Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 51(1), 2024, Pages 15–20, DOI: 10.52846/ami.v51i1.1648
ISSN: 1223-6934

Existence of Positive Solutions for an Elliptic System with Sign-Changing Weights

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Abstract. Using the method of sub-super solutions and comparison principle, we study the existence of positive solutions for a class of quasilinear elliptic systems with sign-changing weights.

2010 Mathematics Subject Classification. Primary 35D05; Secondary 35J60.
Key words and phrases. Sub-Super solutions; Comparison principle; Sign-changing Weights;Elliptic systems .

1. Introduction

In this paper, we consider the existence of positive solutions for a class of quasilinear elliptic systems of the form

$$\begin{cases} -\Delta_p u = a(x)\left(\lambda u^\alpha v^\gamma + \mu f(u)\right), \quad x \in \Omega, \\ -\Delta_q v = b(x)\left(\lambda u^\delta v^\beta + \mu g(v)\right), \quad x \in \Omega, \\ u = v = 0, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in R^N, $1 < p, q < N$, $\alpha, \beta, \gamma, \delta$ are constants, λ and μ are positive parameters.

Problems involving the p-Laplacian arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see [1]) as well as from various problems in mathematical physics notably the flow of non Newtonian fluids. The structure of positive solutions for quasilinear reaction-diffusion systems (nonlinear Newtonian filtration systems) and semilinear reaction-diffusion systems (Newtonian filtration systems) is a front topic in the study of static electric fields in dielectric media, in which the potential is described by the boundary value problem of a static non-Newtonian filtration system, called the PoissonBoltzmann problem. This kind of problems also appears in the study of the nonNewtonian or Newtonian turbulent filtration in porous media and so on, which have extensive engineering background.

In this paper, we denote by $W_0^{1,r}(\Omega)$ $(1 \leq r < \infty)$ the completion of $C_0^\infty(\Omega)$, with respect to the norm

$$\|u\|_r = \left(\int_\Omega |\nabla u|^r \, dx\right)^{\frac{1}{r}}.$$

Received Jun 9, 2022.
This research work has been supported by a research grant from Islamic Azad University-Ayatollah Amoli Branch.

Let us consider the following eigenvalue problem for the r-Laplace operator $-\Delta_r u$, see [4, 5]:

$$\begin{cases} -\Delta_r u = \lambda |u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } x \in \partial\Omega. \end{cases} \tag{2}$$

Let $\phi_{1,r} \in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,r}$ of (2) such that $\phi_{1,r} > 0$ in Ω and $\|\phi_{1,r}\|_\infty = 1$. It can be shown that $\frac{\partial \phi_{1,r}}{\partial \nu} < 0$ on $\partial\Omega$ and hence, depending on Ω, there exist positive constants m, η, σ_r such that

$$\begin{cases} |\nabla \phi_{1,r}|^r - \lambda_{1,r}\phi_{1,r}^r \geq m & \text{on } \overline{\Omega}_\eta, \\ \phi_{1,r} \geq \sigma_r & \text{on } x \in \Omega \backslash \overline{\Omega}_\eta, \end{cases} \tag{3}$$

where $\overline{\Omega}_\eta := \{x \in \Omega : \ d(x, \partial\Omega) \leq \eta\}$.

We also consider the unique solution $e_r \in W_0^{1,r}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta_r e_r = 1 & \text{in } \Omega, \\ e_r = 0 & \text{on } x \in \partial\Omega \end{cases} \tag{4}$$

to discuss our result. It is known that $e_r > 0$ in Ω and $\frac{\partial e_r}{\partial \nu} < 0$ on $\partial\Omega$.

Here we assume that the weight functions $a(x)$ and $b(x)$ take negative values in a subset of $\overline{\Omega}_\eta$, but require $a(x)$ and $b(x)$ be strictly positive in $\Omega - \overline{\Omega}_\eta$. To be precise we assume that there exist positive constants a_0, a_1, b_0 and b_1 such that $a(x) \geq -a_0$, $b(x) \geq b_0$ on $\overline{\Omega}_\eta$ and $a(x) \geq a_1$, $b(x) \geq b_1$ on $\Omega - \overline{\Omega}_\eta$.

2. Existence of positive solutions

We will study the existence of positive solutions by using the method of sub- and supersolutions. A pair of functions (ψ_1, ψ_2) is said to be a subsolution of problem (1) if it is in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that

$$\int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 \cdot \nabla w \, dx \leq \int_\Omega a(x) \left(\lambda \psi_1^\alpha \psi_2^\gamma + \mu f(\psi_1)\right) w \, dx, \quad \forall w \in W,$$

and

$$\int_\Omega |\nabla \psi_2|^{q-2} \nabla \psi_2 \cdot \nabla w \, dx \leq \int_\Omega b(x) \left(\lambda \psi_1^\delta \psi_2^\beta + \mu g(\psi_2)\right) w \, dx, \quad \forall w \in W,$$

where $W := \{w \in C_0^\infty(\Omega) : \ w \geq 0 \text{ in } \Omega\}$. A pair of functions $(z_1, z_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is said to be a supersolution if

$$\int_\Omega |\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla w \, dx \geq \int_\Omega a(x) \left(\lambda z_1^\alpha z_2^\gamma + \mu f(z_1)\right) w \, dx, \quad \forall w \in W,$$

and

$$\int_\Omega |\nabla z_2|^{q-2} \nabla z_2 \cdot \nabla w \, dx \geq \int_\Omega b(x) \left(\lambda z_1^\delta z_2^\beta + \mu g(z_2)\right) w \, dx, \quad \forall w \in W.$$

Lemma 2.1 (see [3]). *Suppose that there exist sub and super-solutions (ψ_1, ψ_2) and (z_1, z_2) respectively of problem (1) such that $(\psi_1, \psi_2) \leq (z_1, z_2)$. Then problem (1) has a solution (u, v) such that $\psi_1 \leq u \leq z_1$ and $\psi_2 \leq v \leq z_2$.*

Let us make the following hypothesis on problem (1):

$(H1)$ $\alpha, \beta \geq 0$, $\gamma, \delta > 0$ and $(p-1-\alpha)(q-1-\beta) > \gamma\delta$;

$(H2)$ There exist positive constants a_0, a_1, b_0 and b_1 such that $a(x) \geq -a_0$, $b(x) \geq b_0$ on $\overline{\Omega}_\eta$ and $a(x) \geq a_1$, $b(x) \geq b_1$ on $\Omega - \overline{\Omega}_\eta$;

$(H3)$ Suppose that there exists $\epsilon > 0$ such that

$$\min\left\{\frac{m}{2a_0\epsilon^{d_1-1}}, \frac{m}{2b_0\epsilon^{d_2-1}}, \frac{1}{\|a\|_\infty}, \frac{1}{\|b\|_\infty}\right\} \leq \max\left\{\frac{\lambda_{1,p}}{2c_1\epsilon^{d_1-1}}, \frac{\lambda_{1,q}}{2c_2\epsilon^{d_2-1}}\right\},$$

and

$$\min\left\{\frac{m\epsilon}{2a_0 f\left(\epsilon^{\frac{1}{p-1}}\right)}, \frac{m\epsilon}{2b_0 g\left(\epsilon^{\frac{1}{q-1}}\right)}, \frac{1}{\|a\|_\infty}, \frac{1}{\|b\|_\infty}\right\} \geq$$

$$\geq \max\left\{\frac{\lambda_{1,p}\epsilon}{2a_1 f\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)}, \frac{\lambda_{1,q}\epsilon}{2b_1 g\left(\frac{q-1}{q}\epsilon^{\frac{1}{q-1}}\sigma_q^{\frac{q}{q-1}}\right)}\right\}.$$

$(H4)$ $f, g \in C^1([0,+\infty),[0,+\infty))$ are increasing and homomorphism such that

$$\lim_{t\to+\infty} f(t) = \lim_{t\to+\infty} g(t) = +\infty;$$

$(H5)$ $\lim_{t\to+\infty}\frac{f(t)}{t^{p-1}} = \lim_{t\to+\infty}\frac{g(t)}{t^{q-1}} = 0.$

Now we are ready to state our existence result.

Theorem 2.2. *Let $(H1)$-$(H5)$ hold. Then there exists a positive solution of (1) for every $\lambda \in [\underline{\lambda}(\epsilon), \overline{\lambda}(\epsilon)]$ and $\mu \in [\underline{\mu}(\epsilon), \overline{\mu}(\epsilon)]$, where*

$$\overline{\lambda}(\epsilon) = \min\left\{\frac{m}{2a_0\epsilon^{d_1-1}}, \frac{m}{2b_0\epsilon^{d_2-1}}, \frac{1}{\|a\|_\infty}, \frac{1}{\|b\|_\infty}\right\},$$

$$\underline{\lambda}(\epsilon) = \max\left\{\frac{\lambda_{1,p}}{2c_1\epsilon^{d_1-1}}, \frac{\lambda_{1,q}}{2c_2\epsilon^{d_2-1}}\right\},$$

$$\overline{\mu}(\epsilon) = \min\left\{\frac{m\epsilon}{2a_0 f\left(\epsilon^{\frac{1}{p-1}}\right)}, \frac{m\epsilon}{2b_0 g\left(\epsilon^{\frac{1}{q-1}}\right)}, \frac{1}{\|a\|_\infty}, \frac{1}{\|b\|_\infty}\right\},$$

$$\underline{\mu}(\epsilon) = \max\left\{\frac{\lambda_{1,p}\epsilon}{2a_1 f\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)}, \frac{\lambda_{1,q}\epsilon}{2b_1 g\left(\frac{q-1}{q}\epsilon^{\frac{1}{q-1}}\sigma_q^{\frac{q}{q-1}}\right)}\right\}.$$

Proof. Let

$$(\psi_1, \psi_2) = \left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\phi_{1,p}^{\frac{p}{p-1}}, \frac{q-1}{q}\epsilon^{\frac{1}{q-1}}\phi_{1,q}^{\frac{q}{q-1}}\right),$$

We shall verify that (ψ_1, ψ_2) is a sub-solution of problem (1). Let $w \in W$, Then a calculation shows that

$$\begin{aligned}\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w\,dx &= \epsilon\int_\Omega |\nabla\phi_{1,p}|^{p-2}\phi_{1,p}\nabla\phi_{1,p}\cdot\nabla w\,dx\\ &= \epsilon\left[\int_\Omega |\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\cdot\nabla(\phi_{1,p}w)\,dx - \int_\Omega |\nabla\phi_{1,p}|^p w\,dx\right]\\ &= \epsilon\int_\Omega \left[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p\right] w\,dx. \qquad (5)\end{aligned}$$

A similar calculation shows that

$$\int_\Omega |\nabla \psi_2|^{q-2}\nabla\psi_1 \cdot \nabla w\, dx = \epsilon \int_\Omega \left[\lambda_{1,q}\phi_{1,q}^q - |\nabla \phi_{1,q}|^q\right] w\, dx. \tag{6}$$

First we consider the case when $x \in \overline{\Omega}_\eta$. We have $|\nabla\phi_{1,r}|^r - \lambda_{1,r}\phi_{1,r}^r \le m$ on $\overline{\Omega}_\eta$ for $r = p, q$. Since $\lambda \le \overline{\lambda}(\epsilon)$ and $\mu \le \overline{\mu}(\epsilon)$ we have $\lambda \le \frac{m}{2a_0\epsilon^{d_1-1}}$ and $\mu \le \frac{m\epsilon}{2a_0 f(\epsilon^{\frac{1}{p-1}})}$.
Hence,

$$-\frac{m\epsilon}{2} \le -\lambda a_0 \epsilon^{\frac{\alpha}{p-1}+\frac{\gamma}{q-1}}, \quad -\frac{m\epsilon}{2} \le -\mu a_0 f(\epsilon^{\frac{1}{p-1}})$$

and

$$\begin{aligned}
&\epsilon\left[\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q\right] \le -m\epsilon\\
&\le -a_0\left(\lambda\epsilon^{\frac{\alpha}{p-1}+\frac{\gamma}{q-1}} + \mu f(\epsilon^{\frac{1}{p-1}})\right)\\
&\le -a_0\left\{\lambda\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}||\phi_{1,p}||_\infty^{\frac{p}{p-1}}\right)^\alpha\left(\frac{q-1}{q}\epsilon^{\frac{1}{q-1}}||\phi_{1,q}||_\infty^{\frac{q}{q-1}}\right)^\gamma + \mu f\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\phi_{1,p}^{\frac{p}{p-1}}\right)\right\}\\
&\le a(x)\left(\lambda\psi_1^\alpha\psi_2^\gamma + \mu f(\psi_1)\right).
\end{aligned}\tag{7}$$

A similar argument shows that

$$\epsilon\left[\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q\right] \le b(x)\left(\lambda\psi_1^\delta\psi_2^\beta + \mu g(\psi_2)\right). \tag{8}$$

On the other hand, on $\Omega - \overline{\Omega}_\eta$, we note that $\phi_{1,r} \ge \sigma_r$ for $r = p, q$, $a(x) \ge a_1$, $b(x) \ge b_1$. Since $\lambda \ge \underline{\lambda}(\epsilon)$ and $\mu \ge \underline{\mu}(\epsilon)$ we have $\lambda \ge \frac{\lambda_{1,p}}{2c_1\epsilon^{d_1-1}}$ and $\mu \ge \frac{\lambda_{1,p}\epsilon}{2a_1 f\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)}$.

Hence,

$$\begin{aligned}
&\frac{1}{2}\epsilon\lambda_{1,p} \le \lambda a_1\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)^\alpha\left(\frac{q-1}{q}\epsilon^{\frac{1}{q-1}}\sigma_q^{\frac{q}{q-1}}\right)^\gamma,\\
&\frac{1}{2}\epsilon\lambda_{1,p} \le \mu a_1 f\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\epsilon\left[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p\right] \le \epsilon\lambda_{1,p}\phi_{1,p}^p\\
&\le \epsilon\lambda_{1,p}||\phi_{1,p}||_\infty^p\\
&\le \epsilon\lambda_{1,p}\\
&\le \lambda a_1\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)^\alpha\left(\frac{q-1}{q}\epsilon^{\frac{1}{q-1}}\sigma_q^{\frac{q}{q-1}}\right)^\gamma + \mu a_1 f\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)\\
&\le a(x)\left(\lambda\psi_1^\alpha\psi_2^\gamma + \mu f(\psi_1)\right)
\end{aligned}\tag{9}$$

on $\Omega - \overline{\Omega}_\eta$. A similar argument shows that

$$\epsilon\left[\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q\right] \le b(x)\left(\lambda\psi_1^\delta\psi_2^\beta + \mu g(\psi_2)\right) \tag{10}$$

on $\Omega - \overline{\Omega}_\eta$.

From (5)-(10) we have

$$\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w\, dx \le \int_\Omega a(x)\left(\lambda\psi_1^\alpha\psi_2^\gamma + \mu f(\psi_1)\right) w\, dx$$

and

$$\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w\,dx \le \int_\Omega b(x)\left(\lambda\psi_1^\delta\psi_2^\beta+\mu g(\psi_2)\right)w\,dx,\quad \forall w\in W,$$

where $W := \{w\in C_0^\infty(\Omega):\ w\ge 0 \text{ in } \Omega\}$. Therefore, (ψ_1,ψ_2) is a sub-solution of problem (1).

Now, we will construct a supersolution (z_1,z_2) of problem (1). We denote $z_1(x)=Ae_p(x)$ and $z_2(x)=Be_q(x)$, where $A,B>1$ are large and to be chosen later. It is clear that

$$-\Delta_p z_1 = A,\quad -\Delta_q z_2 = B,\quad x\in\Omega. \tag{11}$$

Then we have

$$\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w\,dx = A^{p-1}\int_\Omega w\,dx, \tag{12}$$

$$\int_\Omega |\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w\,dx = B^{q-1}\int_\Omega w\,dx. \tag{13}$$

Since $\lambda\le\overline{\lambda}(\epsilon)$ and $\mu\le\overline{\mu}(\epsilon)$ we have $\lambda\le\frac{1}{\|a\|_\infty}$, $\lambda\le\frac{1}{\|b\|_\infty}$, $\mu\le\frac{1}{\|a\|_\infty}$ and $\mu\le\frac{1}{\|b\|_\infty}$. Let $l_p=\|e_p\|_\infty$ and $l_q=\|e_q\|_\infty$. Since $\theta>0$, there exist positive large constants $A,B>1$ such that

$$\frac{1}{2}A^{p-1-\alpha}\ge B^\gamma l_p^\alpha l_q^\gamma \tag{14}$$

and

$$\frac{1}{2}B^{q-1-\beta}\ge A^\delta l_p^\delta l_q^\beta. \tag{15}$$

Moreover, from the hypothesis $(H2)$, for $A,B>1$ large enough we have

$$\frac{1}{2}A^{p-1}\ge f(Al_p) \tag{16}$$

and

$$\frac{1}{2}B^{q-1}\ge g(Bl_q). \tag{17}$$

Then we can chose $A,B>1$ large enough such that

$$\begin{aligned} A^{p-1} &\ge (Al_p)^\alpha(Bl_q)^\gamma + f(Al_p)\\ &\ge \lambda\|a\|_\infty(Al_p)^\alpha(Bl_q)^\gamma+\mu\|a\|_\infty f(Al_p)\\ &\ge a(x)\left(\lambda z_1^\alpha z_2^\gamma+\mu f(z_1)\right) \end{aligned}\tag{18}$$

and

$$\begin{aligned} B^{q-1} &\ge (Al_p)^\delta(Bl_q)^\beta + g(Bl_q)\\ &\ge \lambda\|a\|_\infty(Al_p)^\delta(Bl_q)^\beta+\mu\|a\|_\infty g(Bl_q)\\ &\ge b(x)\left(\lambda z_1^\delta z_2^\beta+\mu g(z_2)\right) \end{aligned}\tag{19}$$

for all $x\in\Omega$.

From (11)-(19) we have

$$\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w\,dx \ge \int_\Omega a(x)\left(\lambda z_1^\alpha z_2^\gamma+\mu f(z_1)\right)w\,dx$$

and

$$\int_\Omega |\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w\,dx \ge \int_\Omega b(x)\left(\lambda z_1^\delta z_2^\beta+\mu g(z_2)\right)w\,dx,\quad \forall w\in W.$$

Then (z_1, z_2) is a supersolution of problem (1). Obviously, $\psi_i(x) \leq z_i(x)$ for all $x \in \Omega$, $i = 1, 2$ with $A, B > 1$ large enough. Thus, by the comparison principle, there exists a solution (u, v) of problem (1)such that $\psi_1 \leq u \leq z_1$ and $\psi_2 \leq v \leq z_2$. This completes the proof of Theorem 2.2. □

References

[1] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations* **51** (1984), 126-150.

[2] G.A. Afrouzi, E. Graily, On positive weak solutions for a class of quasilinear elliptic systems, *Int. Journal of Math. Analysis* **2** (2008), no. 20, 999-1004.

[3] A. Cañada, P. Drábek, J.L. Gámez, Existence of positive solutions for some problems with nonlinear diffusion, *Trans. Amer. Math. Soc.* **349** (1997), 4231-4249.

[4] M. Chhetri, D.D. Hai, R. Shivaji, On positive solutions for classes of p-Laplacian semipositone systems, *Discrete Continuous Dyn. Syst.* **9** (2003), no. 4, 1063-1071.

[5] D.D. Hai, R. Shivaji, An existence result on positive solutions for a class of p-Laplacian systems, *Nonlinear Anal. (TMA)* **56** (2004), 1007-1010.

[6] D. He, Z. Yang, On positive solutions for a class of quasilinear elliptic systems with sign-changing weights, *Differential Equations & Applications* **6** (2014), no. 2, 267-274.

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