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Estimating $Pr(\frac{X}{Y} < C)$ for two power distributions

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Abstract. We consider two independent random variables, having power distributions, X and Y, representing the strength, respectively stress of a system. We wish to estimate the reliability of the system. One quantity which offers a measure of the reliability is $Pr\left(\frac{X}{Y} < C\right)$. We find two estimators for this quantity, and we compare the performances of the two estimators. The case $C = 1$ has been studied in [1].

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1. The distribution of the ratio X/Y

Let us consider two independent random variables X and Y, representing the strength respectively the stress of a system. Both variables have power distributions of parameters (b_1, δ_1) respectively (b_2, δ_2). In order to evaluate the probability $Pr(\frac{X}{Y} < C)$, first we must determine the distribution of the ratio $\frac{X}{Y}$. We have

$$\rho_{\frac{X}{Y}}(t) = \int_{\mathbb{R}} |x|\rho_{(X,Y)}(tx, x)dx = \int_{\mathbb{R}} |x|\rho_X(tx)\rho_Y(x)dx \tag{1}$$

Now, for any $t \leq 0$ we obviously have $\rho_{\frac{X}{Y}}(t) = 0$. For any $t > 0$, from (1) we get:

$$\begin{aligned} \rho_{\frac{X}{Y}}(t) &= \int_{\mathbb{R}} |x|\delta_1 b_1^{-\delta_1}(tx)^{\delta_1-1}1_{(0,b_1)}(tx)\delta_2 b_2^{-\delta_2}x^{\delta_2-1}1_{(0,b_2)}(x)dx = \\ &= \delta_1 b_1^{-\delta_1}\delta_2 b_2^{-\delta_2}t^{-\delta_1-1}\int_0^{b_2}|x|x^{\delta_1+\delta_2-2}1_{(0,b_1)}(tx)dx = \\ &= \delta_1 b_1^{-\delta_1}\delta_2 b_2^{-\delta_2}t^{\delta_1-1}\int_0^{\min(b_2,\frac{b_1}{t})}|x|x^{\delta_1+\delta_2-2}dx \end{aligned}$$

With the following notation

$$K = \delta_1 b_1^{-\delta_1}\delta_2 b_2^{-\delta_2}t^{\delta_1-1}$$

we get

$$\begin{aligned} \rho_{\frac{X}{Y}}(t) &= K\int_0^{\min(b_2,\frac{b_1}{t})}x^{\delta_1+\delta_2-1}dx = \frac{K}{\delta_1+\delta_2}\left[\min(b_2,\tfrac{b_1}{t})\right]^{\delta_1+\delta_2} = \\ &= \begin{cases} \frac{\delta_1\delta_2}{\delta_1+\delta_2}b_1^{-\delta_1}b_2^{-\delta_2}t^{\delta_1-1}b_2^{\delta_1+\delta_2}, & \text{for } b_2 \leq \frac{b_1}{t} \\ \frac{\delta_1\delta_2}{\delta_1+\delta_2}b_1^{-\delta_1}b_2^{-\delta_2}t^{\delta_1-1}\frac{b_1^{\delta_1+\delta_2}}{t^{\delta_1+\delta_2}}, & \text{for } b_2 > \frac{b_1}{t} \end{cases} = \\ &= \begin{cases} \frac{\delta_1\delta_2}{\delta_1+\delta_2}\left(\frac{b_1}{b_2}\right)^{-\delta_1}t^{\delta_1-1}, & \text{for } t \leq \frac{b_1}{b_2} \\ \frac{\delta_1\delta_2}{\delta_1+\delta_2}\left(\frac{b_1}{b_2}\right)^{\delta_2}t^{-\delta_2-1}, & \text{for } t > \frac{b_1}{b_2} \end{cases} \end{aligned} \tag{2}$$

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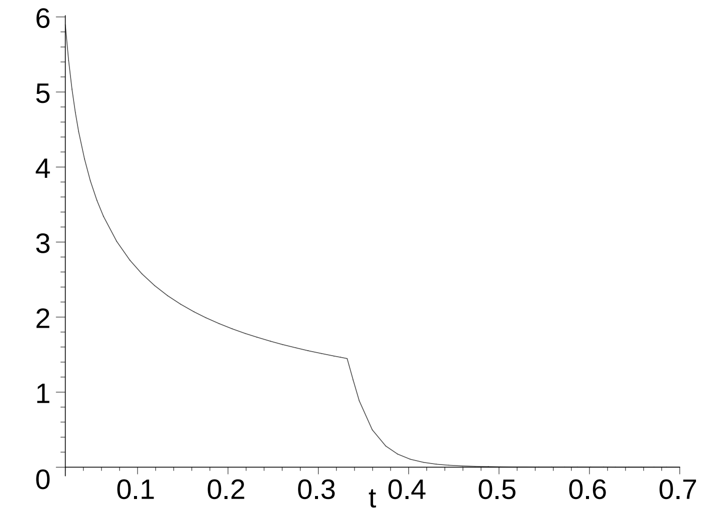
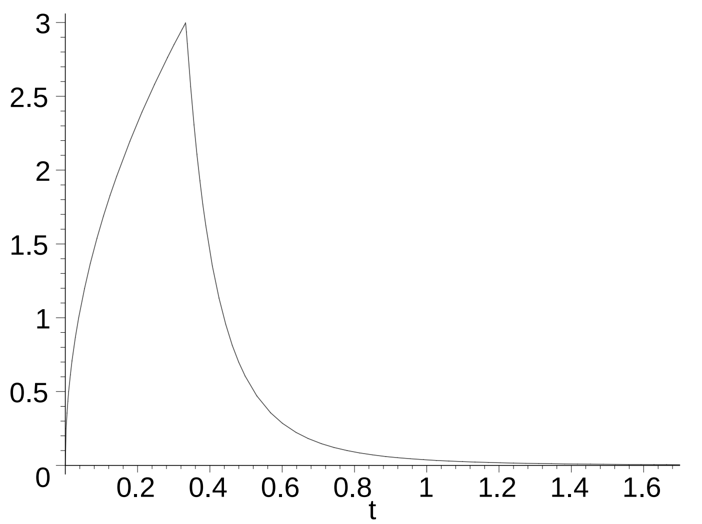
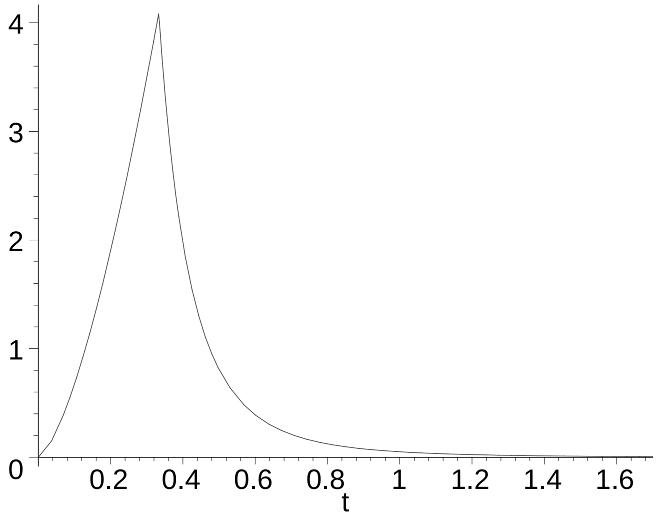
Finally, the expression for the probability density of the X/Y ratio is

$$\rho_{\frac{X}{Y}}(t) = \begin{cases} 0, & t \leq 0 \\ \frac{\delta_1 \delta_2}{\delta_1 + \delta_2} \left(\frac{b_1}{b_2}\right)^{-\delta_1} t^{\delta_1 - 1}, & t \in \left(0, \frac{b_1}{b_2}\right] \\ \frac{\delta_1 \delta_2}{\delta_1 + \delta_2} \left(\frac{b_1}{b_2}\right)^{\delta_2} t^{-\delta_2 - 1}, & t > \frac{b_1}{b_2} \end{cases} \tag{3}$$

Below we present the graphs for the density function in three essential cases: $\delta_1 < 1$, $1 < \delta_1 < 2$ and $\delta_1 > 2$:







2. Estimating the probability $Pr(\frac{X}{Y} < C)$

Now, knowing the probability density for the ratio of the two random variables X and Y, we can evaluate:

$$Pr(\tfrac{X}{Y} < C) = \int_{-\infty}^{C} \rho_{\frac{X}{Y}}(t)dt = \int_{0}^{C} \rho_{\frac{X}{Y}}(t)dt = $$
$$= \begin{cases} 0, \quad C \le 0 \\ \frac{\delta_1\delta_2}{\delta_1+\delta_2}\left(\frac{b_1}{b_2}\right)^{-\delta_1} \int_0^C t^{\delta_1-1}dt, \quad C \in \left(0, \frac{b_1}{b_2}\right] \\ \frac{\delta_1\delta_2}{\delta_1+\delta_2}\left(\frac{b_1}{b_2}\right)^{-\delta_1} \int_0^{\frac{b_1}{b_2}} t^{\delta_1-1}dt + \int_{\frac{b_1}{b_2}}^{C} \frac{\delta_1\delta_2}{\delta_1+\delta_2}\left(\frac{b_1}{b_2}\right)^{\delta_2} t^{-\delta_2-1}dt, \quad C > \frac{b_1}{b_2} \end{cases}$$

and after some more calculations we obtain

$$Pr\left(\frac{X}{Y} < C\right) = \begin{cases} 0, & C \le 0 \\ \frac{\delta_2}{\delta_1+\delta_2}\left(\frac{b_1}{b_2}\right)^{-\delta_1} C^{\delta_1}, & C \in \left(0, \frac{b_1}{b_2}\right] \\ 1 - \frac{\delta_1}{\delta_1+\delta_2}\left(\frac{b_1}{b_2}\right)^{\delta_2} C^{-\delta_2}, & C > \frac{b_1}{b_2} \end{cases} \quad (4)$$

For the case $C = 1$, we notice that

$$Pr(Y < X) = Pr(\frac{X}{Y} > 1) = 1 - Pr(\frac{X}{Y} < 1) = \begin{cases} 1 - \frac{\delta_1}{\delta_1+\delta_2}\left(\frac{b_1}{b_2}\right)^{-\delta_1}, & \text{if } \frac{b_1}{b_2} \ge 1 \\ \frac{\delta_1}{\delta_1+\delta_2}\left(\frac{b_1}{b_2}\right)^{\delta_2}, & \text{if } \frac{b_1}{b_2} < 1 \end{cases} \quad (5)$$

which is exactly what we have found in [1].

3. Numerical results

In what follows we shall estimate the quantity described in the preceding by means of two methods of estimation: a nonparametric estimator (see also [1], [6]) defined by

$$\bar{R}_n = \frac{card\{(X_i, Y_j)|\ X_i/Y_j < C,\ 1 \le i, j \le n\}}{n^2},$$

and a second estimator obtained with the method of moments. First we obtain that

$$\hat{b}_{1,n} = \frac{1+\delta_1}{\delta_1}\bar{X}_n \text{ and } \hat{b}_{2,n} = \frac{1+\delta_2}{\delta_2}\bar{Y}_n.$$

are estimates for the parameters b_1 and b_2 (see [1]), and with the notation

$$r_n = \frac{b_{1,n}}{b_{2,n}} \tag{6}$$

the estimator is $\hat{R}_n = \varphi(r_n)$, where

$$\varphi(\rho) = \begin{cases} 0, & C \leq 0 \\ \frac{\delta_2}{\delta_1+\delta_2}\rho^{-\delta_1}C^{\delta_1}, & C \in (0,\rho] \\ 1 - \frac{\delta_1}{\delta_1+\delta_2}\rho^{\delta_2}C^{-\delta_2}, & C > \rho \end{cases} \tag{7}$$

We have denoted with $(X_i)_{i=\overline{1,n}}$ and $(Y_j)_{j=\overline{1,n}}$ two samples of size n from the two random variables, and by $\bar{X}_n$, $\bar{Y}_n$ the sample means.

Now, we are interested in two quantities which measure the "goodness" of our estimators, the mean bias and the mean square error, and these quantities are defined, for any estimator $\mathbf{R}$, as it follows (see also [1], [6]):

$$MB(\mathbf{R}) = \frac{1}{N}\sum_{i=1}^{N}(Pr(X/Y < C) - \mathbf{R})$$

$$MSE(\mathbf{R}) = \frac{1}{N}\sum_{i=1}^{N}(Pr(X/Y < C) - \mathbf{R})^2$$

For the numerical study, we have used sample sizes of $n = 100$ for both variables, and, in order to estimate the mean square error and the mean bias we have repeated the experiment N times ($N = 1000$). As we have already observed in [1], for power distributions, both estimators work well and the errors are as small as 10^{-2} in the worse cases. Because of the additional parameter C that appears in the model, we have studied the evolution of the MSE over several values of C, namely 0.5, 3, 5 and variable ratio $\frac{b_1}{b_2}$. In the plots below, we have represented the MSE versus the ratio in three cases. The case $C = 1$ has already been studied in [1]. The MSE for $\hat{R}_n$ is represented as crosses, and the MSE for $\bar{R}_n$ is plotted as squares.

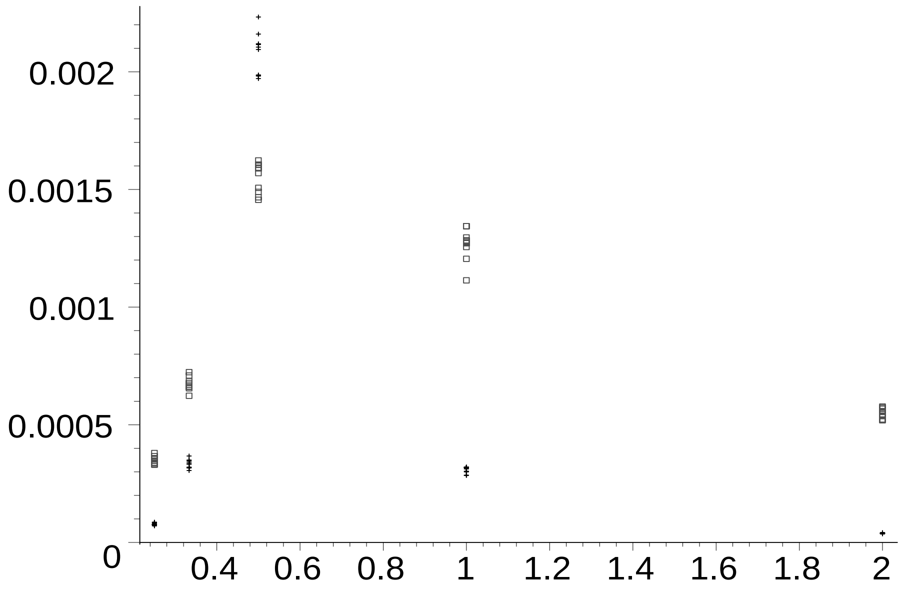


FIGURE 1. $C = 0.5$

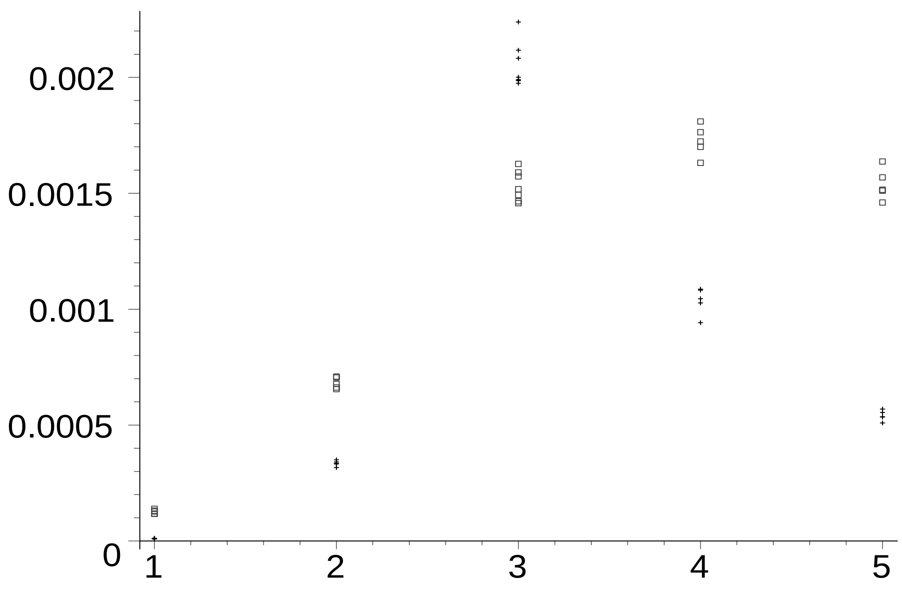


FIGURE 2. $C = 3$

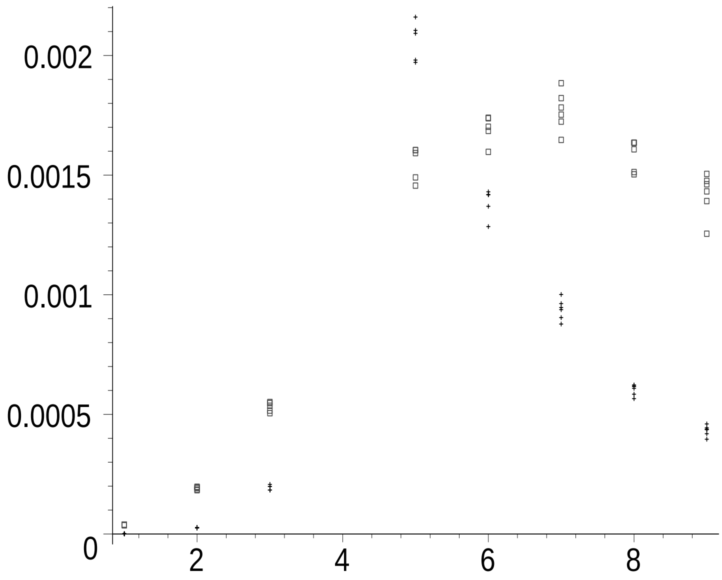


FIGURE 3. $C = 5$

We notice that the estimator $\hat{R}_n$, obtained using the method of moments, has a behavior that does not differ much from a case to another. Meanwhile, the nonparametric estimator $\bar{R}_n$ appears to behave differently for every value of the parameter C. We also observe that in most cases where the ratio $\frac{b_1}{b_2}$ is not equal to C, the best estimator is $\hat{R}_n$. Otherwise, there is a small advantage for the nonparametric estimator, which appears to be more "stable", in the sense that the variation of the MSE is smaller over all the values of the $\frac{b_1}{b_2}$ ratio.

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