Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 50(1), 2023, Pages 189–204, DOI: 10.52846/ami.v50i1.1663
ISSN: 1223-6934

α- Schurer Durrmeyer operators and their approximation properties

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Abstract. The key goal of the present research article is to introduce a new sequence of linear positive operator i.e., α-Schurer Durrmeyer operator and their approximation behaviour on the basis of function $\eta(z)$, where η infinitely differentiable on $[0,1]$, $\eta(z)=0$, $\eta(1)=1$ and $\eta'(z)>0$, for all $z\in[0,1]$. Further, we calculate central moments and basic estimates for the sequence of the operators. Moreover, we discuss the rate of convergence and order of approximation in terms of modulus of continuity, smoothness, Korovkin theorem and Peeter's K-functional. In the subsequent sections local, global and A-Statistical result are studied.

2010 Mathematics Subject Classification. Primary 41A10; Secondary 41A25; 41A36.
Key words and phrases. Schurer-Durrmeyer type operators; local and global approximation; Peeter's K-functional; rate of convergence; Korovkin theorem.

1. Introduction

In 1912 Bernstein [1] introduced a well-known positive linear operators for the continuous functions that defined on the interval $[0,1]$. The Bernstein operators preserve a simpler, more easy and more constructive way that proves the first Weierstrass approximation theorem for the case $[0,1]$ as follows:

$$\mathcal{B}_l(g;u)=\sum_{m=0}^{l}P_{l,m}(u)g\left(\frac{m}{l}\right), \tag{1}$$

where $P_{l,m}(u)=\binom{l}{m}u^m(1-u)^{l-m},\ u\in[0,1]$.
In 1962, Schurer [2] presented the following linear positive operators;

$$T_{l,q}:C[0,1+l]\to[0,1],$$

define for all $f\in[0,1+l]$ by

$$T_{l,q}(f:y)=\sum_{i=0}^{l+q}f\left(\frac{i}{m}\right)S_l,q,i(y),\ y\in[0,1], \tag{2}$$

where $S_l,q,i(y)$ is a basis Bernstein-Schurer polynomials given by

$$S_l,q,i(y)=\binom{l+q}{i}y^i(1-y)^{l+q-i}\ (i=0,1,......,l+q).$$

Received July 12, 2022. Accepted August 9, 2022.

Remark: If $q = 0$ the sequence (2) becomes the classical Bernstein operator. Recently, Chen et al. [3] presented a generalization of the Bernstein operators based on positive parameter α $(0 \leq \alpha \leq 1)$ as follows:

$$\mathcal{S}_n^{(\alpha)}(f;u) = \sum_{t=0}^{n} \mathcal{P}_{m,t}^{(\alpha)}(u) g\left(\frac{t}{n}\right), u \in I,\ I = [0,1], \tag{3}$$

where $\mathcal{P}_{m,t}^{(\alpha)}(u) = \left[\binom{m-2}{k}(1-\alpha)u + \binom{m-2}{t-2}(1-\alpha)(1-u) + \binom{m}{t}\alpha u(1-u)\right] u^{t-1}(1-u)^{m-t-1}$ and $m \geq 2$.
For the special case: $\alpha = 1$, the operator reduces to the well-known Bernstein operators.
Many modifications are investigated for the above sequences, (1), (2) and (3) viz. Acu and Radu [4], Kajla and Acar [5], Agrawal et al. [6], Acu and Cetin [7], Aral and Erbay [8], Cai and Xu [9], Çetin [10], Aslan and İzge [11], Mishra and Sharma [19], Mishra et al. ([30], [16], [31]), Gandhi et al. [32], Gairola et al. ([33], [34],[35]), Mohiuddine et al. [12], Rao et al. ([13], [14]), Raiz et al.([15], [17]).
Recently, Kajla and Acar [18] construct a Durrmeyer type modification of the operators (3) as follows:

$$\mathcal{M}_m^{(\alpha)}(g(t);u) = (m+1)\sum_{t=0}^{m} Q_{m,t}^{(\alpha)}(u)\int_0^1 Q_{m,t}(z)g(z)dz, \tag{4}$$

where

$$Q_{m,t}^{(\alpha)}(u) = \left[\binom{m-2}{t}(1-\alpha)u + \binom{m-2}{t-2}(1-\alpha)(1-u) + \binom{m}{k}\alpha\, u(1-u)\right] u^{t-1}(1-u)^{m-t-1},$$

where $m \geq 2$ and studied some approximation properties. Recently Agrawal et al. [22] introduced Bernstein-Durrmeyer type operator (4) by means of an infinitely differentiable function $\eta(u)$ on I satisfying the properties $\eta(0) = 0,\ \eta(1) = 1$ and $\eta'(u) > 0$, for $u \in I$ as follows.
For $g \in C(I)$,

$$\mathcal{M}_{m,\eta}^{(\alpha)}(g(s);u) = (m+1)\sum_{t=0}^{m} Q_{m,t}^{(\alpha)}(\eta(u))\int_0^1 Q_{m,t}^{(\alpha)}(\eta(u))\big(g\ o\ \eta^{-1}\big)(z)dz. \tag{5}$$

Inspired by the above developments, We introduce a new sequence of operators e.g., $\alpha-$Schurer Durrmeyer operator (5) as follows:

$$T_{m+l,\eta}^{(\alpha)}(h(t);u) = ((m+l)+1)\sum_{j=0}^{m+l} \mathcal{P}_{m+l,j}^{(\alpha)}(\eta(u))L_{m+l}, \tag{6}$$

where $L_{m+l} = \int_0^1 Q_{m+l,j}(z)(h\ o\ \eta^{-1})zdz$.
In the remaining sections, we study some direct approximation theorems via modulus of continuity, rate of convergence, order of approximation locally and globally for better approximation results.
In the following $||\cdot||$ means the uniform norm on $C[I]$. Let $e_i(t) = t^i,\ i \in \mathbb{N}\cup\{0\}$. By simple calculation one has

$$\int_0^1 Q_{m+l,j}^{(\alpha)}(z)z^i dz = \int_0^1 \binom{m+l}{j} z^{j+i}(1-z) = \frac{(m+l)!(j+i)!}{j!((m+l)+i+1)!}. \tag{7}$$

2. Preliminary results

Lemma 2.1. [20] *for the operators* (3), *we have*

(1) $\mathcal{S}_n^{(\alpha)}(e_0;u)=1$;
(2) $\mathcal{S}_n^{(\alpha)}(e_1;u)=u$;
(3) $\mathcal{S}_n^{(\alpha)}(e_2;u)=u^2+\frac{(m+2)(1-\alpha)}{m^2}u(1-u)$;
(4) $\mathcal{S}_n^{(\alpha)}(e_3;u)=u^3+\frac{3(m+2)(1-\alpha)}{m^2}u^2(1-u)+\frac{(m+6(1-\alpha))}{m^3}u(1-u)(1-2u)$.

In the light of the above Lemma, we have

Lemma 2.2. *For the operator* $T^{(\alpha)}_{m+l,\eta}(h;u)$. *Given by* (6) *for each* $u\in I$, *we get*

(1) $\mathcal{M}^{(\alpha)}_{m+l,\eta}(1;u)=1$;
(2) $\mathcal{M}^{(\alpha)}_{m+l,\eta}(\eta(t);u)=\frac{1}{((m+l)+2)}\big[(m+l)\eta(u)+1\big]$;
(3) $\mathcal{M}^{(\alpha)}_{m+l,\eta}(\eta^2(t);u)=\frac{1}{((m+l)+2)((m+l)+1)}\big[(m+l)^2\eta^2(u)+((m+l)+2(1-\alpha))\eta(u)(1-\eta(u))+3(m+l)\eta(u)+2\big]$;
(4) $\mathcal{M}^{(\alpha)}_{m+l,\eta}(\eta^3(t);u)=\frac{1}{((m+l)+2)((m+l)+3)((m+l)+4)}\big[\eta^3(u)\big((m+l)^3-3((m+l)^2-4(m+l)+6(m+l)\alpha-12\alpha+12\big)+\eta^2(u)\big(9(m+l)^2-6(m+l)\alpha-3(m+l)+30\alpha-30\big)+\eta(u)\big(18(m+l)-18\alpha+18\big)+6\big]$;
(5) $\mathcal{M}^{(\alpha)}_{m+l,\eta}(\eta^4(t);u)=\frac{1}{((m+l)+2)((m+l)+3)((m+l)+4)}\big[\eta^4(u)\big((m+l)^4-6(m+l)^3-(m+l)^2+54(m+l)+12(m+l)^2\alpha-60(m+l)\alpha+72\alpha-72\big)+\eta^3(u)\big(16(m+l)^3-36(m+l)^2-12(m+l)^2\alpha-124(m+l)+156(m+l)\alpha-264\alpha+264\big)+\eta^2(u)\big(72(m+l)^2+24(m+l)-96(m+l)\alpha+336\alpha-336+\eta(u)(96(m+l)-144\alpha+144\big)+24\big]$.

Proof. In the light of (7) and Lemma 2.1 a is obvious, now for b

$$\begin{aligned}\mathcal{M}^{(\alpha)}_{m+l,\eta}(\eta(z);u)&=((m+l)+1)\sum_{j=0}^{n}Q^{(\alpha)}_{(m+l),j}(\eta(u))\int_0^1 Q_{(m+l),j}(z)dz\\&=\frac{1}{((m+l)+2)}\left[\sum_{j=0}^{m+l}Q^{\alpha}_{m+l,j}(\eta(u))\cdot j+\sum_{j=0}^{m+l}Q^{(\alpha)}_{m+l,j}(\eta(u))\cdot 1\right]\\&=\frac{1}{((m+l)+2)}\big[(m+l)\eta(u)+1\big],\end{aligned}$$

$$\mathcal{M}^{(\alpha)}_{m+l,\eta}(\eta^2(z);u)=((m+l)+1)\sum_{j=0}^{m+l}Q^{(\alpha)}_{(m+l),\eta}(\eta(u))\int_0^1 Q_{m+l,j}(z)z^2dz$$
$$=\frac{\left[\sum_{j=0}^{m+l}Q^{(\alpha)}_{m+l,j}(\eta(u))\cdot j^2+2\sum_{j=0}^{m+l}Q^{(\alpha)}_{m+l,j}(\eta(u))+3\sum_{j=0}^{m+l}Q^{(\alpha)}_{m+l,j}(\eta(u))\cdot j\right]}{((m+l)+2)((m+l)+3)}.$$

∎

Using Lemma 2.1, we have

$$\mathcal{M}^{(\alpha)}_{m+l,\eta}(\eta^2(z);u) = \frac{1}{((m+l)+2)((m+l)+3)}\Big[(m+l)^2(\eta(u))^2 + (m+l) + 2(1-\alpha)\eta(u)(1-\eta(u)) + 2 + 3(m+l)\eta(u)\Big].$$

Similarly, remaining parts can be easily proved so we quit the detail.

Lemma 2.3. *For the operators* (6), *we have*

(1) $\mathcal{M}^{(\alpha)}_{m+l,\eta}(\eta(z)-\eta(u);u) = \frac{1-2\eta(u)}{(m+l)+2}$;

(2) $\mathcal{M}^{(\alpha)}_{m+l,\eta}((\eta(z)-\eta(u))^2;u) = \frac{(2(m+l)-2\alpha-u)\eta(u)(1-\eta(u)+2)}{((m+l)+2((m+l)+3)}$;

(3) $\mathcal{M}^{(\alpha)}_{m+l,\eta}((\eta(z)-\eta(u))^4;u) = \frac{1}{((m+l)+2)((m+l)+3)((m+l)+4)((m+l)+5)}\Big[\eta^4(u)\big(12(m+l)^2 - 228(m+l) + 24(m+l)^2\alpha - 24(m+l)\alpha + 552\alpha - 432\big) + \eta^3(u)\big(24(m+l)^2 + 476(m+l) + 48(m+l)\alpha - 110\alpha - 864\big) + \eta^4(u)\big(12(m+l)^2 - 30(m+l) - 24(m+l)\alpha + 426\alpha - 186\big) + \eta(u)\big(32(m+l) - 144\alpha - 176\big) + 24\Big]$.

Suppose $h \in C(I)$. Then, the norm of the function h is given by

$$||h|| = \sup_{u\in I}|h(u)|.$$

Lemma 2.4. *For* $h \in C(I)$, *we have* $||\mathcal{M}^{(\alpha)}_{m+l,\eta}(h;\cdot)|| \le ||h||$.

Proof. By Lemma 2.2, $Q^{(\alpha)}_{m+l,j}(\eta(u)) \ge 0,\ for\ all\ u \in I,$

$$\begin{aligned}\left|\mathcal{M}^{(\alpha)}_{m+l,\eta}(h;u)\right| &\le ((m+l)+1)\sum_{j=0}^{m+l}\left|Q^{(\alpha)}_{m+l,j}(\eta(u))\right|\int_0^1 Q_{m+l,\eta}(z)|h(\eta^{-1}(z))|dz\\ &\le ||f||((m+l)+1)\sum_{j=0}^{m+l}Q^{(\alpha)}_{m+l,j}(\eta(u))\int_0^1 Q_{m+l,j}(z)dz\\ &= ||f||\mathcal{M}^{(\alpha)}_{m+l,\eta}(1;u) = ||h||,\end{aligned}$$

for all $u \in I$. $\sup_{u\in I}\left|\mathcal{M}^{(\alpha)}_{m+l,\eta}(h;u)\right| \le ||h||$.
Hence, completes the proof of the Lemma 2.4. □

Lemma 2.5. *For any* $m+l \in \mathbb{N}$, *one has*

$$\mathcal{M}^{(\alpha)}_{m+l,\eta}((\eta(z)-\eta(u))^2;u) \le \frac{2}{((m+l)+2)}\lambda^2_{m+l,\eta}(u),$$

where $\lambda^2_{m+l,\eta}(u) = \psi^2_\eta(u) + \frac{1}{(m+l)+2}$ *and* $\psi^2_\eta(x) = \eta(u)(1-\eta(u))$.

Proof. By Lemma 2.3, we have

$$\begin{aligned}\mathcal{M}^{(\alpha)}_{m+l,\eta}((\eta(z)-\eta(u))^2;u) &= \frac{(2(m+l)-2\alpha-u)\eta(u)(1-\eta(u))+2}{((m+l)+2)((m+l)+3)}\\ &\le \frac{2}{((m+l)+2)}\left[\eta(u)(1-\eta(u)) + \frac{1}{((m+l)+3)}\right]\end{aligned}$$

$$= \frac{2}{((m+l)+2)} \lambda^2_{m+l,\eta}(u).$$

□

Lemma 2.6. *Using Lemma 2.3, for $0 \leq \alpha \leq 1$ and for each $u \in [0,1]$, we get*

(1) $\lim_{(m+l)\to\infty}(m+l)\left\{\mathcal{M}^{(\alpha)}_{(m+l),\eta}((\eta(z)-\eta(u));u)\right\} = 1-2\eta(u);$

(2) $\lim_{(m+l)\to\infty}(m+l)\left\{\mathcal{M}^{(\alpha)}_{(m+l),\eta}((\eta(z)-\eta(u))^2;u)\right\} = 2\eta(u)(1-\eta(u));$

(3) $\lim_{(m+l)\to\infty}(m+l)^2\left\{\mathcal{M}^{(\alpha)}_{(m+l),\eta}((\eta(z)-\eta(u))^4;u)\right\} = 12\eta^2(u)((1+2\alpha)\eta^2(u) + 2\eta(u)+1.$

Proof. (a) Using Lemma 2.3

$$\begin{aligned}\lim_{(m+l)\to\infty}(m+l)\left\{\mathcal{M}^{(\alpha)}_{(m+l),\eta}((\eta(z)-\eta(u));u)\right\} &= \lim_{(m+l)\to\infty}\left(\frac{1-2\eta(u)}{((m+l)+2)}\right)\\ &= \lim_{(m+l)\to\infty}\frac{(m+l)(1-2\eta(u))}{(m+l)(1+\frac{2}{(m+l)})}\\ &= 1-2\eta(u).\end{aligned}$$

(b)

$$\begin{aligned}\lim_{(m+l)\to\infty}(m+l)\left\{\mathcal{M}^{(\alpha)}_{(m+l),\eta}((\eta(z)-\eta(u))^2;u)\right\} &= \lim_{(m+l)\to\infty} n\left(\frac{(2\eta-2\alpha-4)T_1}{T_2}\right)\\ &= \lim_{(m+l)\to\infty}\frac{(m+l)(2\eta-2\alpha-4)T_3}{(m+l)\left(1+\frac{2}{(m+l)}\right)T_4}\\ &= (2\eta-2\alpha-4)\eta(u)(1-\eta(u))+1).\end{aligned}$$

where $T_1 = \eta(u)(1-\eta(u))+1), T_2 = ((m+l)+2)((m+l)+3), T_3 = \eta(u)(1-\eta(u))+1),$ $T_4 = \left(1+\frac{3}{(m+l)}\right)$. Similarly, (c) can be easily proved. □

3. Rate of convergence

Definition 3.1. For $h \in C[0,\infty)$, the modulus of smoothness for a uniformly continuous function h is presented as follows:

$$\omega(h;\delta) = \sup_{|s-y|\leq\delta}|h(s)-h(y)|,\ \ s,y\in[0,\infty).$$

For a uniformly continuous function h in $C[0,\infty)$ and $\delta>0$, one has,

$$|h(s)-h(y)| \leq \left(1+\frac{(1-s)^2}{\delta^2}\right)\omega(h;\delta).$$

Theorem 3.1. *For $T^{(\alpha)}_{m+l}(\cdot;\cdot)$ the operators defined by (6) with $(m+l)>2$ and for each $h\in C[0,\infty)\cap F$. $T^{(\alpha)}_{m+l}(h;u)\to h$ on each compact subset of $[0,\infty)$, where $E=\left\{h: u\geq 0\frac{h(u)}{1+u^2}\right.$ is convergent as $\left.u\to\infty\right\}$.*

Proof. Using Lemma 2.2, property (iv) of Krovokin-type theorem [21], it is enough to show that $T^{(\alpha)}_{m+l}(e_i;u) \to e_i(u)$ for $i=0,1,2$. For $i=0$. It is obvious, for $i=1$

$$\lim_{(m+l)\to\infty} T^{(\alpha)}_{m+l}(e_1;u) = \lim_{(m+l)\to\infty}\left(\frac{(m+l)\eta(u)+1}{(m+l)+2}\right) = e_1(\eta(u)).$$

Similarly, we can prove for $i=2$,

$$T^{(\alpha)}_{m+l,\eta}(e_2;u) \to e_2(u),$$

which completes the proof of Theorem 3.1. □

Theorem 3.2. [23] *Let $L = C[c,d] \to B[c,d]$ be the positive linear operator. Let γ_u be the function defined by*

$$\gamma_u(v) = |v-u|,\ \ (u,v) \in [c,d]\times[c,d].$$

If $g \in C_B([c,d])$, for any $u \in [a,b]$ and $\delta > 0$, the operator L verifies:

$$\big|(Lg)(u) - g(u)\big| \le |g(u)||(Le_0)(u) - L|(Le_0)(u) + \lambda^{(-1)}\sqrt{(Le_0)(u)(L\beta_u^2(u))\omega_\eta(\lambda)}.$$

Theorem 3.3. *Let $g \in C_B[0,\infty)$. Then for the operator $T^{(\alpha)}_{m+l,\eta}(\cdot;\cdot)$, presented by (6) with $m+l>2$, we get*

$$\left|T^{(\alpha)}_{m+l}(g;u) - g(u)\right| \le 2\omega(g;\lambda),$$

where

$$\lambda = \sqrt{T^{(\alpha)}_{m+l}(\psi_u^2(t);u)}.$$

Proof. In view of Lemma 2.2, 2.3 and Theorem 3.1, we have

$$\left|T^{(\alpha)}_{m+l}(g;u) - g(u)\right| \le \left\{1 + \lambda^{-1}\sqrt{T^{(\alpha)}_{m+l}(\psi(u)^2;u}\right\}\omega(g;\lambda).$$

Hence, completes the proof of theorem 3.1 by choosing

$$\lambda = \sqrt{T^{(\alpha)}_{m+l}(\psi(u)^2;u)}.$$

□

4. Local approximation

The local approximation result $C_B[0,\infty)$, that denotes the space of real valued bounded and continuous functions with norm that is studied here $||g|| = \sup_{0\le u<\infty}|g(u)|$. For each $g \in C_B[0,\infty)$ and $\lambda > 0$, Peetre's K-functional is discussed as:

$$\mathcal{K}_2(g;\lambda) = \inf\{||g-h|| + \lambda||h''|| : h \in C_B^2[0,\infty)\},$$

where

$$C_B^2[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}.$$

By Devore and Lorentz ([24], p. 177 Theorem 2.4) there exits an absolute constant $C>0$ such that

$$\mathcal{K}_2 f(g,h) \le C_B(g,\sqrt{\lambda}). \tag{8}$$

Here $\omega_2(g;\lambda)$ is defined modulus of smoothness of second order as:

$$\omega_2(g;\sqrt{\lambda}) = \sup_{0<g<\sqrt{\lambda}} \sup_{u\in[0,\infty)} |g(u+2g)-2g(u+h)+h(u)|.$$

Now, for $g \in C_B[0,\infty)$, $u \geq 0$ and $(m+l) > 1$. Here we consider the auxiliary operator $\hat{T}^{(\alpha)}_{m+l}(\cdot;\cdot)$ as follows:

$$\hat{T}^{(\alpha)}_{m+l}(g;u) = T^{(\alpha)}_{m+l}(g;u) + g(u) - g\left(\frac{(m+l)\eta(u)+1}{(m+l)+2}\right). \tag{9}$$

Lemma 4.1. *Let $h \in C^2_B[0,\infty)$, then for each $u \geq 0$ and $m+l > 2$, we have*

$$\left|\hat{T}^{(\alpha)}_{m+l}(h;u) - h(u)\right| \leq \phi_{m+l}(u)||h''||,$$

where

$$\phi_{m+l} = \frac{1}{((m+l)+2)((m+l)+3)((m+l)+4)((m+l)+5)} \times \frac{(2(m+l)-2\alpha-4)\eta(u)-(1-\eta(u))+2}{((m+l)+2)((m+l)+3)}.$$

Proof. Using the definition (9), for the estimated operator (5), we have

$$\hat{T}^{(\alpha)}_{m+l}(1;u) = 1,\ \hat{T}^{(\alpha)}_{m+l}(\psi_u;u) = 0 \text{ and } |\hat{T}^{(\alpha)}_{m+l}(g;u)| \leq 3||g||. \tag{10}$$

Taking Taylor's expansion and $g \in C^2_B[0,\infty)$, we get

$$h(t) = h(u) + (t-u)h'(u) + \int_u^t (t-v)h''(v)dv. \tag{11}$$

Applying on $\hat{T}^{(\alpha)}_{m+l}(\cdot;\cdot)$ both sides in above given equations, we get

$$\hat{T}^{(\alpha)}_{m+l}(h;u) - h(u) = h'(u)\hat{T}^{(\alpha)}_{m+l}(t-u;u) + \hat{T}^{(\alpha)}_{m+l} - \left(\int_u^1 (t-v)h''(v)dv\right)$$

$$= \hat{T}^{(\alpha)}_{m+l}\left(\int_u^1 (t-u)h''(v)dv;u\right) - \int_u^{\left(\frac{(m+l)\eta(u)+1}{(m+l)+2}\right)} \left(\frac{(m+l)\eta(u)+1}{(m+l)+2} - v\right)h''(v)dv. \tag{12}$$

Since

$$\left|\int_0^1 (t-v)h''(v)dv\right| \leq (t-u)^2||h''||. \tag{13}$$

Then, we obtain

$$\left|\int_u^{\left(\frac{(m+l)\eta(u)+1}{(m+l)+2}\right)} \left(\frac{((m+l)\eta(u)+1)}{(m+l)+2} - v\right)h''(v)dv\right| \leq \left(\frac{((m+l)\eta(u)+1)}{(m+l)+2} - u\right)^2 ||h''||. \tag{14}$$

Taking (13) and (14) in (12), we have

$$\left|\hat{T}^{(\alpha)}_{m+l}(g;u) - g(u)\right| \leq \left\{\hat{T}^{(\alpha)}_{m+l}((t-u)^2;u) + \left(\frac{((m+l)\eta(u)+1)}{(m+l)+2}\right)\right\}||g''||$$
$$= \psi_{m+l}(u)||g''||.$$

Hence, completes the proof of the Lemma 4.1 □

Theorem 4.2. *Suppose $g \in C_B^2[0,\infty)$, then there exists a positive constant C such that*

$$\left|\hat{T}_{m+l}^{(\alpha)}(g;u) - g(u)\right| \leq C\omega_2(g;\sqrt{(\psi_{m+l})} + \omega(g;\hat{T}_{m+l}^{(\alpha)}(\psi_u;u)),$$

where ψ_{m+l} are discussed in Lemma 4.1.

Proof. In the light of the definition $\hat{T}_{m+l}^{(\alpha)}(\cdot;\cdot)$ for $h \in C_B^2[0,\infty)$ and $g \in C_B[0,\infty)$, we get

$$\begin{aligned}\left|\hat{T}_{m+l}^{(\alpha)}(g;u) - g(u)\right| \leq& \left|\hat{T}_{m+l}^{(\alpha)}(g-h;u)\right| + \left|(g-h)(u)\right| + \left|\hat{T}_{m+l}^{(\alpha)}(h;u) - h(u)\right| \\ &+ \left|g\left(\frac{((m+l)\eta(u)+1)}{(m+l)+2}\right) - g(u)\right| \\ \leq& 4||g-h|| + \psi_{m+l}(u)||h''|| + \omega(g;\hat{T}_{m+l}^{(\alpha)}(\psi_u;u)).\end{aligned}$$

Using definition of Peetre's K-functional

$$\left|\hat{T}_{m+l}^{(\alpha)}(g;u) - g(u)\right| \leq C\omega_2(g;\sqrt{\psi_{m+l}(u)}) + \omega(g;\hat{T}_{m+l}^{(\alpha)}(\psi_u;u)).$$

Hence, we arrived our desired result.
Let $\beta_1 > 0$ and $\beta_2 > 0$, be two fixed real values we recall Lipschitz-type space here [29] as:

$$Lip_M^{\beta_1\beta_2}(\gamma) = \left\{g \in C_B[0,\infty) : |g(t) - g(u)| \leq M\frac{|t-u|^\gamma}{(t+\beta_1 u+\beta_2 u^2)^{\frac{\gamma}{2}} : u,t \in (0,\infty)}\right\},$$

where M is positive constant and $0 < \gamma \leq 1$. □

Theorem 4.3. *Let $g \in Lip_M^{\beta_1\beta_2}(\gamma)$ and $u \in [0,\infty)$. Therefore the operators that have been defined by (6), we have*

$$\left|\hat{T}_{m+l}^{(\alpha)}(g;u) - g(u)\right| \leq M\left(\frac{\nu_{m+l}(u)}{\beta_1 u + \beta_2 u^2}\right)^{\frac{\gamma}{2}}, \tag{15}$$

where $\gamma \in (0,1]$ and $\nu_{m+l}(u) = \hat{T}_{m+l}^{(\alpha)}(\psi_u^2;u)$.

Proof. For $\gamma = 1$ and $u \in [0,\infty)$, we have

$$\begin{aligned}\left|\hat{T}_{m+l}^{(\alpha)}(g;u) - g(u)\right| &\leq \hat{T}_{m+l}^{(\alpha)}(|g(t)-g(u)|;u) \\ &\leq M\hat{T}_{m+l}^{(\alpha)}\left(\frac{|t-u|}{(t+\beta_1(u)+\beta_2 u^2)^{\frac{1}{2}}};u\right).\end{aligned}$$

It is obvious, that

$$\frac{1}{t+\beta_1(u)+\beta_2 u^2} < \frac{1}{(t+\beta_1(u)+\beta_2 u^2)^{\frac{1}{2}}},$$

for each $u \in [0,\infty)$, we get

$$\begin{aligned}\left|\hat{T}_{m+l}^{(\alpha)}(g;u) - g(u)\right| &\leq \frac{M}{(\beta_1(u)+\beta_2 u^2)^{\frac{1}{2}}}(\hat{T}_{m+l}^{(\alpha)}(t-u)^2;u)^{\frac{1}{2}} \\ &\leq M\left(\frac{\nu_{m+l}(u)}{(\beta_1(u)+\beta_2 u^2)}\right)^{\frac{1}{2}}.\end{aligned}$$

From Hölder's inequality, the Theorem 4.3 now holds for $\gamma = 1, \gamma \in [0,\infty)$ and for $\gamma \in [0,\infty)$ with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we get

$$\left|\hat{T}^{(\alpha)}_{m+l}(g;u) - g(u)\right| \leq \left(\hat{T}^{(\alpha)}_{m+l}(|g(t)-h(u)|^{\frac{\gamma}{2}};u)\right)^{\frac{\gamma}{2}}$$
$$\leq M(\hat{T}^{(\alpha)}_{m+l})\left(\frac{|t-u|^2}{(t+\beta_1(u)+\beta_2 u^2)};u\right)^{\frac{\gamma}{2}}.$$

Since $\frac{1}{t+\beta_1(u)+\beta_2 u^2} < \frac{1}{\beta_1(u)+\beta_2 u^2}$ for each $u \in (0,\infty)$, we get

$$\left|\hat{T}^{(\alpha)}_{m+l}(g;u) - g(u)\right| \leq M\left(\frac{\hat{T}^{(\alpha)}_{m+l}(|t-u|^2)}{t+\beta_1(u)+\beta_2 u^2};u\right)^{\frac{\gamma}{2}}$$
$$\leq M\left(\frac{\nu_{m+l}(u)}{\beta_1(u)+\beta_2 u^2}\right)^2.$$

Hence, it completes the proof of the Theorem 4.3. □

Now, we take n^{th} term order of Lipschitz type maximal function suggested by Lenze [25] as:

$$\tilde{\omega}(g;u) = \sup_{t\neq u, t\in(0,\infty)} \frac{|g(v)-f(u)|}{|v-u|^n},\ u \in 0,\infty)\ \text{and}\ n \in (0,1]. \tag{16}$$

Theorem 4.4. *Suppose $g \in C_B[0,\infty)$ and $n \in (0,1]$. For each $u \in [0,\infty)$, we get*

$$\left|\hat{T}^{(\alpha)}_{m+l}(g;u) - g(u)\right| \leq \tilde{\omega}_\eta(g;u)(v_{m+l}(u))^{\frac{\gamma}{2}},$$

Proof. We know that

$$\left|\hat{T}^{(\alpha)}_{m+l}(g;u) - g(u)\right| \leq \hat{T}^{(\alpha)}_{m+l}(|g(v)-g(u)|;u).$$

By equation (16), we get

$$\left|\hat{T}^{(\alpha)}_{m+l}(g;u) - g(u)\right| \leq \tilde{\omega}_\eta\big((g;u)(\hat{T}^{(\alpha)}_{m+l}|v-u|^n;u\big).$$

From Hölder's inequality with $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$, we have

$$\left|\hat{T}^{(\alpha)}_{m+l}(g;u) - g(u)\right| \leq \tilde{\omega}_\eta(g;u)(\hat{T}^{(\alpha)}_{m+l}|v-u|^n;u).$$

□

5. Weighted approximation

From [21], let $B_{1+u^2}[0,\infty) = \{g(u) : |g(u) \leq M_g(1+u^2)\}$ be functional weighted space, M_g is constant determined by g and in $B_{1+u^2}[0,\infty),\ u \in [0,\infty),\ C_{1+u^2}[0,\infty)$ is the space continuous function with norm

$$||g(u)||_{1+u^2} = \sup_{u\in[0,\infty)} \frac{|g(u)|}{1+u^2},$$

and

$$C^k_{1+u^2}[0,\infty) = \left\{ g \subset C_{1+u^2}[0,\infty) : \lim_{|u|\to\infty} \frac{g(u)}{1+u^2} = K \right\}.$$

K a constant depends on g.
The modulus of continuity for the function with $b > 0$ and a closed interval $[0, b]$ as:

$$\omega_b(g;\delta) = \sup_{|v-u|\leq\delta} \sup_{u,0\in[0,b]} \big|g(v) - g(u)\big|. \tag{17}$$

Modulus of continuity goes to zero in equation of $g \in C_{1+u^2}[0,\infty)$.

Theorem 5.1. *Let $g \in C^k_{1+u^2}[0,\infty)$ and its modulus of continuity $\omega_{a+1}(g;\delta)$ defined on $[0, a+1] \in [0,\infty)$, we have*

$$||\hat{T}^{(\alpha)}_{m+l}(g;u) - g(u)||_{C[0,b]} \leq 6M_g(1+a^2)\delta_{m+l}(a) + 2\omega_{a+1}(g;\sqrt{\delta(m+l)a}),$$

where $\hat{T}^{(\alpha)}_{m+l}(a) = \hat{T}^{(\alpha)}_{m+l}(\xi;a)$.

Proof. By ([26], p. 378) for $u \in [a,b]$ and $v \in (0,\infty)$, we obtain

$$|g(v) - g(u)| \leq 6M_g(1+a)(v-u)^2 + \left(1 + \frac{|v-u|}{\delta}\right)\omega_{a+1}(g;\delta).$$

Using $\hat{T}^{(\alpha)}_{m+l}$ both sides, we have

$$\left|\hat{T}^{(\alpha)}_{m+l}g(v) - g(u)\right| \leq 6M_g(1+a^2)\hat{T}^{(\alpha)}_{m+l}(v-u)^2 + \left(\frac{1+\hat{T}^{(\alpha)}_{m+l}(1+\sqrt{|v-u|}}{\delta}\right)\omega_{a+1}(g;\delta).$$

From Lemma 2.3 $u \in [a,b]$, we get

$$\left|\hat{T}^{(\alpha)}_{m+l}(g;u) - g(u)\right| \leq 6M_g(1+a)\delta_{m+l}(a) + \left(\frac{1+\sqrt{\delta_{m+l}(b)}}{\delta}\right)\omega_{a+1}(g;\delta).$$

Choosing $\delta = \delta_{m+l}(a)$.
Hence, it completes proof of the above Theorem. □

Theorem 5.2. *If the operator given by (6), i.e. $\hat{T}^{(\alpha)}_{m+l}(\cdot;\cdot)$, from $C^k_{1+u^2}[0,\infty)$ to $B_{1+u^2}[0,\infty)$ satisfy the conditions*

$$\lim_{m+l\to\infty} ||\hat{T}^{(\alpha)}_{m+l}(\eta(t)^i;u) - u^i||_{1+u^2} = 0,$$

where $i = 0, 1, 2$ for each $g \in C^k_{1+u^2}[0,\infty)$, we get

$$\lim_{m+l\to\infty} ||\hat{T}^{(\alpha)}_{m+l}(g;u) - g||_{1+u^2} = 0.$$

Proof. In order to prove this theorem it is enough to show that

$$\lim_{m+l\to\infty} ||\hat{T}^{(\alpha)}_{m+l}(\eta(t)^i) - u^i||_{1+u^2} = 0.$$

By Lemma 2.2, we get $i = 0$

$$||\hat{T}^{(\alpha)}_{m+l}(\eta(t)^0 - u^0)||_{1+u^2} = \sup_{u\in[0,\infty)} \frac{|\hat{T}^{(\alpha)}_{m+l}(\eta(t);u) - 1|}{1+u^2} = 0.$$

For $i = 1$

$$||\hat{T}^{(\alpha)}_{m+l}(\eta(t)^1;u) - u||_{1+u^2} = \sup_{u\in[0,\infty)} \frac{\left|\frac{(m+l)\eta(u)+1}{((m+l)+2)} - u\right|}{1+u^2}$$
$$= \frac{1}{((m+l)+2)} \sup_{u\in[0,\infty)} \frac{u}{1+u^2} - \frac{1}{((m+p)+2)} \sup_{u\in[0,\infty)} \frac{1}{1+u^2},$$

implies that

$$||\hat{T}^{(\alpha)}_{m+l}(\eta(t);u) - u||_{1+u^2} \to 0, \ m+l \to \infty.$$

For, $i = 2$

$$||\hat{T}^{(\alpha)}_{m+l}(\eta(t)^2;u) - u^2||_{1+u^2}$$
$$= \sup_{u\in[0,\infty)} \frac{\left|\frac{1}{S_1}\left[(m+l)^2\eta^2(u) + ((m+l) + 2(1-\alpha)\eta(u)(1-\eta(u) + 3(m+l)\eta(u) + 2\right]\right|}{1+u^2}$$
$$\leq \frac{(m+l)^2}{S_1} \sup_{u\in[0,\infty)} \frac{\eta^2(u)}{1+u^2} + \frac{((m+l) + 2(1-\alpha)(1-\eta(u))}{S_1} \sup_{u\in[0,\infty)} \frac{\eta(u)}{1+u^2}$$
$$+ \frac{3((m+l))}{S_1} \sup_{u\in[0,\infty)} \frac{\eta(u)}{1+u^2} + \frac{2}{S_1} \sup_{u\in[0,\infty)} \frac{1}{1+u^2},$$

where $S_1 = ((m+l)+2)((m+l)+1)$ This implies that $||T^{(\alpha)}_{m+l}(\eta^2(v) - u^2||_{1+u^2} = 0$ as $m+l \to \infty$,
which completes the proof of the Theorem. □

Theorem 5.3. *Suppose $g \in C^k_{1+u^2}[0,\infty)$ and $\lambda > 0$. Then, we obtain*

$$\lim_{m+l\to\infty} \sup_{u\in[0,\infty)} \frac{|T^{(\alpha)}_{m+l}(f;u) - g(u)|}{(1+u^2)^{1+\lambda}} = 0.$$

Proof. Here u_0 is a fixed number, we have

$$\sup_{u\in[0,\infty)} \frac{|T^{(\alpha)}_{m+l}(g;u) - g(u)|}{(1+u^2)^{1+\lambda}} \leq \sup_{u\leq u_0} \frac{|T^{(\alpha)}_{m+l}(f;u) - f(u)|}{(1+u^2)^{1+\lambda}} + \sup_{u\geq u_0} \frac{|T^{(\alpha)}_{m+l}(g;u) - g(u)|}{(1+u^2)^{1+\lambda}}$$
$$\leq ||T^{(\alpha)}_{m+l}(g.;\cdot) - g||_{C[0,u_0]} + ||g||_{1+u^2} \sup_{u\geq u_0} \frac{|T^{(\alpha)}_{m+l}(1+u^2;u)|}{(1+u^2)^{1+\lambda}} + \sup_{u\geq u_0} \frac{|g(u)|}{(1+u)^{1+u}}$$
$$= B_1 + B_2 + B_3. \tag{18}$$

Since $|g(a)| \leq ||g||_{1+u^2}(1+u^2)$, we have

$$T_3 = \sup_{u\geq u_0} \frac{|g(u)|}{(1+u^2)^{1+\lambda}} \leq \sup_{u\geq u_0} \frac{||g||_{1+u^2}(1+u^2)}{(1+u^2)^{1+\lambda}} \leq \frac{||g||_{1+u^2}}{(1+u^2)^{\lambda}}.$$

Assume $\epsilon > 0$, $(m+l)_1 \geq (m+l)$ implies that

$$B_2 + B_3 < 2\frac{||g||_{1+u^2}}{1+u^2} + \frac{\epsilon}{3}.$$

For a sufficient large value of u_0, we have $\frac{||g||_{1+u^2}}{1+u^2} + \frac{\epsilon}{6}$,

$$T_2 + T_3 < \frac{2\epsilon}{3} \ for \ all \ (m+l)_1 \geq (m+l). \tag{19}$$

By theorem 5.1 $(m+l)_2 > (m+l)$ such that

$$B_1 = ||T^{(\alpha)}_{m+l}(g) - g||_{C[0,u_0]} < \frac{\epsilon}{3} \; for\ all\ (m+l)_2 \geq (m+l). \tag{20}$$

Let $(m+l)_3 = max\left((m+l)_1, (m+l)_2\right)$, combining (18), (19) and (20), we get

$$\sup_{u\in[0,\infty)} \frac{\left|T^{(\alpha)}_{m+l}(g;u) - g(u)\right|}{(1+u^2)^\lambda} < \epsilon.$$

Hence, it completes the proof of the Theorem. □

6. A-Statistical approximation

Gadjiv and Orhan [27] introduced the concept of statistical approximation theorem in operator theory. It is a convergent sequence statistically convergent, as is widely known. However, this is not always the case from [21]. Suppose $A = (a_{rk})$ be an infinite suitability matrix with a non-negative value. For an estimated sequence, $V = (Av)_r$ is defined as:

$$(Av)_{m+l} = \sum_{k=1}^{\infty} a_{(m+l)k} v_k.$$

provided the series converges for each $(m+l)$.
A is said to be regular if $\lim_{m+l}(Av)_{m+l} = L$. Then $v = (v_{m+l})$ is said to be a A-statically convergent to L i.e., $St_A - lim\ v = L$ if for every $\epsilon > 0$,

$$lim_{m+l} \sum_{k:|v_k - L|\geq\epsilon} a_{(m+l)k} = 0.$$

Theorem 6.1. *Let $A = (b_{(m+l)k})$ be a regular positive summability matrix and $v \geq 0$. Then, we get $st_A - \lim_m ||T^{(\alpha)}_{m+l}(g;u) - g||_{1+u^{2+\gamma}} = 0$ for all $g \in C^k_{1+u^{2+\gamma}}[0,\infty)$ and $\gamma > 0$.*

Proof. By ([28], p. 191. Theorem 3) it is sufficient to show that $\lambda = 0$ for this results

$$st_A - \lim_{(m+l)} ||T^{(\alpha)}_{m+l} - g||_{1+u^2} = 0, \tag{21}$$

where $i = \{0,1,2\}$
In view of Lemma 2.3, we obtain

$$\begin{aligned} ||T^{(\alpha)}_{m+l}(e_i;u) - u||_{1+u^2} &= \sup_{u\in(0,\infty)} \frac{1}{1+u^2}\left|\frac{(m+l)\eta(u)+1}{((m+l)+2)} - u\right|. \\ &\leq \frac{(m+l)\eta - ((m+l)+2)+1}{((m+l)+2}. \end{aligned}$$

Given $\epsilon > 0$, we define as:

$$L_1 := \left\{(m+l) : ||T^{(\alpha)}_{m+l}(e_i;u) - u|| \geq \epsilon\right\}.$$

$$L_2 := \left\{(m+l) : \frac{(m+l)_\eta - ((m+l)+2)+1}{((m+l)+2)} < \epsilon\right\}.$$

This implies that $L_1 \subset L_2$, which means

$$\Big(\sum_{k\in L_1} a_{(m+l)k} \leq \sum_{k\in L_2} a_{(m+l)k}\Big).$$

From this, we get

$$st_A - \lim_{m+l} ||T^{(\alpha)}_{m+l}(e_i;u) - u||_{1+u^2} = 0.$$

For $i = 2$ and using Lemma 2.2, we have

$$||T^{(\alpha)}_{m+l}(e_2;u) - u||_{1+u^2} = \sup_{u\in[0,\infty)} \frac{1}{1+u^2}\Bigg|\frac{1}{((m+l)+2)((m+l)+1)}\Big\{(m+l)^2\eta^2(u)$$
$$+ ((m+l)+2(1-\alpha))\eta(u)(1-\eta(u)+3((m+l)\eta(u)+2-u^2\Big\}\Bigg|$$
$$\leq \frac{m+l[(m+l)\eta^2(u)+3\eta(u)-6u^2]}{((m+l)+2)(m+l)+1)} + \frac{((m+l)+2(1-\alpha)\eta(u)(1-\eta(u)}{((m+l)+2)((m+l)+1)}$$
$$+\frac{2}{((m+l)+2)((m+l)+1)}.$$

Given $\epsilon > 0$ then, we have the following sets:

$$J_1 := \Big\{m+l : ||T^{(\alpha)}_{m+l}(e_i;u) - u^2|| \geq \epsilon\Big\}.$$

$$J_2 := \Big\{m+l : \frac{(m+l)[(m+l)\eta^2+3\eta-6]}{((m+l)+2)((m+l)+1)} \geq \frac{\epsilon}{3}\Big\}.$$

$$J_3 := \Big\{m+l : \frac{((m+l)+2(1-\alpha)\eta(u)(1-\eta(u))}{((m+l)+2)((m+l)+1)} \geq \frac{\epsilon}{3}\Big\}.$$

$$J_4 := \Big\{m+l : \frac{2}{((m+l)+2)((m+l)+1)} \geq \frac{\epsilon}{3}\Big\}.$$

Therefore, $J_1 \subset J_2 \cup J_3 \cup J_4$ and thus, we get

$$\sum_{k\in J_1} a_{nk} \leq \sum_{k\in J_2} a_{nk} + \sum_{k\in J_3} a_{nk} + \sum_{k\in J_4} a_{nk}.$$

As $n \to \infty$, we get

$$st_A - \lim_{n} ||T^{(\alpha)}_{m+l}(e_2;u) - u^2||_{1+u^2} = 0.$$

□

Theorem 6.2. *Let $h \in C^2_B[0,\infty)$, we get*

$$st_A - \lim_{m+l} ||T^{(\alpha)}_{m+l}(g) - g||_{C^2_B[0,\infty)} = 0.$$

Proof. Using Taylor's expansion, we obtain

$$g(t) - g(u) + g'(u)(t-u) + \frac{1}{2}g''(\xi)(t-u^2),$$

where $t \leq \psi \leq u$. Applying on both sides $T^{(\alpha)}_{m+l}(\cdot;\cdot)$, we get

$$T^{(\alpha)}_{m+l}(g;u) - g(u) = g'(u)T^{(\alpha)}_{m+l}(\lambda_u;u) + \frac{1}{2}g''(\psi)T^{(\alpha)}_{m+l}(\lambda^2_{m+l};u).$$

This implies that

$$||T^{(\alpha)}_{m+l}(g)-g||_{C_B[0,\infty)}||T^{(\alpha)}_{m+l}(e_1;\cdot)||_{C_B[0,\infty)}+||g''||_{C_B[0,\infty)}||T^{(\alpha)}_{m+l}(e_1;\cdot)^2||_{C_B[0,\infty)} = S_1+S_2. \tag{22}$$

By (21), we obtain

$$\lim_{m+l}\sum_{k\in N:||T^{(\alpha)}_{m+l}(g)-g||_{C_B[0,\infty)}} a_{(m+l)_k} \leq \lim_{m+l}\sum_{k\in N:S_1\geq\frac{\epsilon}{3}} a_{(m+l)_k} + \lim_{m+l}\sum_{k\in N:S_2\geq\frac{\epsilon}{3}} a_{(m+l)_k}.$$

Then,

$$st_A-\lim_{m+l}||T^{(\alpha)}_{m+l}(g)-g||_{C_B[0,\infty)}\to 0 \ as\ m\to\infty.$$

This completes the proof of the above theorem. □

Theorem 6.3. *Let $g\in C_B[0,\infty)$, we have*

$$||T^{(\alpha)}_{m+l}(g)-g||_{C_B[0,\infty)}\leq M\omega_2(g;\sqrt{\delta}),$$

where

$$\delta=||T^{(\alpha)}_{m+l}((e_1-\cdot);\cdot)||_{C_B[0,\infty)}+||T^{(\alpha)}_{m+l}((e_1-\cdot)^2;\cdot)||_{C_B[0,\infty)}.$$

Proof. Let $g\in C^2_B[0,\infty)$ and using (22), we get

$$||T^{(\alpha)}_{m+l}(g)-g||_{C_B[0,\infty)}\leq ||g'||_{C_B[0,\infty)}||T^{(\alpha)}_{m+l}((e_1-\cdot);\cdot)||_{C_B[0,\infty)}+\frac{1}{2}||g''||_{C_B[0,\infty)}$$
$$||T^{(\alpha)}_{m+l}((e_1-\cdot)^2;\cdot)||_{C_B[0,\infty)}\leq\delta||g||_{C^2_B[0,\infty)}. \tag{23}$$

For all $g\in C_B[0,\infty)$ and $h\in C^2_B[0,\infty)$ from (23), we have

$$\begin{aligned}||T^{(\alpha)}_{m+l}(h)-g||_{C_B[0,\infty)} &\leq ||T^{(\alpha)}_{m+l}(g)-T^{(\alpha)}_{m+l}(h)||_{C_B[0,\infty)}+||T^{(\alpha)}_{m+l}(h)-h||_{C_B[0,\infty)}\\ &+||(h)-g||_{C_B[0,\infty)}\leq 2||h-g||_{C_B[0,\infty)}+||T^{(\alpha)}_{m+l}(h)-h||_{C_B[0,\infty)}\\ &\leq 2||(h)-g||_{C_B[0,\infty)}+\delta||h||_{C^2_B[0,\infty)}.\end{aligned}$$

By definition of Peetre's K-functional, we get $||T^{(\alpha)}_{m+l}(g)-g||_{C_B[0,\infty)}\leq 2K_2(g;\delta)$. Using (8), we get

$$||T^{(\alpha)}_{m+l}(g)-g||_{C_B[0,\infty)}\leq M\left\{\omega_2(g;\sqrt{\delta})+min(1,\delta)||g||_{C_B[0,\infty)}\right\},$$

using (21), we get

$$st_A-\lim_{m+l}\delta=0,\ then\ st_A-\lim_{m+l}\omega(g;\sqrt{\delta})=0.$$

Considering above results, we can easily calculate rate of A-statical convergence $T^{(\alpha)}_{m+l}(g;u)$ to $g(u)$ in the space $C_B[0,\infty)$. Hence, we arrive at our desired result. □

Acknowledgements. The authors would like to express their deep gratitude to the anonymous learned referee(s) and the editor for their valuable suggestions and constructive comments, which resulted in the subsequent improvement of this research article. The authors are also grateful to all the editorial board members and reviewers of esteemed journal i.e. Annals of the University of Craiova - Mathematics and Computer Science Series. The second author RSR is thankful to Department of Mathematics Faculty of Science, Indira Gandhi National Tribal University, Lalpur

Amarkantak 484887, Madhya Pradesh, India to carry out her research work under the supervision of Prof. Vishnu Narayan Mishra. The first author MR design the problem at IGNTU Amarakantak, Madhya Pradesh India. Each author contributed equally in the development of the manuscript. VNM conceived of the study and participated in its design and coordination. All authors read and approved the final version of manuscript. The authors declare that there is no conflict of interests regarding the publication of this research paper.

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