Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 50(1), 2023, Pages 171–188, DOI: 10.52846/ami.v50i1.1647
ISSN: 1223-6934

Periodic Gabor frames on positive half line

Owais Ahmad

Abstract. In this paper, we introduce the concept of periodic Gabor frames on positive half line. Firstly, we establish a necessary and sufficient condition for a periodic Gabor system to be a Gabor frame. Then, we present some equivalent characterizations of Parseval Gabor frames on positive half line by means of some fundamental equations in time domain.

2010 Mathematics Subject Classification. Primary 42C40; 42C15; Secondary 43A70;11S85, 41A17.
Key words and phrases. Gabor frame, Parseval frame, Periodic set.

1. Introduction

Given a function $g \in L^2(\mathbb{R})$ and constants $a, b > 0$, the associated Gabor system generated by g and the lattice $a\mathbb{Z} \times b\mathbb{Z}$, is a system of the form

$$\mathcal{G}(a,b,g) = \Big\{ E_{mb}T_{na}g(x) =: e^{2\pi imbx}g(x-na) : m,n \in \mathbb{Z} \Big\} \tag{1.1}$$

which is built by the combined action of modulations and translations of a single function, and hence, can be viewed as the set of time-frequency shifts of $g(x) \in L^2(\mathbb{R})$ along the lattice $a\mathbb{Z} \times b\mathbb{Z}$ in $\mathbb{R}^2$. Such systems are also known as *Weyl-Heisenberg systems*. In 1946, Dennis Gabor [11] proposed to study such systems for their usefulness in the analysis of information conveyed by communication channels. The resulting theory led to many applications ranging from auditory signal processing, to pseudodifferential operator analysis, to uncertainty principles. Since the time-frequency shifted versions of the elementary signal do not generally constitute an orthonormal basis, the question is how to obtain the Gabor coefficients and related questions concerning the existence and uniqueness of the Gabor expansions are rather involved (see [24]). Later on, it was recognized that a systematic study of the properties of the Gabor expansions requires the mathematical concept of frames introduced by Duffin and Schaeffer [5]. In [6] Daubechies, Grossmann, and Meyer utilized the mathematical concept of frames for the analysis of the Gabor scheme and showed that Duffin and Schaeffer's definition is an abstraction of a concept given by Gabor for doing signal analysis. The frames introduced by Gabor now are called *Gabor frames* or *Weyl-Heisenberg frames*. The system $\mathcal{G}(a,b,g)$ given by (1.1) is called a *Gabor frame* if there exist constants $0 < A, B < \infty$ such that

$$A\big\|f\big\|_2^2 \le \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \big|\big\langle f, E_{mb}T_{na}g\big\rangle\big|^2 \le B\big\|f\big\|_2^2, \tag{1.2}$$

holds for every $f \in L^2(\mathbb{R})$, and we call the optimal constants A and B the *lower frame bound* and the *upper frame bound*, respectively. A *tight Gabor frame* refers to the case when $A = B$, and a *Parseval frame* refers to the case when $A = B = 1$. Gabor systems that form frames for $L^2(\mathbb{R})$ have a wide variety of applications. One of the most important problem in practice is to determine conditions for Gabor systems to be frames. In practice, once the window function has been chosen, the first question to investigate is to find the values of the time-frequency parameters a, b such that $\mathcal{G}(a, b, g)$ is a frame. Therefore, the product ab will decide whether the system $\mathcal{G}(a, b, g)$ constitutes a frame or even complete for $L^2(\mathbb{R})$ or not. A useful tool in this context is the Ron and Shen [17] criterion. By using this criterion, Gröchenig et al.[14] have proved that the Gabor system $\mathcal{G}(a, b, g)$ given by (1.1) cannot be a frame for $L^2(\mathbb{R})$ if $ab > 1$. In addition to this, they have also shown that the system $\mathcal{G}(a, b, g)$ will form an Riesz basis for $L^2(\mathbb{R})$ if $ab = 1$.

Gabor analysis is a pervasive signal processing method for decomposing and reconstructing signals from their time frequency projections and also in the context of speech processing, texture segmentation, pattern and object recognition. In order to analyze the dynamic time frequency samples of the signals that contain a wide range of spatial and frequency components, the resolution of which is normally very poor, the single windowed Gabor expansion is not suitable. To address this issue, one of the best choices is multigenerator Gabor system which a set of multiple windows of various time frequency localizations in frame system, the representation of signals of multiple and time-varying frequencies would have their corresponding windowing templates and resolutions to relate to. The concept of multigenerator Gabor system was introduced by Zibulski and Zeevi [26] and they in [27] discussed the frame operator associated with the multigenerator Gabor frame by invoking the concept of piecewise Zak transform. They showed that the Gabor frame operator for Gabor frames for $L^2(\mathbb{R})$ could be factorized into a set of products of $p \times q$ matrices, where p is the redundancy of the frame written as an irreducible fraction. Each q matrix depends continuously on two parameters $r, s \in \mathbb{R}$ and the elements in the matrices are given by elements of the Zak transform of the window. The approach naturally does not work for Gabor systems with irrational oversampling. For more details about Gabor frames, we refer to [1, 3, 4, 19, 21].

During the last decade a lot of research has been carried out in the context of the wavelet and Gabor frames on positive half line. Shah [20] constructed Gabor frame on positive half line and obtained necessary and sufficient conditions for Gabor frames in $L^2(\mathbb{R}^+)$. Recent results on wavelet frames in $L^2(\mathbb{R}^+)$ can be found in [2, 7, 8, 9, 10, 18, 25] and the references therein. In recent years there was a considerable interest in the problem of constructing periodic Gabor frames in Hilbert spaces as most of the signals of practical interest are periodic in nature [12, 23]. All signals resulting from experiments with a finite duration can in principle be modeled as periodic signals, for instance, ECG signals and other medical signals. Therefore, the main purpose of this research article is to introduce the notion of periodic Gabor systems on positive half line via Walsh-Fourier transform and establish certain characterizations of these periodic systems to be frames in $L^2(\mathcal{S}), \mathcal{S} \subset \mathbb{R}^+$. More precisely, we first establish a necessary and sufficient condition for the periodic Gabor system to be a frame in $L^2(\mathcal{S})$. Then, we provide some equivalent characterizations of Parseval Gabor frames on positive half line by means of some fundamental equations in time domain.

The remainder of this paper is organized as follows: In Section 2, we discuss some preliminary facts about Walsh-Fourier analysis and introduce the notion of Gabor systems and periodic Gabor frames in $L^2(\mathbb{R}^+)$. In Section 3, we establish a necessary and sufficient condition for periodic Gabor system to be a frame for $L^2(\mathcal{S})$. Section 4 is devoted to obtaining necessary and sufficient conditions for Parseval Periodic Gabor frames in $L^2(\mathcal{S})$

2. Preliminaries on Walsh-Fourier analysis

As usual, let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$. Denote by $[x]$ the integer part of x. Let p be a fixed natural number greater than 1. For $x \in \mathbb{R}^+$ and any positive integer j, we set

$$x_j = [p^j x] (\operatorname{mod} p), \qquad x_{-j} = [p^{1-j} x] (\operatorname{mod} p), \tag{2.1}$$

where $x_j, x_{-j} \in \{0, 1, \dots, p-1\}$. Clearly, x_j and x_{-j} are the digits in the p-expansion of x:

$$x = \sum_{j<0} x_{-j} p^{-j-1} + \sum_{j>0} x_j p^{-j}.$$

Moreover, the first sum on the right is always finite. Besides,

$$[x] = \sum_{j<0} x_{-j} p^{-j-1}, \qquad \{x\} = \sum_{j>0} x_j p^{-j},$$

where $[x]$ and $\{x\}$ are, respectively, the integral and fractional parts of x.

Consider on $\mathbb{R}^+$ the addition defined as follows:

$$x \oplus y = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j},$$

with $\zeta_j = x_j + y_j (\operatorname{mod} p)$, $j \in \mathbb{Z} \setminus \{0\}$, where $\zeta_j \in \{0, 1, \dots, p-1\}$ and x_j, y_j are calculated by (2.1). Clearly, $[x \oplus y] = [x] \oplus [y]$ and $\{x \oplus y\} = \{x\} \oplus \{y\}$. As usual, we write $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

Let $\varepsilon_p = \exp(2\pi i/p)$, we define a function $r_0(x)$ on $[0, 1)$ by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/p) \\ \varepsilon_p^\ell, & \text{if } x \in \left[\ell p^{-1}, (\ell+1) p^{-1}\right), \quad \ell = 1, 2, \dots, p-1. \end{cases}$$

The extension of the function r_0 to $\mathbb{R}^+$ is given by the equality $r_0(x+1) = r_0(x), \forall\, x \in \mathbb{R}^+$. Then, the system of *generalized Walsh functions* $\{w_m(x) : m \in \mathbb{Z}^+\}$ on $[0, 1)$ is defined by

$$w_0(x) \equiv 1 \quad \text{and} \quad w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j$, $\mu_j \in \{0, 1, \dots, p-1\}$, $\mu_k \neq 0$. They have many properties similar to those of the Haar functions and trigonometric series, and form a complete orthogonal system. Further, by a Walsh polynomial we shall mean a finite linear combination of Walsh functions. For $x, y \in \mathbb{R}^+$, let

$$\chi(x, y) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j)\right), \tag{2.2}$$

where x_j, y_j are given by equation (2.1).

We observe that

$$\chi\left(x, \frac{m}{p^n}\right) = \chi\left(\frac{x}{p^n}, m\right) = w_m\left(\frac{x}{p^n}\right), \qquad \forall\, x \in [0, p^n),\ m, n \in \mathbb{Z}^+,$$

and

$$\chi(x \oplus y, z) = \chi(x, z)\,\chi(y, z), \quad \chi(x \ominus y, z) = \chi(x, z)\,\overline{\chi(y, z)},$$

where $x, y, z \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational. It is well known that systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(\cdot, \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1]$ (See Golubov et al.[13]).

The *Walsh-Fourier transform* of a function $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^+} f(x)\,\overline{\chi(x, \xi)}\,dx, \tag{2.3}$$

where $\chi(x, \xi)$ is given by (2.2). The Walsh-Fourier operator $\mathcal{F} : L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, $\mathcal{F}f = \hat{f}$, extends uniquely to the whole space $L^2(\mathbb{R}^+)$. The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [13, 22]). In particular, if $f \in L^2(\mathbb{R}^+)$, then $\hat{f} \in L^2(\mathbb{R}^+)$ and

$$\left\|\hat{f}\right\|_{L^2(\mathbb{R}^+)} = \|f\|_{L^2(\mathbb{R}^+)}.$$

Moreover, if $f \in L^2[0, 1]$, then we can define the Walsh-Fourier coefficients of f as

$$\hat{f}(n) = \int_0^1 f(x)\overline{w_n(x)}\,dx.$$

The series $\sum_{n \in \mathbb{Z}^+} \hat{f}(n) w_n(x)$ is called the *Walsh-Fourier series* of f. Therefore, from the standard L^2-theory, we conclude that the Walsh-Fourier series of f converges to f in $L^2[0, 1]$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_0^1 |f(x)|^2 dx = \sum_{n \in \mathbb{Z}^+} \left|\hat{f}(n)\right|^2. \tag{2.4}$$

By p-adic interval $I \subset \mathbb{R}^+$ of range n, we mean intervals of the form

$$I = I_n^k = \left[kp^{-n}, (k+1)p^{-n}\right),\ k \in \mathbb{Z}^+.$$

The p-adic topology is generated by the collection of p-adic intervals and each p-adic interval is both open and closed under the p-adic topology (see [13]). The family $\left\{[0, p^{-j}) : j \in \mathbb{Z}\right\}$ forms a fundamental system of the p-adic topology on $\mathbb{R}^+$. Therefore, the generalized Walsh functions $w_j(x), 0 \le j \le p^n - 1$, assume constant values on each p-adic interval I_n^k and hence continuous on these intervals. Thus, $w_j(x) = 1$ for $x \in I_n^0$.

Let $\mathcal{E}_n(\mathbb{R}^+)$ be the space of p-adic entire functions of order n, that is, the set of all functions which are constant on all p-adic intervals of range n. Thus, for every $f \in \mathcal{E}_n(\mathbb{R}^+)$, we have

$$f(x) = \sum_{k \in \mathbb{Z}^+} f(p^{-n}k)\chi_{I_n^k}(x), \quad x \in \mathbb{R}^+.$$

Clearly each Walsh function of order up to p^{n-1} belongs to $\mathcal{E}_n(\mathbb{R}^+)$. The set $\mathcal{E}(\mathbb{R}^+)$ of p-adic entire functions on $\mathbb{R}^+$ is the union of all the spaces $\mathcal{E}_n(\mathbb{R}^+)$. It is clear that $\mathcal{E}(\mathbb{R}^+)$ is dense in $L^p(\mathbb{R}^+), 1 \leq p < \infty$ and each function in $\mathcal{E}(\mathbb{R}^+)$ is of compact support. Thus, we consider the following set of functions

$$\mathcal{E}^0(\mathbb{R}^+) = \left\{ f \in \mathcal{E}(\mathbb{R}^+) : supp\ f \subset \mathbb{R}^+ \setminus \{0\} \right\}.$$

We now state some fundamental results about frames in a Hilbert space $\mathbb{H}$.

Definition 2.1. Let $\mathbb{H}$ be a separable Hilbert space. A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathbb{H}$ is called a *frame* for $\mathbb{H}$ if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\big\|f\big\|_2^2 \leq \sum_{k=1}^{\infty} \big|\langle f, f_k\rangle\big|^2 \leq B\big\|f\big\|_2^2$$

for all $f \in \mathbb{H}$. The largest constant A and the smallest constant B satisfying the above inequalities are called the *upper* and the *lower frame bound*, respectively. The sequence $\{f_k\}_{k=1}^{\infty}$ is called a *Parseval frame* for $\mathbb{H}$ if the upper frame bound A and the lower frame bound B coincide.

A measurable set $\mathcal{S}$ in $\mathbb{R}^+$ is said to be a-periodic if $\mathcal{S} + na = \mathcal{S}$, for every $n \in \mathbb{Z}^+$. Let $\mathcal{S}$ be a-periodic subset of $\mathbb{R}^+$, then it is obvious that $\mathcal{S}$ is also av-periodic for every $v \in \mathbb{N}$. We denote $\mathcal{S}^0 = [0, p) \cap \mathcal{S}$ and define

$$L^2(\mathcal{S}) = \Big\{ f \in L^2(\mathbb{R}^+) : \text{supp}(f) \subset \mathcal{S} \Big\}. \tag{2.5}$$

Clearly, $L^2(\mathcal{S})$ is also a Hilbert space with an inner product of $L^2(\mathbb{R}^+)$.

Definition 2.2. For $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\} \subseteq L^2(\mathcal{S})$. A system of the form

$$\mathfrak{G}(\Psi, p, q) := \Big\{ E_{mq_\ell} T_{np} \psi_\ell =: w_{mq_\ell}(x) \psi_\ell\big(x \ominus np\big) : n, m \in \mathbb{Z}^+, 1 \leq \ell \leq L \Big\}, \tag{2.6}$$

is called *multi-generator Gabor system* in $L^2(\mathcal{S})$, where L is a fixed integer and p, q_ℓ are fixed elements in $\mathbb{R}^+ \setminus \{0\}$. The Gabor system $\mathfrak{G}(\Psi, p, q)$ is a *multi-generator Gabor frame* for $L^2(\mathcal{S})$ if there exist frame constants $0 < C \leq D < \infty$ such that for every $f \in L^2(\mathbb{R}^+)$, we have

$$C\big\|f\big\|^2 \leq \sum_{1 \leq \ell \leq L} \sum_{m \in \mathbb{Z}^+} \sum_{n \in \mathbb{Z}^+} \big|\langle f, E_{mq_\ell} T_{np_\ell} \psi_\ell\rangle\big|^2 \leq D\big\|f\big\|^2. \tag{2.7}$$

The above definition extends the corresponding one in [20]. In case of 1-generator Gabor systems, the following result from [20] shows the Reisz basis property.

Lemma 2.1. *Let $g \in L^2(\mathbb{R}^+)$ and $p, q \in \mathbb{R}^+ \setminus \{0\}$ be given. Then the following hold:*
(i) If $|pq| > 1$, then $E_{mq} T_{np} g_{m,n \in \mathbb{Z}^+}$ is not a frame for $L^2(\mathbb{R}^+)$.
(ii) If $E_{mq} T_{np} g_{m,n \in \mathbb{Z}^+}$ is a frame, then $|pq| = 1$ if and only if $E_{mq} T_{np} g_{m,n \in \mathbb{Z}^+}$ is a Riesz basis.

The following two lemmas are very important in proving the results of this paper whose proof can be found in [4].

Lemma 2.2. *Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of elements in a Hilbert space $\mathcal{H}$ and that there exists a constant $B > 0$ such that*

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2$$

for all f in a dense subset $\mathcal{V}$ of $\mathcal{H}$. Then $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence with bound B.

Lemma 2.3. *Suppose that* $\{f_k\}_{k=1}^{\infty}$ *is a sequence of elements in* $\mathcal{H}$ *and that there exist constants* $A, B > 0$ *such that*

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2$$

for all f *in a dense subset* $\mathcal{V}$ *of* $\mathcal{H}$. *Then* $\{f_k\}_{k=1}^{\infty}$ *is a frame for* $\mathcal{H}$ *with bounds* A, B.

The following lemmas 2.4, 2.5, and 2.6 are based on the corresponding results for Gabor frames for $L^2(\mathbb{R})$ [4].

Lemma 2.4. *Let* $f, \psi_\ell \in L^2(\mathbb{R}^+), p, q_\ell \in \mathbb{R}^+ \setminus \{0\}$ *and* $k \in \mathbb{Z}^+$ *be given. Then the series*

$$\sum_{n\in\mathbb{Z}^+} f(x \ominus np)\psi_\ell(x \ominus np \ominus q_\ell^{-1}k), \;\; x \in \mathbb{R}^+$$

converges absolutely for almost all $x \in \mathbb{R}^+$. *Furthermore, for any* $m \in \mathbb{Z}^+$, *we have*

$$\langle f, E_{mq_\ell}T_{np}\psi_\ell \rangle = \int_0^{q_\ell^{-1}} \sum_{k\in\mathbb{Z}^+} f(x \ominus q_\ell^{-1}k)\overline{\psi_\ell(x \ominus np \ominus q_\ell^{-1}k)}\overline{w_{mq_\ell}(x)}dx.$$

Proof. Since $f, T_{q_\ell^{-1}k}\psi_\ell(x) \in L^2(\mathbb{R}^+)$, we have $f\overline{T_{q_\ell^{-1}k}\psi_\ell(x)} \in L^1(\mathbb{R}^+)$. Thus

$$\int_0^{q_\ell^{-1}} \sum_{n\in\mathbb{Z}^+} |f(x \ominus q_\ell^{-1}k)\overline{\psi_\ell(x \ominus np \ominus q_\ell^{-1}k)}|dx$$
$$= \int_{\mathbb{R}^+} \left| f(x \ominus q_\ell^{-1}k)\overline{\psi_\ell(x \ominus q_\ell^{-1}k)} \right| dx < \infty,$$

and hence

$$\sum_{n\in\mathbb{Z}^+} |f(x \ominus q_\ell^{-1}k)\overline{\psi_\ell(x \ominus np \ominus q_\ell^{-1}k)}| < \infty, \; \forall \; x \in [0, q_\ell^{-1}).$$

Since the series $\sum_{n\in\mathbb{Z}^+} f(x \ominus np)\psi_\ell(x \ominus np \ominus q_\ell^{-1}k)$ converges a.e on $[0.q_\ell^{-1})$, therefore, it converges absolutely a.e on $\mathbb{R}^+$ and defines a periodic function. Further, we have

$$\langle f, E_{mq_\ell}T_{np}\psi_\ell \rangle = \int_{\mathbb{R}^+} f(x)\overline{\psi_\ell(x \ominus np)w_{mq_\ell}(x)}dx$$
$$= \int_0^{q_\ell^{-1}} \sum_{k\in\mathbb{Z}^+} f(x \ominus q_\ell^{-1}k)\overline{\psi_\ell(x \ominus np \ominus q_\ell^{-1}k)}\overline{w_{mq_\ell}(x)}dx.$$

This completes the proof. □

Lemma 2.5. *Suppose that* f *is a bounded measurable function with compact support and that the function* $\sum_{n\in\mathbb{Z}^+} |\psi(x \ominus np)|^2$ *is bounded. Then*

$$\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+} |\langle f, E_{mq}T_{np}\psi \rangle|^2 = \frac{1}{q}\int_{\mathbb{R}^+} |f(x)|^2 \sum_{n\in\mathbb{Z}^+} |\psi(x \ominus np)|^2 dx$$
$$+ \frac{1}{q}\sum_{k\in\mathbb{Z}^+\setminus\{0\}} \int_{\mathbb{R}^+} \overline{f(x)}f(x \ominus q^{-1}k) \sum_{n\in\mathbb{Z}^+} \psi(x \ominus np)\overline{\psi(x \ominus np \ominus q^{-1}k)}dx.$$

Proof. Let $n \in \mathbb{Z}^+$ and consider the q^{-1}-periodic function F_n defined as

$$F_n(x) = \sum_{k\in\mathbb{Z}^+} f(x \ominus q^{-1}k)\overline{\psi(x \ominus np \ominus q^{-1}b)}$$

For a given $x \in \mathbb{R}^+$, the compact support of f implies that $f(x \ominus q^{-1}k)$ only can be non-zero for finitely many k-values. The number of k-values for which $f(x \ominus q^{-1}k) \neq 0$ is uniformly bounded, i.e., there is a constant C such that atmost C k- values appear independently of the chosen x. It follows that F_n is bounded, so $F_n \in L^1(0, q^{-1}) \cap L^2(0, q^{-1})$. By Lemma 2.4, for all $m.n \in \mathbb{Z}^+$,

$$\langle f, E_{mq}T_{np}\psi\rangle = \int_0^{q^{-1}} F_n(x)\overline{w_{mq}(x)}. \tag{2.8}$$

Parseval's theorem on F_n gives

$$\sum_{n\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \int_0^{q^{-1}} F_n(x)\overline{w_{mq}(x)}dx \right|^2 = \frac{1}{q}\int_0^{q^{-1}} |F_n(x)|^2 dx. \tag{2.9}$$

The assumption on f being a bounded measurable function with compact support will justify all interchanges of integration and summation in the final calculation. This follows from the observation that

$$\sum_{k\in\mathbb{Z}^+\setminus\{0\}} \int_{\mathbb{R}^+} \overline{f(x)} f(x \ominus q^{-1}k) \sum_{n\in\mathbb{Z}^+} \psi(x \ominus np)\overline{\psi(x \ominus np \ominus q^{-1}k)}dx < \infty.$$

By virtue of (2.8) and (2.9), we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+} |\langle f, E_{mq}T_{np}\psi\rangle|^2 &= \sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+} \left| \int_0^{q^{-1}} F_n(x)\overline{w_{mq}(x)}dx \right|^2 \\ &= \frac{1}{q}\int_0^{q^{-1}} |F_n(x)|^2 dx.\end{aligned}$$

Writing

$$|F_n(x)|^2 = F_n(x)\overline{F_n(x)} = \sum_{\ell\in\mathbb{Z}^+} \overline{f(x \ominus q^{-1}\ell)}\psi(x \ominus np \ominus q^{-1}\ell)F_n(x),$$

and using the fact that F_n is q^{-1}- periodic, we have

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+} |\langle f, E_{mq}T_{np}\psi\rangle|^2 = \frac{1}{q}\sum_{n\in\mathbb{Z}^+}\int_0^{q^{-1}} \sum_{\ell\in\mathbb{Z}^+} \overline{f(x \ominus q^{-1}\ell)}\psi(x \ominus np \ominus q^{-1}\ell)F_n(x)dx \\ &= \frac{1}{q}\sum_{n\in\mathbb{Z}^+}\int_0^{q^{-1}} \overline{f(x)}\psi(x \ominus np)F_n(x)dx \\ &= \frac{1}{q}\sum_{n\in\mathbb{Z}^+}\int_0^{q^{-1}} \overline{f(x)}\psi(x \ominus np)\sum_{k\in\mathbb{Z}^+} \overline{f(x \ominus q^{-1}k)}\psi(x \ominus np \ominus q^{-1}k)dx\end{aligned}$$

$$= \frac{1}{q}\int_{\mathbb{R}^+} |f(x)|^2 \sum_{n\in\mathbb{Z}^+} |\psi(x\ominus np)|^2 dx$$
$$+ \frac{1}{q}\sum_{k\in\mathbb{Z}^+\setminus\{0\}} \int_{\mathbb{R}^+} \overline{f(x)} f(x\ominus q^{-1}k) \sum_{n\in\mathbb{Z}^+} \psi(x\ominus np)\overline{\psi(x\ominus np\ominus q^{-1}k)}dx.$$

□

Lemma 2.6. *Let $\{f_n\}_{n=1}^{\infty}$ be a family of functions in $L^2(\mathbb{R}^+)$ and suppose that for every $q \in \mathbb{R}^+ \setminus \{0\}$, we have*

$$B = \frac{1}{|q|}\sup_{x\in\mathbb{R}^+} \sum_{k\in\mathbb{Z}^+} \left| \sum_{n\in\mathbb{Z}^+} f_n(x)\overline{f_n\left(x\ominus q^{-1}k\right)} \right| < \infty. \tag{2.10}$$

Then $\{E_{mq}f_n : m, n \in \mathbb{Z}^+\}$ constitutes a Bessel sequences with upper bound B for $L^2(\mathbb{R}^+)$. Furthermore, if

$$A = \frac{1}{|q|}\inf_{x\in\mathbb{R}^+} \left\{ \sum_{n\in\mathbb{Z}^+} \left|f_n(x)\right|^2 - \sum_{k\in\mathbb{N}} \left| \sum_{n\in\mathbb{Z}^+} f_n(x)\overline{f_n\left(x\ominus q^{-1}k\right)} \right| \right\} > 0, \tag{2.11}$$

then $\{E_{mq}f_n : m, n \in \mathbb{Z}^+\}$ forms a frame in $L^2(\mathbb{R}^+)$ with bounds A and B.

Proof. Consider $f_n \in L^2(\mathbb{R}^+)$ that is continuous and has compact support. By Lemma 2.5,we have

$$\sum_{m\in\mathbb{Z}^+} \left|\left\langle f_n, E_{mq}f_n \right\rangle\right|^2 = \frac{1}{|q|}\int_{\mathbb{R}^+} |f_n(x)|^2 \sum_{n\in\mathbb{Z}^+} |f_n(x)|^2 dx$$
$$+ \frac{1}{|q|}\sum_{k\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \overline{f_n(x)} f_n(x\ominus q^{-1}k) \sum_{n\in\mathbb{Z}^+} f_n(x)\overline{f_n(x\ominus q^{-1}k)}dx.$$

For $k \in \mathbb{Z}^+$, let $B_k(x) = \sum_{n\in\mathbb{Z}^+} f_n(x)\overline{T_{q^{-1}k}f_n(x)}$. It is clear that $B_k(x)$ is well defined by Lemma 2.4. Now

$$\sum_{k\in\mathbb{N}} \left|T_{-kq^{-1}}B_k(x)\right| = \sum_{k\in\mathbb{N}} \left| T_{-kq^{-1}} \sum_{n\in\mathbb{Z}^+} f_n(x)\overline{T_{q^{-1}k}f_n(x)} \right|$$
$$= \sum_{k\in\mathbb{N}} \left| \sum_{n\in\mathbb{Z}^+} T_{-kq^{-1}}f_n(x)\overline{f_n(x)} \right|$$

Replacing k with $-k$ (which is allowed because we sum over all k) and complex conjugating all terms, we arrive at

$$\sum_{k\in\mathbb{N}} \left|T_{-kq^{-1}}B_k(x)\right| = \sum_{k\in\mathbb{N}} \left| \sum_{n\in\mathbb{Z}^+} T_{-kq^{-1}}f_n(x)\overline{f_n(x)} \right|$$
$$= \sum_{k\in\mathbb{N}} \left| \sum_{n\in\mathbb{Z}^+} \overline{T_{kq^{-1}}f_n(x)} f_n(x) \right|$$
$$= \sum_{k\in\mathbb{N}} \left|B_k(x)\right|.$$

Now,

$$\begin{aligned}
&\left|\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^+}\overline{f_n(x)}f_n(x\ominus q^{-1}k)\sum_{n\in\mathbb{Z}^+}f_n(x)\overline{f_n(x\ominus q^{-1}k)}dx\right| \\
&\qquad\qquad\le \sum_{k\in\mathbb{N}}\int_{\mathbb{R}^+}|f_n(x)|\left|T_{kq^{-1}}f_n(x)\right||B_k(x)|dx \\
&\qquad\qquad= \sum_{k\in\mathbb{N}}\int_{\mathbb{R}^+}|f_n(x)|\sqrt{|B_k(x)|}\left|T_{kq^{-1}}f_n(x)\right|\sqrt{|B_k(x)|}dx \\
&\qquad\qquad= (\star).
\end{aligned}$$

Using Cauchy's-Schwarz inequality twice, first on the integral, and then on the sum over k,

$$\begin{aligned}
(\star) &\le \sum_{k\in\mathbb{N}}\left\{\int_{\mathbb{R}^+}|f_n(x)|^2|B_k(x)|dx\right\}^{1/2}\left\{\int_{\mathbb{R}^+}\left|T_{kq^{-1}}f_n(x)\right|^2|B_k(x)|dx\right\}^{1/2} \\
&\le \left\{\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^+}|f_n(x)|^2|B_k(x)|dx\right\}^{1/2}\left\{\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^+}\left|T_{kq^{-1}}f_n(x)\right|^2|B_k(x)|dx\right\}^{1/2} \\
&= \left\{\int_{\mathbb{R}^+}|f_n(x)|^2\sum_{k\in\mathbb{N}}|B_k(x)|dx\right\}^{1/2}\left\{\int_{\mathbb{R}^+}\left|T_{-kq^{-1}}f_n(x)\right|^2\sum_{k\in\mathbb{N}}|B_k(x)|dx\right\}^{1/2} \\
&= \int_{\mathbb{R}^+}|f_n(x)|^2\sum_{k\in\mathbb{N}}|B_k(x)|dx.
\end{aligned}$$

Now we have

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^+}|\langle f_n, E_{mq}f_n\rangle|^2 \\
&\qquad\le \frac{1}{|q|}\int_{\mathbb{R}^+}\left\{|f_n(x)|^2\left[\sum_{n\in\mathbb{Z}^+}|f_n(x)|^2+\sum_{k\in\mathbb{N}}\left|\sum_{n\in\mathbb{Z}^+}f_n(x)\overline{f_n(x\ominus q^{-1}k)}\right|\right]\right\}dx \\
&\qquad= \frac{1}{|q|}\int_{\mathbb{R}^+}|f_n(x)|^2\sum_{k\in\mathbb{N}}\left|\sum_{n\in\mathbb{Z}^+}f_n(x)\overline{f_n(x\ominus q^{-1}k)}\right|dx \\
&\qquad\le B\|f_n\|^2.
\end{aligned}$$

Because this estimate holds on a dense subset of $L^2(\mathbb{R}^+)$, it holds on $L^2(\mathbb{R}^+)$ by Lemma 2.2. This proves the first part. If also (2.11) is satisfied, we again consider f_n which are continuous with compact support and obtain that

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^+}|\langle f_n, E_{mq}f_n\rangle|^2 \\
&\qquad\ge \frac{1}{|q|}\int_{\mathbb{R}^+}\left\{|f_n(x)|^2\left[\sum_{n\in\mathbb{Z}^+}|f_n(x)|^2-\sum_{k\in\mathbb{N}}\left|\sum_{n\in\mathbb{Z}^+}f_n(x)\overline{f_n(x\ominus q^{-1}k)}\right|\right]\right\}dx \\
&\qquad\ge A\|f_n\|^2.
\end{aligned}$$

By Lemma 2.3, the lower frame condition actually holds for all $f \in L^2(\mathbb{R}^+)$. This completed the proof of the lemma. □

3. Necessary and sufficient conditions for periodic Gabor frames on positive half line

In this Section, we obtain a necessary and sufficient conditions for the periodic Gabor system $\mathfrak{G}(\Psi, p, q)$ defined by (2.6) to be frame for $L^2(\mathcal{S})$. In order to establish the main results, we shall first drive a relationship between the Gabor frames of $L^2(\mathbb{R}^+)$ and its counterpart periodic sets in $L^2(\mathcal{S})$ in the form of the following remark.

Remark 3.1. Let $\Psi = \{\psi_1, \dots, \psi_L\} \subseteq L^2(\mathcal{S})$ and $p, q_\ell \in \mathbb{R}^+ \setminus \{0\}$. If the Gabor system $\mathfrak{G}(\Psi, p, q)$ given by (2.6) is a Bessel sequence for $L^2(\mathcal{S})$ with upper bound B, then it is a Bessel sequence for $L^2(\mathbb{R}^+)$ with the same upper bound.

We now proceed to derive a sufficient condition for the Gabor system $\mathfrak{G}(\Psi, p, q)$ to be frame for $L^2(\mathcal{S})$.

Theorem 3.1. *Let $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\} \subseteq L^2(\mathcal{S})$ and $p, q \in \mathbb{R}^+ \setminus \{0\}$ suppose that*

$$B = \frac{1}{|q_\ell|} \sup_{x\in[0,q^{-1})} \left| \sum_{1\le\ell\le L} \sum_{n\in\mathbb{Z}^+} \sum_{k\in\mathbb{Z}^+} T_{np}\psi_\ell(x) \overline{T_{np}\psi_\ell\big(x \ominus q_\ell^{-1}k\big)} \right| < \infty. \tag{3.2}$$

Then, the periodic Gabor system $\mathfrak{G}(\Psi, p, q)$ is a Bessel sequence with upper bound B. Furthermore, if

$$A = \frac{1}{|q_\ell|} \inf_{x\in[0,q^{-1})} \left\{ \left| \sum_{1\le\ell\le L} \sum_{n\in\mathbb{Z}^+} T_{np}\psi_\ell(x) \right|^2 - \left| \sum_{1\le\ell\le L} \sum_{n\in\mathbb{Z}^+} \sum_{k\in\mathbb{Z}^+} T_{np}\psi_\ell(x) \overline{T_{np}\psi_\ell\big(x \ominus q_\ell^{-1}k\big)} \right| \right\} > 0. \tag{3.3}$$

Then, the Periodic Gabor system $\mathfrak{G}(\Psi, p, q)$ constitutes a frame for $L^2(\mathcal{S})$ with frame bounds A and B.

Proof. We define

$$\Phi_1(x) = \left| \sum_{1\le\ell\le L} \sum_{n\in\mathbb{Z}^+} \sum_{k\in\mathbb{Z}^+} T_{np}\psi_\ell(x) \overline{T_{np}\psi_\ell\big(x \ominus q_\ell^{-1}k\big)} \right|$$

$$\Phi_2(x) = \left| \sum_{1\le\ell\le L} \sum_{n\in\mathbb{Z}^+} T_{np}\psi_\ell(x) \right|^2 - \left| \sum_{1\le\ell\le L} \sum_{n\in\mathbb{Z}^+} \sum_{k\in\mathbb{Z}^+} T_{np}\psi_\ell(x) \overline{T_{np}\psi_\ell\big(x \ominus q_\ell^{-1}k\big)} \right|.$$

Then, it is clear that both $\Phi_1(x)$ and $\Phi_2(x)$ are periodic functions and, hence we obtain

$$B = \frac{1}{|q_\ell|} \sup_{x\in\mathbb{R}^+} \left| \sum_{1\le\ell\le L} \sum_{n\in\mathbb{Z}^+} \sum_{k\in\mathbb{Z}^+} T_{(n)p}\psi_\ell(x) \overline{T_{np}\psi_\ell\big(x \ominus q_\ell^{-1}k\big)} \right| < \infty, \tag{3.4}$$

$$A = \frac{1}{|q_\ell|} \inf_{x\in\mathbb{R}^+} \Bigg\{ \Big| \sum_{1\le\ell\le L} \sum_{n\in\mathbb{Z}^+} T_{np}\psi_\ell(x) \Big|^2 - \Big| \sum_{1\le\ell\le L} \sum_{n\in\mathbb{Z}^+} \sum_{k\in\mathbb{Z}^+} T_{np}\psi_\ell(x) \overline{T_{np}\psi_\ell\big(x\ominus q_\ell^{-1}k\big)} \Big| \Bigg\} > 0. \tag{3.5}$$

For each $1 \le \ell \le L$ and $n = \ell \oplus sk, k \in \mathbb{Z}^+$, we consider

$$f_n(x) = T_{kp}\psi_\ell(x). \tag{3.6}$$

Then, it follows from (3.4) and (3.5) that

$$B = \frac{1}{|q_\ell|} \sup_{x\in\mathbb{R}^+} \Big| \sum_{n\in\mathbb{Z}^+} \sum_{k\in\mathbb{Z}^+} f_n(x) \overline{f_n\big(x\ominus q_\ell^{-1}k\big)} \Big| < \infty,$$

$$A = \frac{1}{|q_\ell|} \inf_{x\in\mathbb{R}^+} \Bigg\{ \sum_{n\in\mathbb{Z}^+} \Big| f_n(x) \Big|^2 - \sum_{n\in\mathbb{Z}^+} \sum_{k\in\mathbb{Z}^+} \Big| f_n(x) \overline{f_n\big(x\ominus q_\ell^{-1}k\big)} \Big| \Bigg\} > 0.$$

Invoking Lemma 2.6, and the fact $L^2(\mathcal{S}) \subset L^2(\mathbb{R}^+)$, we obtain the desired result. □

Next, we establish a necessary condition for the periodic Gabor system $\mathfrak{G}(\Psi, p, q)$ to be a frame for $L^2(\mathcal{S})$, which depend upon the interplay among the generating functions ψ_ℓ, and the parameters $p, q_1, \dots, q_L$.

Theorem 3.2. *Let $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\} \subseteq L^2(\mathcal{S})$, and $p, q_1, q_2, \dots, q_L \in \mathbb{R}^+ \setminus \{0\}$. If the system $\mathfrak{G}(\Psi, p, q)$ as defined by (2.6) is a Gabor frame for $L^2(\mathcal{S})$ with bounds A and B, then*

$$A\,\chi_{\mathcal{S}}(x) \le \sum_{1\le\ell\le L} \Bigg\{ \frac{1}{|q_\ell|} \sum_{n\in\mathbb{Z}^+} \Big|\psi_\ell\big(x\ominus np\big)\Big|^2 \Bigg\} \le B\,\chi_{\mathcal{S}}(x), \quad a.e.. \tag{3.7}$$

Proof. Since $\mathcal{S}$ is a a-periodic measurable subset of $\mathbb{R}^+$, therefore the family $\psi_\ell(x\ominus np)$ belong to $L^2(\mathcal{S})$ for all $n \in \mathbb{N}_0$ and for each $1 \le \ell \le L$. From this, we infer that

$$\sum_{1\le\ell\le L} \Bigg\{ \frac{1}{|q_\ell|} \sum_{n\in\mathbb{Z}^+} \Big|\psi_\ell\big(x\ominus np\big)\Big|^2 \Bigg\} = 0, \quad a.e.\ x \in \mathbb{R}^+ \setminus \mathcal{S}.$$

We establish our desired result by contradiction. Assume that the upper bound in (3.7) is not true on $\mathcal{S}$. As such, there exist a measurable subset $H \subset \mathcal{S}$ with positive measure such that

$$\sum_{1\le\ell\le L} \Bigg\{ \frac{1}{|q_\ell|} \sum_{n\in\mathbb{Z}^+} \Big|\psi_\ell\big(x\ominus np\big)\Big|^2 \Bigg\} > B, \quad a.e.\ x \in H.$$

Since H is of positive measure, so for each fixed $\ell = 1, 2, \ldots, L$, we contain the set E in a sphere F with diameter $|q_\ell|^{-1}$. Setting

$$H_0 = \left\{ x \in H : \frac{1}{|q_\ell|} \sum_{n \in \mathbb{Z}^+} \left|\psi_\ell\big(x \ominus np\big)\right|^2 \geq \frac{1}{|q_\ell|} + B \right\}$$

$$H_k = \left\{ x \in H : \frac{1}{|q_\ell|(k+1)} + B \leq \frac{1}{|q_\ell|} \sum_{n \in \mathbb{Z}^+} \left|\psi_\ell\big(x \ominus np\big)\right|^2 < \frac{1}{|q_\ell| k} + B \right\}, \ k \in \mathbb{N}.$$

From the above setting, it is clear that each $H_k, k \in \mathbb{Z}^+$ is measurable and forms a partition of E. Since E is of positive measure, hence, there exist atleast one $H_s, s \in \mathbb{Z}^+$ with positive measure. If $f(x) = \chi_{H_s}(x)$, for some characteristic function χ_{H_s}, then $\|f\|^2 = |H_s|$ and the function $f(x)\overline{T_{np}\psi_\ell}(x)$ has a compact support in H_s, for each $1 \leq \ell \leq L$. Using the fact that H_s is contained in $\mathcal{S}$ and the collection $\left\{ \sqrt{|q_\ell|} w_{mq_\ell}(x) : m \in \mathbb{Z}^+, 1 \leq \ell \leq L \right\}$ forms an orthonormal basis for $L^2(\mathcal{S})$, we have

$$\begin{aligned}
\sum_{m \in \mathbb{Z}^+} \left| \left\langle f, E_{mq_\ell} T_{np} \psi_\ell \right\rangle \right|^2 &= \sum_{m \in \mathbb{Z}^+} \left| \left\langle \overline{T_{np}\psi_\ell} f, E_{mq_\ell} \right\rangle \right|^2 \\
&= \frac{1}{|q_\ell|} \int_{\mathbb{R}^+} \left|f(x)\right|^2 \left|\psi_\ell\big(x \ominus np\big)\right|^2 dx \\
&= \frac{1}{|q_\ell|} \int_{H_s} \left|f(x)\right|^2 dx \sum_{n \in \mathbb{Z}^+} \left|\psi_\ell\big(x \ominus np\big)\right|^2 \\
&\geq \left(B + \frac{1}{|q_\ell|(s+1)} \right) \|f\|_2^2.
\end{aligned}$$

Therefore, we have

$$\sum_{1 \leq \ell \leq L} \sum_{m \in \mathbb{Z}^+} \sum_{n \in \mathbb{Z}^+} \left| \left\langle f, E_{mq_\ell} T_{np} \psi_\ell \right\rangle \right|^2 \geq \sum_{1 \leq \ell \leq L} \left(B + \frac{1}{|q_\ell|(s+1)} \right) \|f\|_2^2,$$

which is a contradiction to the assumption that B is an upper frame bound for the Gabor system $\mathfrak{G}(\Psi, p, q)$. In the similar lines, one can show that if the lower bound in (3.7) is violated, then A cannot be lower bound for Gabor system given by (2.6). This completes the proof of the theorem. □

4. Parseval periodic Gabor frames on positive half line

In this section, we provide the characterization the periodic Gabor system $\mathfrak{G}(\Psi, p, q)$ as defined by (2.6) to be Parseval periodic Gabor frame for $L^2(\mathcal{S})$. For that we first state and prove the following lemmas that will be useful in our main results.

Lemma 4.1. *Let $\mathfrak{G}(\Psi, p, q)$ be the multi-generator Gabor system given by (2.6). Then, for each $\psi_\ell \in L^2(\mathbb{R}^+)$ and $f \in \mathcal{E}^0(\mathbb{R}^+)$, we have*

$$\begin{aligned}
\sum_{m \in \mathbb{Z}^+} \sum_{n \in \mathbb{Z}^+} \left| \left\langle f, E_{mq_\ell} T_{np} \psi_\ell \right\rangle \right|^2 &= \frac{1}{|q_\ell|} \int_{\mathbb{R}^+} \left|f(x)\right|^2 \sum_{n \in \mathbb{Z}^+} \left|\psi_\ell\big(x \ominus np\big)\right|^2 dx \\
&+ \sum_{n \in \mathbb{Z}^+} \frac{1}{|q_\ell|} \int_{\mathbb{R}^+} \overline{f(x)}\, f\big(x \ominus np\big) \sum_{k \in \mathbb{Z}^+} \psi_\ell\Big(x \oplus q_\ell^{-1} k\Big) \overline{\psi_\ell\Big(x \oplus q_\ell^{-1} k \ominus np\Big)}\, dx
\end{aligned}$$

Proof. Since $f \in \mathcal{E}^0(\mathbb{R}^+)$, the compact support of f implies the function $f(x \ominus q_\ell^{-1}k)$ can be non-zero only for finitely many values of k. The number of values of k for which $f(x \ominus q_\ell^{-1}k) \neq 0$ is uniformly bounded i.e., there is a constant C such that atmost C k -values appear independently of the chosen x. Consequently, each $\sum_{n\in\mathbb{Z}^+} f\Big(x\oplus q_\ell^{-1}k\Big)\overline{\psi_\ell\Big(x \oplus q_\ell^{-1}k \ominus np\Big)}$ is bounded, and hence $\sum_{n\in\mathbb{Z}^+} f\Big(x\oplus q_\ell^{-1}k\Big)\overline{\psi_\ell\Big(x \oplus q_\ell^{-1}k \ominus np\Big)} \in L^1[0, q_\ell^{-1}) \cap L^2[0, q_\ell^{-1})$. By Lemma 2.4, for $m, n \in \mathbb{Z}^+$, we have

$$\left|\Big\langle f, E_{mq_\ell}T_{np}\psi_\ell\Big\rangle\right|^2 = \int_0^{q_\ell^{-1}} \sum_{n\in\mathbb{Z}^+} f\Big(x \oplus q_\ell^{-1}k\Big)\overline{\psi_\ell\Big(x \oplus q_\ell^{-1}k \ominus np\Big)}\,\overline{w_{mq_\ell}(x)}dx.$$

Applying Parseval's theorem and using the fact that $\left\{q_\ell^{\frac{1}{2}} w_{mq_\ell}(x) : m \in \mathbb{Z}^+\right\}$ forms an orthonormal basis for $L^2[0, q_\ell^{-1})$, we obtain

$$\sum_{m\in\mathbb{Z}^+}\left|\int_0^{q_\ell^{-1}} \sum_{n\in\mathbb{Z}^+} f\Big(x \oplus q_\ell^{-1}k\Big)\overline{\psi_\ell\Big(x \oplus q_\ell^{-1}k \ominus np\Big)}\,\overline{w_{mq_\ell}(x)}dx\right|^2$$
$$= \frac{1}{|q_\ell|}\int_0^{q_\ell^{-1}} \left|\sum_{n\in\mathbb{Z}^+} f\Big(x \oplus q_\ell^{-1}k\Big)\overline{\psi_\ell\Big(x \oplus q_\ell^{-1}k \ominus np\Big)}\right|^2 dx$$

Now we have

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\Big\langle f, E_{mq_\ell}T_{np}\psi_\ell\Big\rangle\right|^2 \\
&= \sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\int_0^{q_\ell^{-1}}\left|f(x)\,\overline{\psi_\ell(x \ominus np)}\,\overline{w_{mq_\ell}(x)}\,dx\right|^2 \\
&= \sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\int_0^{q_\ell^{-1}}\left|f\big(x \oplus np\big)\,\overline{\psi_\ell(x)}\,\overline{w_{mq_\ell}(x)}\,dx\right|^2 \\
&= \sum_{n\in\mathbb{Z}^+}\frac{1}{|q_\ell|}\int_{\mathbb{R}^+}\overline{f\big(x \oplus np\big)}\,\psi_\ell(x)\sum_{m\in\mathbb{Z}^+} f\Big(x \oplus q_\ell^{-1}m \oplus np\Big)\overline{\psi_\ell\Big(x \oplus q_\ell^{-1}m\Big)}dx \\
&= \sum_{n\in\mathbb{Z}^+}\frac{1}{|q_\ell|}\int_{\mathbb{R}^+}\left|f\big(x \oplus np\big)\right|^2 |\psi_\ell(x)|^2\,dx \\
&+ \sum_{n\in\mathbb{Z}^+}\frac{1}{|q_\ell|}\int_{\mathbb{R}^+}\overline{f\big(x \oplus np\big)}\,\psi_\ell(x)\sum_{m\in\mathbb{Z}^+} f\Big(x \oplus q_\ell^{-1}m \oplus np\Big)\overline{\psi_\ell\Big(x \oplus q_\ell^{-1}m\Big)}dx \\
&= \frac{1}{|q_\ell|}\int_{\mathbb{R}^+}\left|f(x)\right|^2 dx\sum_{n\in\mathbb{Z}^+}\left|\psi_\ell\big(x \ominus np\big)\right|^2 dx \\
&+ \sum_{k\in\mathbb{Z}^+}\frac{1}{|q_\ell|}\int_{\mathbb{R}^+}\overline{f(x)}\,f\big(x \ominus np\big)\sum_{n\in\mathbb{N}}\psi_\ell\Big(x \oplus q_\ell^{-1}k\Big)\overline{\psi_\ell\Big(x \oplus q_\ell^{-1}k \ominus np\Big)}\,dx.
\end{aligned}$$

□

Theorem 4.2. *If the Gabor system $\mathfrak{G}(\Psi, p, q)$ as defined by (2.6) constitutes a Parseval frame for $L^2(\mathcal{S})$ with frame bound $A = 1$, then*

$$\sum_{1\le \ell\le L} \frac{1}{|q_\ell|}\left\{\sum_{n\in\mathbb{Z}^+} \left|\psi_\ell\big(x\ominus np\big)\right|^2\right\} = \chi_{\mathcal{S}}, \ \ a.e. \tag{4.1}$$

Moreover, if $q_1 = q_2 = \cdots = q_L = q$, then

$$\sum_{1\le \ell\le L} \left\{\sum_{n\in\mathbb{Z}^+} \left|\psi_\ell\big(x\ominus np\big)\right|^2\right\} = |q|\,\chi_{\mathcal{S}}, \ \ a.e. \tag{4.2}$$

$$\sum_{1\le \ell\le L} \left\{\sum_{k\in\mathbb{N}} \psi_\ell\Big(x\oplus q_\ell^{-1}m\Big)\overline{\psi_\ell\Big(x\oplus q_\ell^{-1}k\ominus np\Big)}\right\} = 0. \tag{4.3}$$

Proof. To establish our result, we define $\nu^1 = \min\left\{|b_\ell|^{-1} : 1\le \ell\le L\right\}$. For this choice of ν^1, we consider

$$\mathcal{D} = \Big\{f : f\in L^2(\mathcal{S}) \text{ and } \operatorname{supp} f\subset \big([0,\nu^1)\cap\mathcal{S}\big)\Big\}.$$

For all $f\in\mathcal{D}$ and any fixed ℓ, we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\Big\langle f, E_{mq_\ell}T_{np}\psi_\ell\Big\rangle\right|^2 &= \sum_{m\in\mathbb{Z}^+}\left|\int_{\mathbb{R}^+} f(x)\overline{\psi_\ell\big(x\ominus np\big)}\,\overline{w_{mq_\ell}(x)}\,dx\right|^2\\ &= \frac{1}{|q_\ell|}\int_0^{\nu^1}\left|f(x)\overline{\psi_\ell\big(x\ominus np\big)}\right|^2 dx.\end{aligned}$$

Therefore,

$$\frac{1}{|q_\ell|}\int_0^{\nu^1}\left|f(x)\right|^2 dx = \int_0^{\nu^1}\left|f(x)\right|^2 \sum_{1\le \ell\le L}\frac{1}{|q_\ell|}\left\{\sum_{n\in\mathbb{Z}^+}\left|\psi_\ell\big(x\ominus np\big)\right|^2\right\} dx$$

for any $f\in\mathcal{D}$. This implies that that

$$\sum_{1\le \ell\le L} \frac{1}{|q_\ell|}\left\{\sum_{n\in\mathbb{Z}^+} \left|\psi_\ell\big(x\ominus na\big)\right|^2\right\} = 1 \text{ a.e. } [0,\nu^1)\cap\mathcal{S}. \tag{4.4}$$

which gives the desired result (4.1) and it's particular case (4.2).

Next we proceed to prove (4.3). Using Lemma 4.1 for any fixed $\ell, 1\le \ell\le L$, we get

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\Big\langle f, E_{mq_\ell}T_{np}\psi_\ell\Big\rangle\right|^2\\ &= \frac{1}{|q_\ell|}\int_{\mathbb{R}^+}\left|f(x)\right|^2 dx\sum_{n\in\mathbb{Z}^+}\left|\psi_\ell\big(x\ominus np\big)\right|^2 + \sum_{n\in\mathbb{Z}^+}\frac{1}{|q_\ell|}\int_{\mathbb{R}^+}\overline{f(x)}\,\psi_\ell\big(x\ominus np\big)\,dx\\ &\qquad\times\sum_{k\in\mathbb{Z}^+}\psi_\ell\Big(x\oplus q_\ell^{-1}k\Big)\overline{\psi_\ell\Big(x\oplus q_\ell^{-1}k\ominus np\Big)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_{1\le \ell\le L}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\Big|\Big\langle f, E_{mq_\ell}T_{np}\psi_\ell\Big\rangle\Big|^2\\
&=\int_{\mathbb{R}^+}\big|f(x)\big|^2dx\sum_{1\le\ell\le L}\frac{1}{|q_\ell|}\sum_{n\in\mathbb{Z}^+}\big|\psi_\ell\big(x\ominus np\big)\big|^2\\
&\qquad\qquad+\sum_{1\le\ell\le L}\sum_{n\in\mathbb{Z}^+}\frac{1}{|b_\ell|}\int_{\mathbb{R}^+}\overline{f(x)}\,\psi_\ell\big(x\ominus np\big)\,dx\\
&\qquad\qquad\qquad\times\sum_{k\in\mathbb{Z}^+}\psi_\ell\Big(x\oplus q_\ell^{-1}k\Big)\overline{\psi_\ell\Big(x\oplus q_\ell^{-1}k\ominus np\Big)}\\
&=I_1+I_2. \qquad (4.5)
\end{aligned}$$

Since the system $\mathfrak{G}(\Psi,p,q)$ given by (2.6) is a tight frame with frame bound $A=1$ and using (4.1), it follows that $I_2=0$. Also, by changing k to $\ominus k$, we observe that the contribution in I_2 for any value of k is the complex conjugate of the contribution from the value $\ominus k$. Therefore, we can write

$$\sum_{k\in\mathbb{Z}^+}\mathrm{Re}\left\{\frac{1}{|q_\ell|}\int_{\mathbb{R}^+}\overline{f(x)}\,\psi_\ell\big(x\ominus np\big)\,F_k(x)\,dx\right\}=0 \qquad (4.6)$$

where

$$F_k(x)=\sum_{1\le\ell\le L}\left(\sum_{n\in\mathbb{Z}^+}\psi_\ell\big(x\oplus q_\ell^{-1}k\big)\overline{\psi_\ell\big(x\oplus q_\ell^{-1}k\ominus np\big)}\right)=0. \qquad (4.7)$$

To establish the required result, we consider three cases. Firstly, we consider the case when $x\in\mathcal{S}$. Since $\mathcal{S}$ is a-periodic set, so $(x\ominus np)\in\mathcal{S}$ for all $n\in\mathbb{Z}^+$ and hence, $\psi_\ell(x\ominus na)=0$, for all $n\in\mathbb{Z}^+, 1\le\ell\le L$. Thus, $F_k(x)=0,\ \forall\ k\in\mathbb{N}$. For case second, we consider $x\ominus q^{-1}k\notin\mathcal{S}$ for fixed $k\in\mathbb{N}$. Then, it is obvious that $(x\ominus np\ominus q^{-1}k)\notin\mathcal{S}$ for all $n\in\mathbb{Z}^+$ and as a result, we have $F_k(x)=0,\ \forall\ k\in\mathbb{N}$. The third case is when $x\in\mathcal{S}$ and $(x\ominus q^{-1}k)\in\mathcal{S}$ for any fixed $k\in\mathbb{N}$. Let $\mathcal{J}$ be any interval in $\mathbb{R}^+$ of atmost $|q|^{-1}$. Denote $\mathcal{J}\cap\mathcal{S}$ by $\mathcal{J}^0$ and $(\mathcal{J}\ominus q_\ell^{-1}k)\cap(\mathcal{S}\oplus q_\ell^{-1}k$ by $\mathcal{J}'$. If the measure of $\mathcal{J}\cap\mathcal{J}'$ is zero, then $x\notin\mathcal{J}^0$ a.e. or $x\notin\mathcal{J}\ominus q^{-1}k$ a.e and hence, $F_k(x)=0,\ \forall\ k\in\mathbb{N}$. Now, if the measure of $\mathcal{J}^0\cap\mathcal{J}'$ is positive. Then, we define a new function $f\in L^2(\mathcal{S})$ by

$$f(x)=\begin{cases}\exp\big\{\ominus\arg F_{k_0}(x)\big\}, & x\in\mathcal{J}^0\cap\mathcal{J}'\\ 1, \quad x\in\mathcal{J}^0\cap\mathcal{J}'\ominus q_\ell^{-1}k\\ 0 \quad \text{otherwise.}\end{cases} \qquad (4.8)$$

Using (4.6), we observe that

$$\int_{\mathcal{J}^0\cap\mathcal{J}'}\big|F_{k_0}(x)\big|\,dx=0.$$

Since $\mathcal{J}$ is an arbitrary interval of length atmost $|q|^{-1}$, so $F_{k_0}(x)=0, a.e$ in $\mathcal{S}$. Moreover, it can be easily verified by a simple computation that $F_{\ominus k_0}(x)=F_{k_0}(x\oplus q_\ell^{-1}u(k_0)),\ 1\le\ell\le L$. This proves the theorem completely. □

Next, we shall establish a sufficient condition for the Gabor system (2.6) to be a Parseval frame for $L^2(\mathcal{S})$. To do so, we define the following:

$$\nu^1 = \min_{1\le \ell \le L}\left\{\frac{1}{|q_\ell|}\right\}$$
$$\nu^j = \min_{1\le \ell \le L}\left\{\frac{1}{|q_\ell|} : |q_\ell| < \frac{1}{\nu^{j-1}}\right\}, \quad j \ge 2$$
$$\mathcal{I}_j = \left\{\ell : |q_\ell| = \frac{1}{\nu^j}, 1 \le \ell \le L\right\}.$$

Furthermore, for a unique positive number $j_0 \in \mathbb{N}$, we have

$$\mathcal{I}_j \ne \emptyset, 1 \le j \le j_0, \quad \mathcal{I}_{j_1} \cap \mathcal{I}_{j_2} = \emptyset,\ j_1 \ne j_2, \quad \text{and} \bigcup_{j=1}^{j_0} \mathcal{I}_j = \{1, 2, \cdots, L\}. \tag{4.9}$$

Theorem 4.3. *Let j_0 be the unique positive integer satisfying (4.9). Suppose that $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\} \subseteq L^2(\mathcal{S})$, and $p, q_1, q_2, \ldots, q_L \in \mathbb{R}^+ \setminus \{0\}$ satisfy*

$$\sum_{1\le \ell \le L}\left\{\sum_{n\in\mathbb{Z}^+}\big|\psi_\ell(x \ominus np)\big|^2\right\} = \chi_{\mathcal{S}} \tag{4.10}$$

$$\sum_{\ell\in\mathcal{I}_j}\left\{\sum_{m\in\mathbb{N}}\psi_\ell\Big(x \oplus q_\ell^{-1}m\Big)\overline{\psi_\ell\Big(x \oplus q_\ell^{-1}k \ominus mp\Big)}\right\} = 0,\ k \in \mathbb{N}, 1 \le j \le j_0. \tag{4.11}$$

Then, the multi-generator Gabor system $\mathfrak{G}(\Psi, p, q)$ as defined by (2.6) is a Parseval frame for $L^2(\mathcal{S})$.

Proof. By Lemma 4.1, we have

$$\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\Big|\Big\langle f, E_{mq_\ell}T_{np}\psi_\ell\Big\rangle\Big|^2 = \frac{1}{|q_\ell|}\int_{\mathbb{R}^+}\big|f(x)\big|^2dx\sum_{n\in\mathbb{Z}^+}\big|\psi_\ell(x\ominus np)\big|^2$$
$$+\sum_{n\in\mathbb{Z}^+}\frac{1}{|q_\ell|}\int_{\mathbb{R}^+}\overline{f(x)}\,\psi_\ell(x\ominus np)\,dx\sum_{k\in\mathbb{Z}^+}\psi_\ell\Big(x\oplus q_\ell^{-1}k\Big)\overline{\psi_\ell\Big(x\oplus q_\ell^{-1}k\ominus np\Big)},$$

which implies that

$$\sum_{1\le\ell\le L}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\Big|\Big\langle f, E_{mq_\ell}T_{np}\psi_\ell\Big\rangle\Big|^2$$
$$= \int_{\mathbb{R}^+}\big|f(x)\big|^2dx\sum_{1\le\ell\le L}\frac{1}{|q_\ell|}\sum_{n\in\mathbb{Z}^+}\big|\psi_\ell(x\ominus np)\big|^2 + \mathcal{R} \tag{4.12}$$

where

$$\mathcal{R} = \sum_{1\le\ell\le L}\sum_{n\in\mathbb{Z}^+}\frac{1}{|q_\ell|}\int_{\mathbb{R}^+}\overline{f(x)}\,\psi_\ell(x\ominus np)\,dx\sum_{k\in\mathbb{Z}^+}\psi_\ell\Big(x\oplus q_\ell^{-1}k\Big)\overline{\psi_\ell\Big(x\oplus q_\ell^{-1}k\ominus np\Big)}.$$

By implementing conditions (4.12) and (4.11) in (4.12), it follows that

$$\sum_{1\le\ell\le L}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\Big|\Big\langle f, E_{mq_\ell}T_{np}\psi_\ell\Big\rangle\Big|^2 = \int_{\mathbb{R}^+}\big|f(x)\big|^2dx = \big\|f\big\|, \quad \forall\ f \in L^2(\mathcal{S}).$$

Hence, the system $\mathfrak{G}(\Psi, p, q)$ forms a Parseval frame for $L^2(\mathcal{S})$.

□

Theorems 4.2 and 4.3 provides a necessary and sufficient condition for the Gabor system $\mathfrak{G}(\Psi, p, q)$ to be a Parseval frame for $L^2(\mathcal{S})$ in the following way.

Theorem 4.4. *The Gabor system $\mathfrak{G}(\Psi, p, q)$ given by (2.6) constitutes a Parseval frame for $L^2(\mathcal{S})$ if and only if*

$$\sum_{1\le\ell\le L} \frac{1}{|q_\ell|}\left\{\sum_{n\in\mathbb{Z}^+} \left|\psi_\ell\big(x\ominus np\big)\right|^2\right\} = \chi_{\mathcal{S}},$$

$$\sum_{1\le\ell\le L}\left\{\sum_{m\in\mathbb{N}} \psi_\ell\Big(x\oplus q_\ell^{-1}m\Big)\overline{\psi_\ell\Big(x\oplus q_\ell^{-1}k\ominus mp\Big)}\right\} = 0,\ a.e\ x\in\mathcal{S}.$$

References

[1] O. Ahmad, F.A. Shah, and N.A. Sheikh, Gabor frames on non-Archimedean fields, *International Journal of Geometric Methods in Modern Physics* **15** (2018), Art. ID 1850079 (17 pages).

[2] O. Ahmad, M.Y. Bhat, and N.A. Sheikh, Characterization of Wavelets associated with AB-MRA on $L^2(\mathbb{R}^n)$, *Annals of the University of Craiova, Mathematics and Computer Science Series*, **48** (2021), no. 2, 293-306.

[3] A. Akan and L.F. Chaparro, Multi-window Gabor expansion for evolutionary spectral analysis, *IEEE Trans. on Sig. Proc.* **63** (1997), no. 3, 249-262.

[4] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2015.

[5] R.J. Duffin and A.C. Shaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.* **72** (1952), 341–366.

[6] I. Daubechies, A. Grossmann, and Y. Meyer, Painless non-orthogonal expansions, *J. Math. Phys.* **27** (1986), no. 5, 1271–1283.

[7] Yu.A. Farkov, Orthogonal p-wavelets on $\mathbb{R}^+$, In: *Proc. Int. Conf. Wavelets and Splines*, St. Peterberg State University, (2005), 4–26.

[8] Yu.A. Farkov and E.A. Rodionov, Nonstationary wavelets related to the Walsh functions, *Amer. J. Comput. Math.* **2** (2012), 82–87.

[9] Yu.A. Farkov, On wavelets related to Walsh series, *J. Approx. Theory* **161** (2009), 259–279.

[10] Yu.A. Farkov, A.Yu. Maksimov, and S.A. Stoganov, On biorthogonal wavelets related to the Walsh functions, *Int. J. Wavelets Multiresolut. Inf. Process.* **9** (2011), no. 3, 485–499.

[11] D. Gabor, Theory of communications, *J. Inst. Elect. Engn.* **93** (1946), 429–457.

[12] J.P. Gabardo and Y.Z. Li, Density results for Gabor systems associated with periodic subsets of the real line, *Jour. of Approx. Theory* **157** (2009), no. 2, 172–192.

[13] B.I. Golubov, A.V. Efimov, and V.A. Skvortsov, *Walsh Series and Transforms: Theory and Applications*, Kluwer, Dordrecht, 1991.

[14] K. Gröchenig, A.J. Janssen, N. Kaiblinger, and GE. Pfander, Note on B-splines, wavelet scaling functions, and Gabor frames, *IEEE Trans. Informat. Theory* **49** (2003), no. 12, 3318–3320.

[15] K. Gröchenig, *Foundation of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[16] K. Gröchenig and M. Leinert, Wiener's lemma for twisted convolution and Gabor frames, *J. Amer. Math. Soc.* **17** (2004), no. 1, 1–18.

[17] A. Ron and Z. Shen, Weyl-Heisenberg frames and Riesz bases in $L^2(\mathbb{R}^d)$, *Duke Math. J.* **89** (1997), 237–282.

[18] Meenakshi, P. Manchanda and A.H. Siddiqi, Wavelets associated with nonuniform multiresolution analysis on positive half-line, *Int. J. Wavelets Multiresolut. Inf. Process.* **10** (2012), no. 2, Art. ID 1250018 (27 pages).

[19] F.A. Shah, O. Ahmad, and A. Rahimi, Frames Associated with Shift Invariant Spaces on Local Fields, *Filomat* **32** (2018), no. 9, 3097–3110.

[20] F.A. Shah, Gabor frames on a half-line, *J. Contemp. Math. Anal.* **47** (2012), no. 5, 251–260.

[21] F. A. Shah, O. Ahmad, and N.A. Sheikh, Orthogonal Gabor Systems on Local fields, *Filomat* **31** (2017), no. 6, 5193–5201.

[22] F. Schipp, W.R. Wade, and P. Simon, *Walsh Series: An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, Bristol and New York, 1990.

[23] Y.Z. Li and Q.F. Lian, Multi-window Gabor frames and oblique Gabor duals on discrete periodic sets, *Sci. China* **54** (2011), no. 5, 987–1010.
[24] J. Wexler and S. Raz, Discrete Gabor expansions, *Signal Process.* **21** (1990) 207–220.
[25] Y. Zhang, Walsh Shift-Invariant Sequences and p-adic Nonhomogeneous Dual Wavelet Frames in $L^2(\mathbb{R}^+)$, *Results Math.* **74** (2019), Art. ID 111 (26 pages).
[26] M. Zibulski and Y.Y. Zeevi, Discrete multiwindow Gabor-type transforms, *IEEE Trans. on Signal Proc.* **45** (1997), no. 6, 1428–1442.
[27] M. Zibulski and Y.Y. Zeevi, Analysis of multiwindow Gabor-type schemes by frame methods, *Appl. and Comput. Harmon. Anal.* **4** (1997), no. 2, 188–221.

(Owais Ahmad) Department of Mathematics, National Institute of Technology, Hazratbal, Srinagar -190 006, Jammu and Kashmir, India
E-mail address: siawoahmad@gmail.com