p_{l+1}$; $6k-3 \leq l \leq 6k$	$(6k-l+3, 6k-l+9, l-6k+5)$

The edge metric codes for the set of edges $E_2 = \{e = p_l q_l | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 2:

TABLE 2. Edge metric codes for the edges present in E_2

Edges	$\gamma'(e)$
p_lq_l; $l=1$	$(3,9,6)$
p_lq_l; $2\leq l\leq 3$	$(1,10-l,l+5)$
p_lq_l; $4\leq l\leq 7$	$(l-1,10-l,l+5)$
p_lq_l; $8\leq l\leq 9$	$(l-1,1,l+5)$
p_lq_l; $10\leq l\leq s-4$	$(l-1,l-7,l+5)$
p_lq_l; $s-3\leq l\leq s+2$	$(l-1,l-7,6k-l-2)$
p_lq_l; $s+3\leq l\leq s+8$	$(6k-l+4,l-7,6k-l-2)$
p_lq_l; $s+9\leq l\leq 6k-5$	$(6k-l+4,6k-l+10,6k-l-2)$
p_lq_l; $6k-4\leq l\leq 6k-3$	$(6k-l+4,6k-l+10,1)$
p_lq_l; $6k-2\leq l\leq 6k$	$(6k-l+4,6k-l+10,l-6k+5)$

The edge metric codes for the set of edges $E_3=\{e=q_lr_l|l=1,2,3,...,m\}$, with respect to the set R_e^* are listed in Table 3:

TABLE 3. Edge metric codes for the edges present in E_3

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
q_lr_l; $l=1$	$(2,10,7)$	q_lr_l; $12\leq l\leq s-4$	$(l,l-6,l+6)$
q_lr_l; $l=2$	$(0,9,8)$	q_lr_l; $s-3\leq l\leq s+2$	$(l,l-6,6k-l-1)$
q_lr_l; $l=3$	$(1,8,9)$	q_lr_l; $s+3\leq l\leq s+8$	$(6k-l+5,l-6,6k-l-1)$
q_lr_l; $l=4$	$(3,7,10)$	q_lr_l; $s+9\leq l\leq 6k-7$	$(6k-l+5,6k-l+11,6k-l-1)$
q_lr_l; $l=5$	$(5,6,11)$	q_lr_l; $l=6k-6$	$(6k-l+5,6k-l+11,4)$
q_lr_l; $l=6$	$(6,4,12)$	q_lr_l; $l=6k-5$	$(6k-l+5,6k-l+11,2)$
q_lr_l; $l=7$	$(7,2,13)$	q_lr_l; $l=6k-4$	$(6k-l+5,6k-l+11,0)$
q_lr_l; $l=8$	$(8,0,14)$	q_lr_l; $l=6k-3$	$(6k-l+5,6k-l+11,1)$
q_lr_l; $l=9$	$(9,1,15)$	q_lr_l; $l=6k-2$	$(6k-l+5,6k-l+11,3)$
q_lr_l; $l=10$	$(10,3,16)$	q_lr_l; $l=6k-1$	$(6k-l+5,6k-l+11,5)$
q_lr_l; $l=11$	$(11,5,17)$	q_lr_l; $l=6k$	$(4,6k-l+11,6)$

The edge metric codes for the set of edges $E_4=\{e=r_lq_{l+1}|l=1,2,3,...,m\}$, with respect to the set R_e^* are listed in Table 4:

Now, from these edge metric codes for the edges of P_m, corresponding to the set R_e^*, we find that $\gamma'(q_lr_l|R_e^*)=\gamma'(r_lq_{l-1}|R_e^*)$, for every $12\leq l\leq 6k-7$ (some other pair of distinct edges may also have the same edge metric codes in P_m). Then, from the remaining edge metric codes for the edges in P_m, we obtain that $\gamma'(e_1|R_e^*)\neq\gamma'(e_2|R_e^*)$ for every pair of distinct edges e_1 and e_2 in P_m, other than the same edge metric codes. Thus, for $R_e=R_e^*\cup\{r_{14},r_{20},...,r_{6k-10}\}$, we obtain that $\gamma'(q_lr_l|R_e^*)\neq\gamma'(r_lq_{l-1}|R_e^*)$, and so $\gamma'(e_1|R_e^*)\neq\gamma'(e_2|R_e^*)$ for any e_1 and e_2 in $E(P_m)$. From this, we find that $|R_e|\leq\lceil\frac{m}{6}\rceil$. Hence, $edim(P_m)\leq\lceil\frac{m}{6}\rceil$ in this case.

Case (II) $m\equiv\ 1\ (mod\ 6)$.
Then, we write $m=6k+1=2s+1$, where $k,s\in\mathbb{N}$ and $k\geq 3$. For this case, we prove that $edim(P_m)\leq\lceil\frac{m}{6}\rceil$. For this, suppose $R_e=\{r_2,r_8,r_{14},...,r_{6k-10},r_{6k-4},r_{6k+1}\}\subset V(P_m)$. We will prove that R_e is an edge resolving set with minimum cardinality for

TABLE 4. Edge metric codes for the edges present in E_4

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
r_lq_{l+1}; $l=1$	$(1,9,8)$	r_lq_{l+1}; $11 \leq l \leq s-5$	$(l+1, l-5, l+7)$
r_lq_{l+1}; $l=2$	$(0,8,9)$	r_lq_{l+1}; $s-4 \leq l \leq s+1$	$(l+1, l-5, 6k-l-2)$
r_lq_{l+1}; $l=3$	$(2,7,10)$	r_lq_{l+1}; $s+2 \leq l \leq s+7$	$(6k-l+4, l-5, 6k-l-2)$
r_lq_{l+1}; $l=4$	$(4,6,11)$	r_lq_{l+1}; $s+8 \leq l \leq 6k-7$	$(6k-l+4, 6k-l+10, 6k-l-2)$
r_lq_{l+1}; $l=5$	$(6,5,12)$	r_lq_{l+1}; $l=6k-6$	$(6k-l+4, 6k-l+10, 3)$
r_lq_{l+1}; $l=6$	$(7,3,13)$	r_lq_{l+1}; $l=6k-5$	$(6k-l+4, 6k-l+10, 1)$
r_lq_{l+1}; $l=7$	$(8,1,14)$	r_lq_{l+1}; $l=6k-4$	$(6k-l+4, 6k-l+10, 0)$
r_lq_{l+1}; $l=8$	$(9,0,15)$	r_lq_{l+1}; $l=6k-3$	$(6k-l+4, 6k-l+10, 2)$
r_lq_{l+1}; $l=9$	$(10,2,16)$	r_lq_{l+1}; $l=6k-2$	$(6k-l+4, 6k-l+10, 4)$
r_lq_{l+1}; $l=10$	$(11,4,17)$	r_lq_{l+1}; $l=6k-1$	$(6k-l+4, 6k-l+10, 6)$
		r_lq_{l+1}; $l=6k$	$(3, 6k-l+10, 7)$

P_m. By total enumeration, one can verify easily that the set R_e is an edge resolving set with minimum cardinality for P_m, whenever k =3, 4, and 5. Next, for $k \geq 6$, we have to prove that the cardinality of minimum edge resolving set R_e for P_m is $\leq \lceil \frac{m}{6} \rceil$. For this, we show that the edge metric codes with respect to the set R_e, are distinct for every two distinct members of $E(P_m)$.

Suppose $R_e^* = \{r_2, r_8, r_{6k+1}\}$. Next, we give edge metric codes for every edge of P_m; $m \geq 26$, corresponding to the set R_e^*. The edge metric codes for the set of edges $E_1 = \{e = p_lp_{l+1} | l = 1,2,3,...,m\}$, with respect to the set R_e^* are listed in Table 5:

TABLE 5. Edge metric codes for the edges present in E_1

Edges	$\gamma'(e)$
p_lp_{l+1}; $1 \leq l \leq 3$	$(2, 9-l, l+1)$
p_lp_{l+1}; $4 \leq l \leq 7$	$(l-1, 9-l, l+1)$
p_lp_{l+1}; $l=8$	$(7,2,9)$
p_lp_{l+1}; $9 \leq l \leq s$	$(l-1, l-7, l+1)$
p_lp_{l+1}; $s+1 \leq l \leq s+2$	$(l-1, l-7, 6k-l+2)$
p_lp_{l+1}; $s+3 \leq l \leq s+8$	$(6k-l+4, l-7, 6k-l+2)$
p_lp_{l+1}; $s+9 \leq l \leq 6k+1$	$(6k-l+4, 6k-l+10, 6k-l+2)$
p_lp_{l+1}; $l=6k+2$	$(6k-l+4, 6k-l+10, 2)$

The edge metric codes for the set of edges $E_2 = \{e = p_lq_l | l = 1,2,3,...,m\}$, with respect to the set R_e^* are listed in Table 6:

The edge metric codes for the set of edges $E_3 = \{e = q_lr_l | l = 1,2,3,...,m\}$, with respect to the set R_e^* are listed in Table 7:

The edge metric codes for the set of edges $E_4 = \{e = r_lq_{l+1} | l = 1,2,3,...,m\}$, with respect to the set R_e^* are listed in Table 8:

Now, from these edge metric codes for the edges of P_m, corresponding to the set R_e^*, we find that $\gamma'(q_lr_l|R_e^*) = \gamma'(r_lq_{l-1}|R_e^*)$, for every $12 \leq l \leq 6k-2$ (some other pair of distinct edges may also have the same edge metric codes in P_m). Then, from the remaining edge metric codes for the edges in P_m, we obtain that $\gamma'(e_1|R_e^*) \neq \gamma'(e_2|R_e^*)$ for every pair of distinct edges e_1 and e_2 in P_m, other than the same edge metric codes. Thus, for $R_e = R_e^* \cup \{r_{14}, r_{20}, ..., r_{6k-10}, r_{6k-4}\}$, we obtain

Table 6. Edge metric codes for the edges present in E_2

Edges	$\gamma'(e)$
p_lq_l; $l=1$	$(3,9,1)$
p_lq_l; $2\leq l\leq 3$	$(1,10-l,l+1)$
p_lq_l; $4\leq l\leq 7$	$(l-1,10-l,l+1)$
p_lq_l; $8\leq l\leq 9$	$(l-1,1,l+1)$
p_lq_l; $10\leq l\leq s$	$(l-1,l-7,l+1)$
p_lq_l; $s+1\leq l\leq s+2$	$(l-1,l-7,6k-l+3)$
p_lq_l; $s+3\leq l\leq s+8$	$(6k-l+5,l-7,6k-l+3)$
p_lq_l; $s+9\leq l\leq 6k+1$	$(6k-l+5,6k-l+11,6k-l+3)$
p_lq_l; $l=6k+2$	$(6k-l+5,6k-l+11,1)$

Table 7. Edge metric codes for the edges present in E_3

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
q_lr_l; $l=1$	$(2,10,1)$	q_lr_l; $l=10$	$(10,3,12)$
q_lr_l; $l=2$	$(0,9,3)$	q_lr_l; $l=11$	$(11,5,13)$
q_lr_l; $l=3$	$(1,8,5)$	q_lr_l; $12\leq l\leq s$	$(l,l-6,l+2)$
q_lr_l; $l=4$	$(3,7,6)$	q_lr_l; $s+1\leq l\leq s+2$	$(l,l-6,6k-l+4)$
q_lr_l; $l=5$	$(5,6,7)$	q_lr_l; $s+3\leq l\leq s+8$	$(6k-l+6,l-6,6k-l+4)$
q_lr_l; $l=6$	$(6,4,8)$	q_lr_l; $s+9\leq l\leq 6k-1$	$(6k-l+6,6k-l+12,6k-l+4)$
q_lr_l; $l=7$	$(7,2,9)$	q_lr_l; $l=6k$	$(6k-l+6,6k-l+12,4)$
q_lr_l; $l=8$	$(8,0,10)$	q_lr_l; $l=6k+1$	$(6k-l+6,6k-l+12,2)$
q_lr_l; $l=9$	$(9,1,11)$	q_lr_l; $l=6k+2$	$(4,6k-l+12,0)$

Table 8. Edge metric codes for the edges present in E_4

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
r_lq_{l+1}; $l=1$	$(1,9,2)$	r_lq_{l+1}; $l=9$	$(10,2,12)$
r_lq_{l+1}; $l=2$	$(0,8,4)$	r_lq_{l+1}; $l=10$	$(11,4,13)$
r_lq_{l+1}; $l=3$	$(2,7,6)$	r_lq_{l+1}; $11\leq l\leq s-1$	$(l+1,l-5,l+3)$
r_lq_{l+1}; $l=4$	$(4,6,7)$	r_lq_{l+1}; $s\leq l\leq s+1$	$(l+1,l-5,6k-l+3)$
r_lq_{l+1}; $l=5$	$(6,5,8)$	r_lq_{l+1}; $s+2\leq l\leq s+7$	$(6k-l+5,l-5,6k-l+3)$
r_lq_{l+1}; $l=6$	$(7,3,9)$	r_lq_{l+1}; $s+8\leq l\leq 6k-1$	$(6k-l+5,6k-l+11,6k-l+3)$
r_lq_{l+1}; $l=7$	$(8,1,10)$	r_lq_{l+1}; $6k$	$(6k-l+5,6k-l+11,3)$
r_lq_{l+1}; $l=8$	$(9,0,11)$	r_lq_{l+1}; $6k+1$	$(6k-l+5,6k-l+11,1)$
		r_lq_{l+1}; $6k+2$	$(3,6k-l+11,0)$

that $\gamma'(q_lr_l|R_e^*)\neq\gamma'(r_lq_{l-1}|R_e^*)$, and so $\gamma'(e_1|R_e^*)\neq\gamma'(e_2|R_e^*)$ for any e_1 and e_2 in $E(P_m)$. From this, we find that $|R_e|\leq\lceil\frac{m}{6}\rceil$. Hence, $edim(P_m)\leq\lceil\frac{m}{6}\rceil$ in this case as well.

Case (III) $m\equiv\ 2\ (mod\ 6)$.
Then, we write $m=6k+2=2s$, where $k,s\in\mathbb{N}$ and $k\geq 3$. For this case, we prove that $edim(P_m)\leq\lceil\frac{m}{6}\rceil$. For this, suppose $R_e=\{r_2,r_8,r_{14},...,r_{6k-10},r_{6k-4},r_{6k+2}\}\subset V(P_m)$. We will prove that R_e is an edge resolving set with minimum cardinality for P_m. By total enumeration, one can verify easily that the set R_e is an edge resolving

set with minimum cardinality for P_m, whenever k =3, 4, and 5. Next, for $k \geq 6$, we have to prove that the cardinality of minimum edge resolving set R_e for P_m is $\leq \lceil \frac{m}{6} \rceil$. For this, we show that the edge metric codes with respect to the set R_e, are distinct for every two distinct members of $E(P_m)$.

Suppose $R_e^* = \{r_2, r_8, r_{6k+2}\}$. Next, we give edge metric codes for every edge of P_m; $m \geq 15$, corresponding to the set R_e^*. The edge metric codes for the set of edges $E_1 = \{e = p_l p_{l+1} | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 9:

TABLE 9. Edge metric codes for the edges present in E_1

Edges	$\gamma'(e)$
$p_l p_{l+1}$; $1 \leq l \leq 3$	$(2, 9 - l, l + 1)$
$p_l p_{l+1}$; $4 \leq l \leq 7$	$(l - 1, 9 - l, l + 1)$
$p_l p_{l+1}$; $l = 8$	$(7, 2, 9)$
$p_l p_{l+1}$; $9 \leq l \leq s$	$(l - 1, l - 7, l + 1)$
$p_l p_{l+1}$; $s + 1 \leq l \leq s + 2$	$(l - 1, l - 7, 6k - l + 3)$
$p_l p_{l+1}$; $s + 3 \leq l \leq s + 8$	$(6k - l + 5, l - 7, 6k - l + 3)$
$p_l p_{l+1}$; $s + 9 \leq l \leq 6k + 1$	$(6k - l + 5, 6k - l + 11, 6k - l + 3)$
$p_l p_{l+1}$; $l = 6k + 2$	$(6k - l + 5, 6k - l + 11, 2)$

The edge metric codes for the set of edges $E_2 = \{e = p_l q_l | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 10:

TABLE 10. Edge metric codes for the edges present in E_2

Edges	$\gamma'(e)$
$p_l q_l$; $l = 1$	$(3, 9, 1)$
$p_l q_l$; $2 \leq l \leq 3$	$(1, 10 - l, l + 1)$
$p_l q_l$; $4 \leq l \leq 7$	$(l - 1, 10 - l, l + 1)$
$p_l q_l$; $8 \leq l \leq 9$	$(l - 1, 1, l + 1)$
$p_l q_l$; $10 \leq l \leq s$	$(l - 1, l - 7, l + 1)$
$p_l q_l$; $s + 1 \leq l \leq s + 2$	$(l - 1, l - 7, 6k - l + 4)$
$p_l q_l$; $s + 3 \leq l \leq s + 8$	$(6k - l + 6, l - 7, 6k - l + 4)$
$p_l q_l$; $s + 9 \leq l \leq 6k + 1$	$(6k - l + 6, 6k - l + 12, 6k - l + 4)$
$p_l q_l$; $l = 6k + 2$	$(6k - l + 6, 6k - l + 12, 1)$

The edge metric codes for the set of edges $E_3 = \{e = q_l r_l | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 11:

The edge metric codes for the set of edges $E_4 = \{e = r_l q_{l+1} | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 12:

Now, from these edge metric codes for the edges of P_m, corresponding to the set R_e^*, we find that $\gamma'(q_l r_l | R_e^*) = \gamma'(r_l q_{l-1} | R_e^*)$, for every $12 \leq l \leq 6k - 1$ (some other pair of distinct edges may also have the same edge metric codes in P_m). Then, from the remaining edge metric codes for the edges in P_m, we obtain that $\gamma'(e_1 | R_e^*) \neq \gamma'(e_2 | R_e^*)$ for every pair of distinct edges e_1 and e_2 in P_m, other than the same edge metric codes. Thus, for $R_e = R_e^* \cup \{r_{14}, r_{20}, ..., r_{6k-10}, r_{6k-4}\}$, we obtain that $\gamma'(q_l r_l | R_e^*) \neq \gamma'(r_l q_{l-1} | R_e^*)$, and so $\gamma'(e_1 | R_e^*) \neq \gamma'(e_2 | R_e^*)$ for any e_1 and e_2 in $E(P_m)$. From this, we find that $|R_e| \leq \lceil \frac{m}{6} \rceil$. Hence, $edim(P_m) \leq \lceil \frac{m}{6} \rceil$ in this case as

TABLE 11. Edge metric codes for the edges present in E_3

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
q_lr_l; $l=1$	$(2,10,1)$	q_lr_l; $l=10$	$(10,3,12)$
q_lr_l; $l=2$	$(0,9,3)$	q_lr_l; $l=11$	$(11,5,13)$
q_lr_l; $l=3$	$(1,8,5)$	q_lr_l; $12\le l\le s$	$(l,l-6,l+2)$
q_lr_l; $l=4$	$(3,7,6)$	q_lr_l; $s+1\le l\le s+2$	$(l,l-6,6k-l+5)$
q_lr_l; $l=5$	$(5,6,7)$	q_lr_l; $s+3\le l\le s+8$	$(6k-l+7,l-6,6k-l+5)$
q_lr_l; $l=6$	$(6,4,8)$	q_lr_l; $s+9\le l\le 6k-1$	$(6k-l+7,6k-l+13,6k-l+5)$
q_lr_l; $l=7$	$(7,2,9)$	q_lr_l; $l=6k$	$(6k-l+7,6k-l+13,4)$
q_lr_l; $l=8$	$(8,0,10)$	q_lr_l; $l=6k+1$	$(6k-l+7,6k-l+13,2)$
q_lr_l; $l=9$	$(9,1,11)$	q_lr_l; $l=6k+2$	$(4,6k-l+13,0)$

TABLE 12. Edge metric codes for the edges present in E_4

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
r_lq_{l+1}; $l=1$	$(1,9,2)$	r_lq_{l+1}; $l=9$	$(10,2,12)$
r_lq_{l+1}; $l=2$	$(0,8,4)$	r_lq_{l+1}; $l=10$	$(11,4,13)$
r_lq_{l+1}; $l=3$	$(2,7,6)$	r_lq_{l+1}; $11\le l\le s-1$	$(l+1,l-5,l+3)$
r_lq_{l+1}; $l=4$	$(4,6,7)$	r_lq_{l+1}; $s\le l\le s+1$	$(l+1,l-5,6k-l+4)$
r_lq_{l+1}; $l=5$	$(6,5,8)$	r_lq_{l+1}; $s+2\le l\le s+7$	$(6k-l+6,l-5,6k-l+4)$
r_lq_{l+1}; $l=6$	$(7,3,9)$	r_lq_{l+1}; $s+8\le l\le 6k-1$	$(6k-l+6,6k-l+12,6k-l+4)$
r_lq_{l+1}; $l=7$	$(8,1,10)$	r_lq_{l+1}; $6k$	$(6k-l+6,6k-l+12,3)$
r_lq_{l+1}; $l=8$	$(9,0,11)$	r_lq_{l+1}; $6k+1$	$(6k-l+6,6k-l+12,1)$
		r_lq_{l+1}; $6k+2$	$(3,6k-l+12,0)$

well.

Case (IV)$m\equiv\ 3\ (mod\ 6)$.
Then, we write $m=6k+3=2s+1$, where $k,s\in\mathbb{N}$ and $k\ge 3$. For this case, we prove that $edim(P_m)\le\lceil\frac{m}{6}\rceil$. For this, suppose $R_e=\{r_2,r_8,r_{14},...,r_{6k-10},r_{6k-4},r_{6k+2}\}\subset V(P_m)$. We will prove that R_e is an edge resolving set with minimum cardinality for P_m. By total enumeration, one can verify easily that the set R_e is an edge resolving set with minimum cardinality for P_m, whenever k =3, 4, and 5. Next, for $k\ge 6$, we have to prove that the cardinality of minimum edge resolving set R_e for P_m is $\le\lceil\frac{m}{6}\rceil$. For this, we show that the edge metric codes with respect to the set R_e, are distinct for every two distinct members of $E(P_m)$.

Suppose $R_e^*=\{r_2,r_8,r_{6k+2}\}$. Next, we give edge metric codes for every edge of P_m; $m\ge 33$, corresponding to the set R_e^*. The edge metric codes for the set of edges $E_1=\{e=p_lp_{l+1}|l=1,2,3,...,m\}$, with respect to the set R_e^* are listed in Table 13:

The edge metric codes for the set of edges $E_2=\{e=p_lq_l|l=1,2,3,...,m\}$, with respect to the set R_e^* are listed in Table 14:

The edge metric codes for the set of edges $E_3=\{e=q_lr_l|l=1,2,3,...,m\}$, with respect to the set R_e^* are listed in Table 15:

The edge metric codes for the set of edges $E_4=\{e=r_lq_{l+1}|l=1,2,3,...,m\}$, with respect to the set R_e^* are listed in Table 16:

Now, from these edge metric codes for the edges of P_m, corresponding to the set R_e^*, we find that $\gamma'(q_lr_l|R_e^*)=\gamma'(r_lq_{l-1}|R_e^*)$, for every $12\le l\le 6k-1$ (some

TABLE 13. Edge metric codes for the edges present in E_1

Edges	$\gamma'(e)$
p_lp_{l+1}; $1 \le l \le 3$	$(2, 9-l, l+2)$
p_lp_{l+1}; $4 \le l \le 7$	$(l-1, 9-l, l+2)$
p_lp_{l+1}; $l = 8$	$(7, 2, 10)$
p_lp_{l+1}; $9 \le l \le s-1$	$(l-1, l-7, l+2)$
p_lp_{l+1}; $s \le l \le s+1$	$(l-1, l-7, 6k-l+3)$
p_lp_{l+1}; $s+2 \le l \le s+8$	$(6k-l+6, l-7, 6k-l+3)$
p_lp_{l+1}; $s+9 \le l \le 6k+1$	$(6k-l+6, 6k-l+12, 6k-l+3)$
p_lp_{l+1}; $6k+2 \le l \le 6k+3$	$(6k-l+6, 6k-l+12, 2)$

TABLE 14. Edge metric codes for the edges present in E_2

Edges	$\gamma'(e)$
p_lq_l; $l = 1$	$(3, 9, 3)$
p_lq_l; $2 \le l \le 3$	$(1, 10-l, l+2)$
p_lq_l; $4 \le l \le 7$	$(l-1, 10-l, l+2)$
p_lq_l; $8 \le l \le 9$	$(l-1, 1, l+2)$
p_lq_l; $10 \le l \le s$	$(l-1, l-7, l+2)$
p_lq_l; $s+1 \le l \le s+3$	$(l-1, l-7, 6k-l+4)$
p_lq_l; $s+4 \le l \le s+9$	$(6k-l+7, l-7, 6k-l+4)$
p_lq_l; $s+10 \le l \le 6k+1$	$(6k-l+7, 6k-l+13, 6k-l+4)$
p_lq_l; $6k+2 \le l \le 6k+3$	$(6k-l+7, 6k-l+13, 1)$

TABLE 15. Edge metric codes for the edges present in E_3

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
q_lr_l; $l = 1$	$(2, 10, 3)$	q_lr_l; $l = 10$	$(10, 3, 13)$
q_lr_l; $l = 2$	$(0, 9, 5)$	q_lr_l; $l = 11$	$(11, 5, 14)$
q_lr_l; $l = 3$	$(1, 8, 6)$	q_lr_l; $12 \le l \le s$	$(l, l-6, l+3)$
q_lr_l; $l = 4$	$(3, 7, 7)$	q_lr_l; $s+1 \le l \le s+3$	$(l, l-6, 6k-l+5)$
q_lr_l; $l = 5$	$(5, 6, 8)$	q_lr_l; $s+4 \le l \le s+9$	$(6k-l+8, l-6, 6k-l+5)$
q_lr_l; $l = 6$	$(6, 4, 9)$	q_lr_l; $s+9 \le l \le 6k-1$	$(6k-l+8, 6k-l+14, 6k-l+5)$
q_lr_l; $l = 7$	$(7, 2, 10)$	q_lr_l; $l = 6k$	$(6k-l+8, 6k-l+14, 4)$
q_lr_l; $l = 8$	$(8, 0, 11)$	q_lr_l; $l = 6k+1$	$(6k-l+8, 6k-l+14, 2)$
q_lr_l; $l = 9$	$(9, 1, 12)$	q_lr_l; $l = 6k+2$	$(6k-l+8, 6k-l+13, 0)$
		q_lr_l; $l = 6k+3$	$(4, 6k-l+13, 1)$

other pair of distinct edges may also have the same edge metric codes in P_m). Then, from the remaining edge metric codes for the edges in P_m, we obtain that $\gamma'(e_1|R_e^*) \neq \gamma'(e_2|R_e^*)$ for every pair of distinct edges e_1 and e_2 in P_m, other than the same edge metric codes. Thus, for $R_e = R_e^* \cup \{r_{14}, r_{20}, ..., r_{6k-10}, r_{6k-4}\}$, we obtain that $\gamma'(q_lr_l|R_e^*) \neq \gamma'(r_lq_{l-1}|R_e^*)$, and so $\gamma'(e_1|R_e^*) \neq \gamma'(e_2|R_e^*)$ for any e_1 and e_2 in $E(P_m)$. From this, we find that $|R_e| \le \lceil \frac{m}{6} \rceil$. Hence, $edim(P_m) \le \lceil \frac{m}{6} \rceil$ in this case as well.

Case (V)$m \equiv\ 4\ (mod\ 6)$.

TABLE 16. Edge metric codes for the edges present in E_4

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
r_lq_{l+1}; $l=1$	$(1,9,4)$	r_lq_{l+1}; $l=10$	$(11,4,14)$
r_lq_{l+1}; $l=2$	$(0,8,6)$	r_lq_{l+1}; $11 \leq l \leq s-1$	$(l+1,l-5,l+4)$
r_lq_{l+1}; $l=3$	$(2,7,7)$	r_lq_{l+1}; $s \leq l \leq s+2$	$(l+1,l-5,6k-l+4)$
r_lq_{l+1}; $l=4$	$(4,6,8)$	r_lq_{l+1}; $s+3 \leq l \leq s+8$	$(6k-l+7,l-5,6k-l+4)$
r_lq_{l+1}; $l=5$	$(6,5,9)$	r_lq_{l+1}; $s+9 \leq l \leq 6k-1$	$(6k-l+7,6k-l+13,6k-l+4)$
r_lq_{l+1}; $l=6$	$(7,3,10)$	r_lq_{l+1}; $6k$	$(6k-l+7,6k-l+13,3)$
r_lq_{l+1}; $l=7$	$(8,1,11)$	r_lq_{l+1}; $6k+1$	$(6k-l+7,6k-l+13,1)$
r_lq_{l+1}; $l=8$	$(9,0,12)$	r_lq_{l+1}; $6k+2$	$(6k-l+7,6k-l+13,0)$
r_lq_{l+1}; $l=9$	$(10,2,13)$	r_lq_{l+1}; $6k+3$	$(3,6k-l+13,2)$

Then, we write $m = 6k+4 = 2s$, where $k, s \in \mathbb{N}$ and $k \geq 3$. For this case, we prove that $edim(P_m) \leq \lceil \frac{m}{6} \rceil$. For this, suppose $R_e = \{r_2, r_8, r_{14}, ..., r_{6k-10}, r_{6k-4}, r_{6k+2}\} \subset V(P_m)$. We will prove that R_e is an edge resolving set with minimum cardinality for P_m. By total enumeration, one can verify easily that the set R_e is an edge resolving set with minimum cardinality for P_m, whenever k =3, 4, and 5. Next, for $k \geq 6$, we have to prove that the cardinality of minimum edge resolving set R_e for P_m is $\leq \lceil \frac{m}{6} \rceil$. For this, we show that the edge metric codes with respect to the set R_e, are distinct for every two distinct members of $E(P_m)$.

Suppose $R_e^* = \{r_2, r_8, r_{6k+2}\}$. Next, we give edge metric codes for every edge of P_m; $m \geq 33$, corresponding to the set R_e^*. The edge metric codes for the set of edges $E_1 = \{e = p_lp_{l+1} | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 17:

TABLE 17. Edge metric codes for the edges present in E_1

Edges	$\gamma'(e)$
p_lp_{l+1}; $1 \leq l \leq 3$	$(2,9-l,l+3)$
p_lp_{l+1}; $4 \leq l \leq 7$	$(l-1,9-l,l+3)$
p_lp_{l+1}; $l=8$	$(7,2,11)$
p_lp_{l+1}; $9 \leq l \leq s-2$	$(l-1,l-7,l+3)$
p_lp_{l+1}; $s-1 \leq l \leq s+2$	$(l-1,l-7,6k-l+3)$
p_lp_{l+1}; $s+3 \leq l \leq s+8$	$(6k-l+7,l-7,6k-l+3)$
p_lp_{l+1}; $s+9 \leq l \leq 6k+1$	$(6k-l+7,6k-l+13,6k-l+3)$
p_lp_{l+1}; $6k+2 \leq l \leq 6k+3$	$(6k-l+7,6k-l+13,2)$
p_lp_{l+1}; $l=4$	$(6k-l+7,6k-l+13,3)$

The edge metric codes for the set of edges $E_2 = \{e = p_lq_l | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 18:

The edge metric codes for the set of edges $E_3 = \{e = q_lr_l | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 19:

The edge metric codes for the set of edges $E_4 = \{e = r_lq_{l+1} | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 20:

Now, from these edge metric codes for the edges of P_m, corresponding to the set R_e^*, we find that $\gamma'(q_lr_l|R_e^*) = \gamma'(r_lq_{l-1}|R_e^*)$, for every $12 \leq l \leq 6k-1$ (some other pair of distinct edges may also have the same edge metric codes in P_m).

TABLE 18. Edge metric codes for the edges present in E_2

Edges	$\gamma'(e)$
p_lq_l; $l=1$	$(3,9,4)$
p_lq_l; $2\le l\le 3$	$(1,10-l,l+3)$
p_lq_l; $4\le l\le 7$	$(l-1,10-l,l+3)$
p_lq_l; $8\le l\le 9$	$(l-1,1,l+3)$
p_lq_l; $10\le l\le s-2$	$(l-1,l-7,l+3)$
p_lq_l; $s-1\le l\le s+2$	$(l-1,l-7,6k-l+4)$
p_lq_l; $s+3\le l\le s+8$	$(6k-l+8,l-7,6k-l+4)$
p_lq_l; $s+9\le l\le 6k+1$	$(6k-l+8,6k-l+14,6k-l+4)$
p_lq_l; $6k+2\le l\le 6k+3$	$(6k-l+8,6k-l+14,1)$
p_lq_l; $l=6k+4$	$(6k-l+8,6k-l+14,3)$

TABLE 19. Edge metric codes for the edges present in E_3

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
q_lr_l; $l=1$	$(2,10,5)$	q_lr_l; $l=11$	$(11,5,15)$
q_lr_l; $l=2$	$(0,9,6)$	q_lr_l; $12\le l\le s-2$	$(l,l-6,l+4)$
q_lr_l; $l=3$	$(1,8,7)$	q_lr_l; $s-1\le l\le s+2$	$(l,l-6,6k-l+5)$
q_lr_l; $l=4$	$(3,7,8)$	q_lr_l; $s+3\le l\le s+7$	$(6k-l+9,l-6,6k-l+5)$
q_lr_l; $l=5$	$(5,6,9)$	q_lr_l; $s+8\le l\le 6k-1$	$(6k-l+9,6k-l+15,6k-l+5)$
q_lr_l; $l=6$	$(6,4,10)$	q_lr_l; $l=6k$	$(6k-l+9,6k-l+15,4)$
q_lr_l; $l=7$	$(7,2,11)$	q_lr_l; $l=6k+1$	$(6k-l+9,6k-l+15,2)$
q_lr_l; $l=8$	$(8,0,12)$	q_lr_l; $l=6k+2$	$(6k-l+9,6k-l+15,0)$
q_lr_l; $l=9$	$(9,1,13)$	q_lr_l; $l=6k+3$	$(4,6k-l+15,1)$
q_lr_l; $l=10$	$(10,3,14)$	q_lr_l; $l=6k+4$	$(2,6k-l+15,3)$

TABLE 20. Edge metric codes for the edges present in E_4

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
r_lq_{l+1}; $l=1$	$(1,9,6)$	r_lq_{l+1}; $l=10$	$(11,4,15)$
r_lq_{l+1}; $l=2$	$(0,8,7)$	r_lq_{l+1}; $11\le l\le s-3$	$(l+1,l-5,l+5)$
r_lq_{l+1}; $l=3$	$(2,7,8)$	r_lq_{l+1}; $s-2\le l\le s+1$	$(l+1,l-5,6k-l+4)$
r_lq_{l+1}; $l=4$	$(4,6,9)$	r_lq_{l+1}; $s+2\le l\le s+7$	$(6k-l+8,l-5,6k-l+4)$
r_lq_{l+1}; $l=5$	$(6,5,10)$	r_lq_{l+1}; $s+8\le l\le 6k-1$	$(6k-l+8,6k-l+14,6k-l+4)$
r_lq_{l+1}; $l=6$	$(7,3,11)$	r_lq_{l+1}; $6k$	$(6k-l+8,6k-l+14,3)$
r_lq_{l+1}; $l=7$	$(8,1,12)$	r_lq_{l+1}; $6k+1$	$(6k-l+8,6k-l+14,1)$
r_lq_{l+1}; $l=8$	$(9,0,13)$	r_lq_{l+1}; $6k+2$	$(6k-l+8,6k-l+14,0)$
r_lq_{l+1}; $l=9$	$(10,2,14)$	r_lq_{l+1}; $6k+3$	$(6k-l+8,6k-l+14,2)$
		r_lq_{l+1}; $6k+4$	$(3,6k-l+14,4)$

Then, from the remaining edge metric codes for the edges in P_m, we obtain that $\gamma'(e_1|R_e^*)\neq\gamma'(e_2|R_e^*)$ for every pair of distinct edges e_1 and e_2 in P_m, other than the same edge metric codes. Thus, for $R_e=R_e^*\cup\{r_{14},r_{20},...,r_{6k-10},r_{6k-4}\}$, we obtain that $\gamma'(q_lr_l|R_e^*)\neq\gamma'(r_lq_{l-1}|R_e^*)$, and so $\gamma'(e_1|R_e^*)\neq\gamma'(e_2|R_e^*)$ for any e_1 and e_2 in $E(P_m)$. From this, we find that $|R_e|\le\lceil\frac{m}{6}\rceil$. Hence, $edim(P_m)\le\lceil\frac{m}{6}\rceil$ in this case as well.

Case (VI)$m \equiv\ 5\ (mod\ 6)$.
Then, we write $m = 6k+5 = 2s+1$, where $k, s \in \mathbb{N}$ and $k \geq 3$. For this case, we prove that $edim(P_m) \leq \lceil \frac{m}{6} \rceil$. For this, suppose $R_e = \{r_2, r_8, r_{14}, ..., r_{6k-10}, r_{6k-4}, r_{6k+2}\} \subset V(P_m)$. We will prove that R_e is an edge resolving set with minimum cardinality for P_m. By total enumeration, one can verify easily that the set R_e is an edge resolving set with minimum cardinality for P_m, whenever k =3, 4, and 5. Next, for $k \geq 6$, we have to prove that the cardinality of minimum edge resolving set R_e for P_m is $\leq \lceil \frac{m}{6} \rceil$. For this, we show that the edge metric codes with respect to the set R_e, are distinct for every two distinct members of $E(P_m)$.

Suppose $R_e^* = \{r_2, r_8, r_{6k+2}\}$. Next, we give edge metric codes for every edge of P_m; $m \geq 33$, corresponding to the set R_e^*. The edge metric codes for the set of edges $E_1 = \{e = p_l p_{l+1} | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 21:

Table 21. Edge metric codes for the edges present in E_1

Edges	$\gamma'(e)$
$p_l p_{l+1}$; $1 \leq l \leq 3$	$(2, 9-l, l+4)$
$p_l p_{l+1}$; $4 \leq l \leq 7$	$(l-1, 9-l, l+4)$
$p_l p_{l+1}$; $l = 8$	$(7, 2, 12)$
$p_l p_{l+1}$; $9 \leq l \leq s-3$	$(l-1, l-7, l+4)$
$p_l p_{l+1}$; $s-2 \leq l \leq s+2$	$(l-1, l-7, 6k-l+3)$
$p_l p_{l+1}$; $s+3 \leq l \leq s+8$	$(6k-l+8, l-7, 6k-l+3)$
$p_l p_{l+1}$; $s+9 \leq l \leq 6k+1$	$(6k-l+8, 6k-l+14, 6k-l+3)$
$p_l p_{l+1}$; $6k+2 \leq l \leq 6k+3$	$(6k-l+8, 6k-l+14, 2)$
$p_l p_{l+1}$; $l = 6k+4$	$(6k-l+8, 6k-l+14, 3)$
$p_l p_{l+1}$; $l = 6k+5$	$(6k-l+8, 6k-l+14, 4)$

The edge metric codes for the set of edges $E_2 = \{e = p_l q_l | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 22:

Table 22. Edge metric codes for the edges present in E_2

Edges	$\gamma'(e)$
$p_l q_l$; $l = 1$	$(3, 9, 5)$
$p_l q_l$; $2 \leq l \leq 3$	$(1, 10-l, l+4)$
$p_l q_l$; $4 \leq l \leq 7$	$(l-1, 10-l, l+4)$
$p_l q_l$; $8 \leq l \leq 9$	$(l-1, 1, l+4)$
$p_l q_l$; $10 \leq l \leq s-2$	$(l-1, l-7, l+4)$
$p_l q_l$; $s-1 \leq l \leq s+3$	$(l-1, l-7, 6k-l+4)$
$p_l q_l$; $s+4 \leq l \leq s+9$	$(6k-l+9, l-7, 6k-l+4)$
$p_l q_l$; $s+10 \leq l \leq 6k+1$	$(6k-l+9, 6k-l+15, 6k-l+4)$
$p_l q_l$; $6k+2 \leq l \leq 6k+3$	$(6k-l+9, 6k-l+15, 1)$
$p_l q_l$; $l = 6k+4$	$(6k-l+9, 6k-l+15, 3)$
$p_l q_l$; $l = 6k+5$	$(6k-l+9, 6k-l+15, 4)$

The edge metric codes for the set of edges $E_3 = \{e = q_l r_l | l = 1, 2, 3, ..., m\}$, with respect to the set R_e^* are listed in Table 23:

TABLE 23. Edge metric codes for the edges present in E_3

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
q_lr_l; $l=1$	$(2,10,6)$	q_lr_l; $l=11$	$(11,5,16)$
q_lr_l; $l=2$	$(0,9,7)$	q_lr_l; $12\leq l\leq s-2$	$(l,l-6,l+5)$
q_lr_l; $l=3$	$(1,8,8)$	q_lr_l; $s-1\leq l\leq s+3$	$(l,l-6,6k-l+5)$
q_lr_l; $l=4$	$(3,7,9)$	q_lr_l; $s+4\leq l\leq s+9$	$(6k-l+10,l-6,6k-l+5)$
q_lr_l; $l=5$	$(5,6,10)$	q_lr_l; $s+10\leq l\leq 6k-1$	$(6k-l+10,6k-l+16,6k-l+5)$
q_lr_l; $l=6$	$(6,4,11)$	q_lr_l; $l=6k$	$(6k-l+10,6k-l+16,4)$
q_lr_l; $l=7$	$(7,2,12)$	q_lr_l; $l=6k+1$	$(6k-l+10,6k-l+16,2)$
q_lr_l; $l=8$	$(8,0,13)$	q_lr_l; $l=6k+2$	$(6k-l+10,6k-l+16,0)$
q_lr_l; $l=9$	$(9,1,14)$	q_lr_l; $l=6k+3$	$(6k-l+10,6k-l+16,1)$
q_lr_l; $l=10$	$(10,3,15)$	q_lr_l; $l=6k+4$	$(6k-l+10,6k-l+15,3)$
		q_lr_l; $l=6k+5$	$(4,6k-l+15,5)$

TABLE 24. Edge metric codes for the edges present in E_4

Edges	$\gamma'(e)$	Edges	$\gamma'(e)$
r_lq_{l+1}; $l=1$	$(1,9,7)$	r_lq_{l+1}; $11\leq l\leq s-3$	$(l+1,l-5,l+6)$
r_lq_{l+1}; $l=2$	$(0,8,8)$	r_lq_{l+1}; $s-2\leq l\leq s+2$	$(l+1,l-5,6k-l+4)$
r_lq_{l+1}; $l=3$	$(2,7,9)$	r_lq_{l+1}; $s+3\leq l\leq s+8$	$(6k-l+9,l-5,6k-l+4)$
r_lq_{l+1}; $l=4$	$(4,6,10)$	r_lq_{l+1}; $s+9\leq l\leq 6k-1$	$(6k-l+9,6k-l+156k-l+4)$
r_lq_{l+1}; $l=5$	$(6,5,11)$	r_lq_{l+1}; $6k$	$(6k-l+9,6k-l+15,3)$
r_lq_{l+1}; $l=6$	$(7,3,12)$	r_lq_{l+1}; $6k+1$	$(6k-l+9,6k-l+15,1)$
r_lq_{l+1}; $l=7$	$(8,1,13)$	r_lq_{l+1}; $6k+2$	$(6k-l+9,6k-l+15,0)$
r_lq_{l+1}; $l=8$	$(9,0,14)$	r_lq_{l+1}; $6k+3$	$(6k-l+9,6k-l+15,2)$
r_lq_{l+1}; $l=9$	$(10,2,15)$	r_lq_{l+1}; $6k+4$	$(6k-l+9,6k-l+14,4)$
r_lq_{l+1}; $l=10$	$(11,4,16)$	r_lq_{l+1}; $6k+5$	$(3,6k-l+14,6)$

The edge metric codes for the set of edges $E_4=\{e=r_lq_{l+1}|l=1,2,3,...,m\}$, with respect to the set R_e^* are listed in Table 24:

Now, from these edge metric codes for the edges of P_m, corresponding to the set R_e^*, we find that $\gamma'(q_lr_l|R_e^*)=\gamma'(r_lq_{l-1}|R_e^*)$, for every $12\leq l\leq 6k-1$ (some other pair of distinct edges may also have the same edge metric codes in P_m). Then, from the remaining edge metric codes for the edges in P_m, we obtain that $\gamma'(e_1|R_e^*)\neq\gamma'(e_2|R_e^*)$ for every pair of distinct edges e_1 and e_2 in P_m, other than the same edge metric codes. Thus, for $R_e=R_e^*\cup\{r_{14},r_{20},...,r_{6k-10},r_{6k-4}\}$, we obtain that $\gamma'(q_lr_l|R_e^*)\neq\gamma'(r_lq_{l-1}|R_e^*)$, and so $\gamma'(e_1|R_e^*)\neq\gamma'(e_2|R_e^*)$ for any e_1 and e_2 in $E(P_m)$. From this, we find that $|R_e|\leq\lceil\frac{m}{6}\rceil$. Hence, $edim(P_m)\leq\lceil\frac{m}{6}\rceil$ in this case as well.

From all of these cases and by using lemma 3, we find that $edim(P_m)=\lceil\frac{m}{6}\rceil$ for every $m\geq 15$. $\square$

Now, regarding the independence of an edge resolving set for P_m, we have the following result:

Theorem 4. Let P_m be the graph of pentagonal circular ladder. Then, its independent edge metric number is

$$edim(P_m) = \begin{cases} 3, & \text{if } 3 \leq m \leq 14; \\ \lceil \frac{m}{6} \rceil & \text{if } m \geq 15. \end{cases}$$

Proof. Refer Theorem 3, for proof. □

Raza et al. [11] considered three families of the planar graphs viz., D_m (prism graph), A_m (antiprism graph), and R_m (a graph obtained by superimposing the outer cycle of A_m on the inner cycle of vertices of D_m). For these families of the planar graphs they studied mixed metric dimension and raised a problem regarding the edge metric dimension of R_m, that *The edge metric dimension of the planar graph R_m is equal to its mixed metric dimension?* Now, in the next section, we compute the edge metric dimension for the planar graph R_m, and which comes out to be equal to its mixed metric dimension i.e., $mdim(R_m) = 5 = edim(R_m)$.

3. Edge metric dimension of the planar graph R_m

The planar graph R_m [11], comprises of $3m$ number of vertices and $6m$ number of edges. It has $2m$ number of faces each having three sides, m number of faces each having four sides, and two m-sided faces, as shown in Fig. 3. We represent the set of vertices and edges for the planar graph R_m as $V(R_m)$ and $E(R_m)$, respectively. The sets $V(R_m)$ and $E(R_m)$ are as follows:

$$V(R_m) = \{p_l, q_l, r_l : 1 \leq l \leq m\}$$

and

$$E(R_m) = \{p_l q_l, q_l r_l, r_l r_{l+1}, q_l q_{l+1}, p_l p_{l+1}, q_l p_{l+1} : 1 \leq l \leq m\}$$

We name the set $D = \{p_l : 1 \leq l \leq m\}$ of vertices in R_m, as the p-vertices, the set $E = \{q_l : 1 \leq l \leq m\}$ of vertices in R_m, as the q-vertices, and the set $F = \{r_l : 1 \leq l \leq m\}$ of vertices in R_m, as the r-vertices. For our purpose, we take $p_{m+1} = p_1$, $r_{m+1} = r_1$, and $q_{m+1} = q_1$ (whenever necessary). In this section, we study the notion of edge metric dimension for the planar graph R_m. We prove that $edim(R_m) = 3$ for $m \geq 8$. Additionally, we prove that an edge resolving set R_e for R_m is independent. Next, we have the following result regarding the edge metric dimension of R_m.

Theorem 5. Let R_m be the planar graph as defined above. Then, for $m \geq 8$, we have $edim(R_m) = 5$.

Proof. For $8 \leq m \leq 11$, it is easy to verify that the edge metric dimension of a planar graph R_m is 5. Now, for $m \geq 12$, we eagerly consider the resulting two cases relying on the positive integer m i.e., when the positive natural number m is even and when it is odd.

Case (I):the integer m is even.
Then, we write $m = 2s$, where $s \in \mathbb{N}$ and $s \geq 4$. Let $R_e = \{r_1, q_2, q_{s+1}, p_3, p_{s+3}\} \subset$

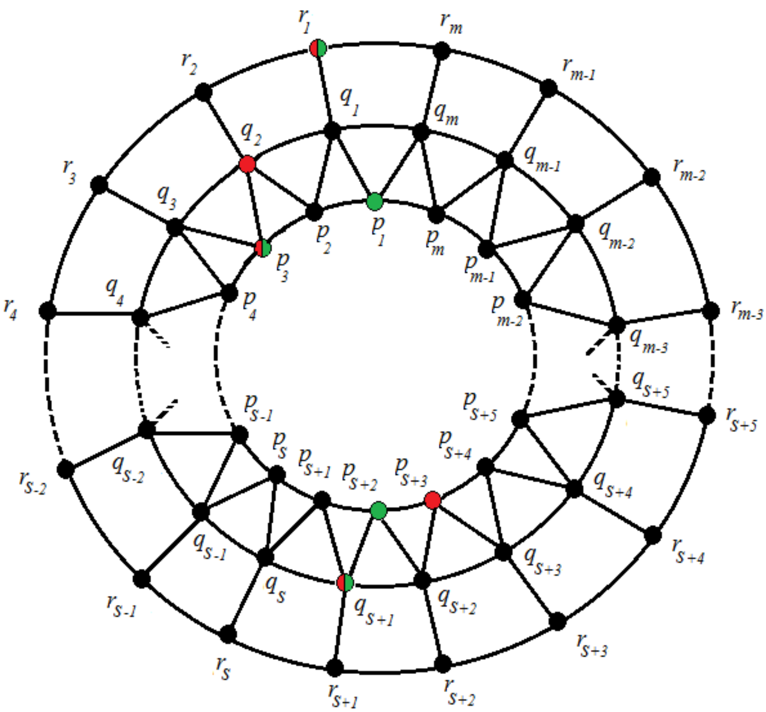

FIGURE 3. The Plane Graph R_m

$V(R_m)$ (for the location of these five edge basis vertices, see vertices in red colour and both red and green colour in figure 3). Now, in order to find that $\mathfrak{L}_E$ is an edge metric generator of the radially symmetrical graph R_m, we give the edge metric representations for each edge of $E(R_m)$ regarding the set $\mathfrak{L}_E$. For edge metric codes of $E(R_m)$ see in [11], proof of Theorem 3.

We notice that no two edges are having the same edge metric codes, suggesting that $edim(R_m) \leq 5$. Now, so as to finish the evidence for this case, we show that $edim(R_m) \geq 5$ by working out that there does not exist an edge metric generator $\mathfrak{L}_E$ such that $|\mathfrak{L}_E| = 4$. Suppose on the contrary that $edim(R_m) = 4$. At that point, we have the accompanying prospects to be talked about.

By the symmetry of the plane graph, R_m different relations can be considered, they will have the same sort of logical inconsistencies. In this manner, the above conversation explains that there is no edge resolving set comprising of four vertices for $V(R_m)$ inferring that $edim(R_m) = 5$ in this case.

Case (II):the integer m is odd.
For this situation, the integer m can be written as $m = 2s + 1$, where $s \in \mathbb{N}$ and $s \geq 4$. Let $R_e = \{r_1, q_{s+1}, p_1, p_3, p_{s+2}\} \subset V(R_m)$ (for the location of these five edge basis vertices, see vertices in green colour and both red and green colour in Figure 3). Now, in order to find that $\mathfrak{L}_E$ is an edge metric generator of the radially symmetrical

graph R_m, we give the edge metric representations for each edge of $E(R_m)$ regarding the set $\mathfrak{L}_E$. For edge metric codes of $E(R_m)$ see in [11], proof of Theorem 3.

Again, we see that no two edges are having the same edge metric codes, suggesting that $edim(R_m) \leq 5$. As appeared before the logical inconsistency table for indeed, even m, same sort of logical inconsistency happen if there should be an occurrence of odd m, so $edim(R_m) \geq 5$. Subsequently, it is demonstrated that $edim(R_m) = 5$. □

Edge resolving sets	Contradictions
$R_e = \{p_1, q_1, r_l, r_{l+1}\}$, $(1 \leq l \leq 2s)$	$\gamma'(p_1p_m\|R_e) = \gamma'(p_1q_m\|R_e)$, for $1 \leq l \leq s-1$ $\gamma'(r_1q_1\|R_e) = \gamma'(q_1q_2\|R_e)$, for $l = s$, and $\gamma'(r_1q_1\|R_e) = \gamma'(q_1q_m\|R_e)$, for $s+1 \leq l \leq 2s-1$.
$R_e = \{p_1, r_1, q_l, q_{l+1}\}$, $(1 \leq l \leq 2s)$	$\gamma'(p_1q_m\|R_e) = \gamma'(p_1p_m\|R_e)$, for $1 \leq l \leq s-1$ $\gamma'(p_3q_3\|R_e) = \gamma'(p_3p_4\|R_e)$, for $l = s$, and $\gamma'(r_mq_m\|R_e) = \gamma'(q_1q_m\|R_e)$, for $s+1 \leq l \leq 2s-1$.
$R_e = \{q_1, r_1, p_l, p_{l+1}\}$, $(1 \leq l \leq 2s)$	$\gamma'(p_1q_m\|R_e) = \gamma'(p_1p_m\|R_e)$, for $1 \leq l \leq s-1$ $\gamma'(q_3p_4\|R_e) = \gamma'(p_3p_4\|R_e)$, for $s \leq l \leq s+2$, and $\gamma'(p_3p_2\|R_e) = \gamma'(q_2p_2\|R_e)$, for $s+3 \leq l \leq 2s-1$.
$R_e = \{p_1, q_1, r_l, r_{s+l}\}$, $(1 \leq l \leq 2s)$	$\gamma'(p_1p_m\|R_e) = \gamma'(q_mp_1\|R_e)$, for $1 \leq l \leq s-1$ $\gamma'(p_{m-1}q_{m-2}\|R_e) = \gamma'(p_{m-1}p_{m-2}\|R_e)$, for $l = s$, and for $s+3 \leq l \leq 2s-1$, it is same as for $1 \leq l \leq s$.
$R_e = \{p_1, r_1, q_l, q_{s+l}\}$, $(1 \leq l \leq 2s)$	$\gamma'(p_1q_m\|R_e) = \gamma'(p_1p_m\|R_e)$, for $1 \leq l \leq s-1$ $\gamma'(p_{m-1}p_{m-2}\|R_e) = \gamma'(p_{m-1}q_{m-2}\|R_e)$, for $l = s$, and for $s+3 \leq l \leq 2s-1$, it is same as for $1 \leq l \leq s$.
$R_e = \{q_1, r_1, p_l, p_{s+l}\}$, $(1 \leq l \leq 2s)$	$\gamma'(q_{m-1}q_m\|R_e) = \gamma'(q_mp_m\|R_e)$, for $1 \leq l \leq s-1$ $\gamma'(p_2q_1\|R_e) = \gamma'(q_1q_2\|R_e)$, for $l = s$, and for $s+3 \leq l \leq 2s-1$, it is same as for $1 \leq l \leq s$.
$R_e = \{p_1, q_2, r_l, p_s\}$, $(1 \leq l \leq 2s)$	$\gamma'(q_mp_1\|R_e) = \gamma'(p_1p_m\|R_e)$, for $1 \leq l \leq 2s-1$ and $\gamma'(q_2q_3\|R_e) = \gamma'(q_2p_3\|R_e)$, for $l = 2s$.
$R_e = \{q_1, r_2, p_l, q_s\}$, $(1 \leq l \leq 2s)$	$\gamma'(q_mq_1\|R_e) = \gamma'(p_1q_1\|R_e)$, for $2 \leq l \leq 2s$ and $\gamma'(p_sp_{s-1}\|R_e) = \gamma'(q_{s-1}p_{s-1}\|R_e)$, for $l = 1$.
$R_e = \{r_1, p_2, q_l, r_s\}$, $(1 \leq l \leq 2s)$	$\gamma'(q_mp_1\|R_e) = \gamma'(p_1p_m\|R_e)$, for $1 \leq l \leq 2s-1$ and $\gamma'(p_sp_{s-1}\|R_e) = \gamma'(q_{s-1}p_{s-1}\|R_e)$, for $l = 2s$.

Now, regarding the independence of an edge resolving set for R_m, we have the following result:

Theorem 6. Let R_m be the planar graph as defined above. Then, its independent edge metric number is 5 for any $m \geq 8$.

Proof. Refer Theorem 5, for proof. □

4. Conclusion

The problem of characterizing the classes of planar graphs with the constant edge metric dimension is of great interest nowadays. In this manuscript, we have studied the notion of edge metric dimension for two families of planar graphs, viz., P_m and R_m. For P_m, we proved that $edim(P_m) = 3$ for $3 \leq m \leq 14$ and $edim(P_m) = \lceil \frac{m}{6} \rceil$ for $m \geq 15$. We additionally find that the edge metric dimension is strictly greater than the metric dimension for the pentagonal circular ladder i.e., $edim(P_m) > dim(P_m)$; for every $m \geq 3$. We also answer the problem raised in the recent past regarding the edge metric dimension of the family of planar graph R_m [11], and we prove for this family R_m, that $edim(R_m) = 5$. Moreover, for both of these families of the planar graphs, we show that the edge metric basis set R_e is independent.

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