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C_{12}-modules via left exact preradicals

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Abstract. In this paper, we study modules with the condition that images of all submodules under a left exact preradical for the category of right modules over a ring can be essentially embedded in direct summands. This new class of modules properly contains the class of C_{12}-modules (and hence also CS-modules and uniform modules). It is shown that any module isomorphic to a direct summand of a module which satisfies the rC_{12} property. In contrast to CS-modules, it is shown that the class of modules with the former property is closed under essential extensions whenever any module in the new class is relative injective with respect to its essential extensions.

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1. Introduction

Throughout this article, all rings are associative with identity and all modules are unitary right modules. Let R be a ring and M an R-module. A submodule N of M is *essential (or large)* in M if for every $0 \neq K$ submodule of M, we have $N \cap K \neq 0$. Given a submodule L of M, by a *complement (submodule)* of L in M, we mean a submodule C of M, maximal with respect to $L \cap C = 0$. A module is said to be CS *(or extending)* or said to satisfy the C_1 *condition* if every submodule is essential in a direct summand. Equivalently, every complement is a direct summand (see, [6], [15]). Several generalizations of the CS notion have been worked out extensively by many authors (see, [1], [2], [3], [4], [5], [11], [14], [16]). Amongst other generalizations in particular, recall that a module is called a C_{12}*-module (or satisfy* C_{12}*)* if every submodule is essentially embedded in a direct summand (see, [13]). In this trend, as the first attempt, Tercan [14] defined ES-module notion in terms of left exact preradicals for a ring R. Recall that a functor r from the category of the right R-modules to itself is called a *left exact preradical* if it has the following properties

(i) $r(M)$ is a submodule of M for every right R-module M,
(ii) $r(N) = N \cap r(M)$ for every submodule N of a right R-module M,
(iii) $\varphi(r(M)) \subseteq r(M^{'})$ for every homomorphism $\varphi : M \to M^{'}$ for right R-modules M, $M^{'}$.

Let r be a left exact preradical in the category of right R-modules. Amongst the aforementioned properties, $r(M_1 \oplus M_2) = r(M_1) \oplus r(M_2)$ holds true for all right R-modules M_1, M_2. Observe that the singular submodule and socle are left exact preradicals and the identity (id) and zero functors are trivial left exact preradicals.

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For a good source of left exact preradicals the reader is referred to [12]. Recently rC_{11}-modules are investigated in [16]. Recall that a module M is called rC_{11}-module provided for every submodule N of M, $r(N)$ has a complement which is a direct summand of M.

In this paper, we deal with a class of modules which properly contains the class of C_{11}-modules and rC_{11}-modules. We define rC_{12}-modules in terms of left exact preradicals for the category of right modules. In other words, we focus on the only images of all submodules of the module under the left exact preradical rather than all submodules of the module in the definition of C_{12}-modules. Formerly, we obtain basic structural properties of rC_{12}-modules and determine connections with the other class of modules.

Since any result including a left exact preradical in the category of right R-modules constructs a framework, we make special choice of left exact preradicals to provide counter examples. Incidentally, we show that rC_{12} property is not inherited by direct summands of a module. In contrast to the latter closure property we prove that direct sums of distributive rC_{12}-modules enjoy with the property under a certain condition.

We use r to denote a left exact preradical in the category of right R-modules. Moreover, let M be a module. Thus $N \leq M$, $SocM$, $Z(M)$, and $r_R(x)$ will stand for N is a submodule of M, socle of M, singular submodule of M, and the right annihilator of an element x in M, respectively. For any other terminology or unexplained notions, we refer to [4], [6], [8], [12], [15].

2. Basic results

This section contains basic structural properties of rC_{12}-modules. Also, we think of connections between rC_{12} condition and the rC_{11} property as well as direct summands of a module satisfying the rC_{12} property. Let us begin with mentioning of the definition of rC_{12} property.

Definition 2.1. A right R-module M is called *rC_{12}-module (or satisfies rC_{12})* for any submodule N of M, there exist a direct summand K of M and a monomorphism $\alpha : r(N) \to K$ such that $\alpha(r(N))$ is essential in K.

Lemma 2.1. *Let M be a module and $L \leq N \leq M$. If L is essential in N then $r(L)$ is essential in $r(N)$.*

Proof. Let $0 \neq x \in r(N)$. Then $x \in N$ and hence there exists $t \in R$ such that $0 \neq xt \in L$. Also $xt \in r(N)$ since $r(N) \leq M$. Thus $0 \neq xt \in r(N) \cap L = r(L)$. It follows that $r(L)$ is essential in $r(N)$. □

Proposition 2.2. *Let M be an rC_{12}-module. Then $r(M)$ is a C_{12}-module.*

Proof. Let N be a submodule of $r(M)$. Note that $r(N) = N$. By assumption, there exist submodules K, K' such that $M = K \oplus K'$ and a monomorphism $\alpha : N \to K$, such that $\alpha(N)$ is essential in K. Then $r(M) = r(K) \oplus r(K')$ and $\alpha(r(N)) \leq r(K) \leq K$. Hence $\alpha(N)$ is essential in $r(K)$. It follows that $r(M)$ is a C_{12}-module. □

Next fact characterizes rC_{12}-modules in terms of complement submodules.

Lemma 2.3. *Let M be a right R-module. Then M satisfies rC_{12} if and only if for every complement submodule N of M there exist a direct summand K of M and a monomorphism $\alpha : r(N) \to K$ such that $\alpha(r(N))$ is essential in K.*

Proof. Assume that M satisfies rC_{12}. Then, M has the stated property by definition. Conversely, let $X \leq M$. Then there exist a complement submodule N of M such that X is essential in N. Thus $r(X)$ is essential in $r(N)$ by Lemma 2.1. By assumption, there exist a direct summand K of M and a monomorphism $\alpha : r(N) \to K$ such that $\alpha(r(N))$ is essential in K. Since α is a monomorphism, $\alpha(r(X))$ is essential in $\alpha(r(N))$ and so, $\alpha(r(X))$ is essential in K. Thus M satisfies rC_{12}. □

It is clear from Lemma 2.3 that any C_{12}-module satisfies rC_{12}. In particular, C_{11}-modules (and hence CS or uniform modules) satisfy rC_{12}. Recall that any indecomposable C_{12}-module over a right Noetherian ring is uniform (see [13, Lemma 1.1]). However, there are indecomposable rC_{12}-modules for special choices of r over right Noetherian rings which are not uniform as the following example illustrates. This example also shows that the class of C_{12}-modules is properly contained in the class of rC_{12}-modules.

Example 2.1. **(i)** The Specker group $\mathbb{Z}^{\mathbb{N}}$ does not satisfy C_{12} (see, [11, Lemma 3.4]). Now, let $r = Z$. Then, from [8, Proposition 1.22], $r(\mathbb{Z}^{\mathbb{N}}) = 0$. It follows that $\mathbb{Z}^{\mathbb{N}}$ is an rC_{12}-module.

(ii) Let R be a principal domain and $r = Soc$. If R is not a complete discrete valuation ring, then there exists an indecomposable torsion-free R-module M of rank 2 [9, Theorem 19]. For M, $r(M) = 0$. Hence M satisfies rC_{12}. It is clear that M_R has uniform dimension 2. Since M is not uniform, M does not satisfy C_{12}.

Now we intend to show that rC_{12} property is not inherited by direct summands. Before doing so, we obtain the following fact. For this recall that a module M satisfies rC_{11} if for every submodule N of M, $r(N)$ has a complement which is a direct summand of M (see, [16]).

Proposition 2.4. *If M satisfies rC_{11}, then M satisfies rC_{12}.*

Proof. Let $N \leq M$. Then there exist direct summands $K, K^{'}$ of M such that $M = K \oplus K^{'}$, $r(N) \cap K^{'} = 0$ and $r(N) \oplus K^{'}$ is essential in M. Let $\pi : M \to K$ be the canonical projection and let $\alpha = \pi_{|_{r(N)}} : r(N) \to K$. So α is a monomorphism. Let $0 \neq k \in K$. Then there exists $r \in R$ such that $0 \neq kr = x + k^{'}$ for some $x \in r(N)$, $k^{'} \in K^{'}$. Since $kr = \pi(kr) = \pi(x + k^{'}) = \pi(x) = \alpha(x)$, $kR \cap \alpha(r(N)) \neq 0$ for all $0 \neq k \in K$. Hence $\alpha(r(N))$ is essential in K. □

The next result which has already pointed out previously brings up a different behavior of the rC_{12} condition from the other extending properties.

Proposition 2.5. *Let M be any module. Then M is isomorphic to a direct summand of a module which satisfies rC_{12}.*

Proof. For any module X, let $E(X)$ denote the injective hull of X. Let $M^{'} = E(E(M) \oplus E(M) \oplus \cdots)$. Note that $M^{'}$ is injective. Let $M^{''} = M \oplus M^{'}$. So, M is isomorphic to $M \oplus 0$ which is a direct summand of $M^{''}$. Let us show that $M^{''}$ satisfies rC_{12}. Note that $E(M^{''}) = E(M) \oplus M^{'} = E(E(M) \oplus E(M) \oplus \cdots)$ which is isomorphic to $M^{'}$ and hence there exists a monomorphism $\beta : M^{''} \to M^{'}$. Let N be a submodule of $M^{''}$. Then $\beta(r(N))$ is a submodule of $M^{'}$. But $M^{'}$ is injective thus there exists a direct summand K of $M^{'}$ and hence of $M^{''}$ such that $\beta(r(N))$ is essential submodule of K. Thus $M^{''}$ satisfies rC_{12}. □

Lemma 2.6. *The Specker group* $\mathbb{Z}^{\mathbb{N}}$ *does not satisfy* rC_{12} *with nonzero image under a left exact preradical* r.

Proof. Let M be the Specker group $\mathbb{Z}^{\mathbb{N}}$ and let N be the subgroup $\overset{\infty}{\underset{i=1}{\oplus}} \mathbb{Z}$ of M. Suppose that there exists a direct summand K of M and a monomorphism $\alpha : r(N) \to K$ such that $\alpha(r(N))$ is essential in K. Note that $r(N)$ is isomorphic to $\alpha(r(N))$. By Nunke's theorem [10, Theorem 5], K is isomorphic to M. This implies that K has uncountable rank. But $\alpha(r(N))$ has countable rank. Thus $\alpha(r(N))$ cannot be essential in K. □

Corollary 2.7. *There exists a* $\mathbb{Z}$*-module* M *satisfying* rC_{12} *such that some direct summand* K *of* M *does not satisfy* rC_{12}.

Proof. By Proposition 2.5 and Lemma 2.6. □

Notice that Corollary 2.7 provides existence of a group (i.e., $\mathbb{Z}$-module) which satisfies rC_{12} but it has a direct summand which does not satisfy the rC_{12} property. Moreover, the following proposition makes it clear that the converse of Proposition 2.4 is not true, in general.

Lemma 2.8. [16, Corollary 3.19]*Let* M *be a module which satisfies* rC_{11}. *Let* N *be a direct summand of* M *such that* M/N *is an injective module. Then* N *satisfies* rC_{11}.

Proposition 2.9. *There exists a* $\mathbb{Z}$*-module* M *which satisfies* rC_{12} *but* M *does not satisfy* rC_{11}.

Proof. By the construction of Proposition 2.5, if M is the Specker group $\mathbb{Z}^{\mathbb{N}}$ then there exists an injective $\mathbb{Z}$-module M' such that $M'' = M \oplus M'$ satisfies rC_{12}. By Lemma 2.8, Proposition 2.4 and Lemma 2.6, M'' does not satisfy rC_{11}. □

3. Direct sums and summands of rC_{12}-modules

In this section, we consider direct sums and direct summands of modules which satisfy the rC_{12} property. We show that the class of distributive rC_{12}-modules is closed under direct sums under a certain condition. On the other hand, Corollary 2.7 shows that the rC_{12} property is not inherited by direct summands. We provide a concrete example which exhibits the failure of the latter closure property.

By modifying Lemma 2.1 in [7] for noncommutative rings, we get the following theorem.

Theorem 3.1. *Let* $M \oplus N$ *be a direct sum of two* R*-modules with the property* rC_{12}. *If* $r_R(x) + r_R(y) = R$ *for all* $x \in M$ *and all* $y \in N$, *then* $M \oplus N$ *is an* rC_{12}*-module.*

Proof. Let X be any submodule of $M \oplus N$. From Lemma [7, Lemma 2.1], $X = A \oplus B$ for some submodule A of M and some submodule B of N. By the rC_{12} assumption, there exist a direct summand K of M, a direct summand L of N, and monomorphisms $\alpha : r(A) \to K$ and $\beta : r(B) \to L$ such that $\alpha(r(A))$ is essential in K and $\beta(r(B))$ is essential in L . Note that $K \oplus L$ is a direct summand of $M \oplus N$ and $r(X) = r(A \oplus B) = r(A) \oplus r(B)$. Define

$$\theta : r(X) \to K \oplus L \text{ by } \theta(x) = \theta(a + b) = \alpha(a) + \beta(b)$$

where $x \in r(X)$, $a \in r(A)$, $b \in r(B)$. It is easy to check that θ is a monomorphism. Furthermore, $\theta(r(X)) = \alpha(r(A)) \oplus \beta(r(B))$ is essential submodule of $K \oplus L$. Thus M satisfies rC_{12}. $\square$

Recall that a submodule X of M is said to be *fully invariant* if $f(X) \leq X$ for all $f \in End_R(M)$. Any direct sum of rC_{12}-modules need not to be an rC_{12}-module. For example, consider the Specker group $\mathbb{Z}^{\mathbb{N}}$. Let $r = id$. Then $\mathbb{Z}^{\mathbb{N}}$ is not an rC_{12}-module by Lemma 2.6, although $\mathbb{Z}_{\mathbb{Z}}$ is an rC_{12}-module (see, [11, Lemma 3.4]). Now we prove that any direct sum of distributive rC_{12}-modules is again an rC_{12}-module under a certain condition. Recall that a module is called *distributive* if its lattice of submodules is a distributive lattice.

Theorem 3.2. *Let $M = \oplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of distributive rC_{12}-modules. If every submodule of M is fully invariant (i.e., M is a duo module), then M is an rC_{12}-module.*

Proof. Let M_λ $(\lambda \in \Lambda)$ be a non-empty collection of distributive modules, each satisfying rC_{12}. Let $M = \oplus_{\lambda\in\Lambda} M_\lambda$. Let N be any submodule of M. Let $\lambda \in \Lambda$. Note that $N \cap M_\lambda$ is a submodule of M_λ and M_λ satisfies rC_{12}. There exists a direct summand K_λ of M_λ and a monomorphism $\alpha : r(N \cap M_\lambda) \to K_\lambda$ such that $\alpha(r(N \cap M_\lambda))$ is essential in K_λ. Let $\Lambda^{'}$ be a non-empty subset of Λ containing λ such that there exists a direct summand $K^{'}$ of $M^{'} = \oplus_{\lambda\in\Lambda^{'}} M_\lambda$ and a monomorphism $\alpha^{'} : r(N \cap M^{'}) \to K^{'}$ such that $\alpha^{'}(r(N \cap M^{'}))$ is essential in $K^{'}$. Suppose $\Lambda^{'} \neq \Lambda$. Let $\mu \in \Lambda$, $\mu \notin \Lambda^{'}$. Now $N \cap M_\mu$ is a submodule of M_μ, so there exists a direct summand K_μ of M_μ and a monomorphism $\alpha^{''} : r(N \cap M_\mu) \to K_\mu$ such that $\alpha^{''}(r(N \cap M_\mu))$ is essential in K_μ. Let $\Lambda^{''} = \Lambda^{'} \cup \{\mu\}$ and $M^{''} = \oplus_{\lambda\in\Lambda^{''}} M_\lambda = M^{'} \oplus M_\mu$. It is clear that $K^{'} \cap K_\mu = 0$. Let $K^{''} = K^{'} \oplus K_\mu$. Note that $K^{''}$ is a direct summand of $M^{''}$.

Consider the submodule $N \cap M^{''}$. Since M is a duo module, it is distributive by [7, Propositon 2.3]. Hence $N \cap M^{''} = N \cap (M^{'} \oplus M_\mu) = (N \cap M^{'}) \oplus (N \cap M_\mu)$. Then $r(N \cap M^{''}) = r(N \cap M^{'}) \oplus r(N \cap M_\mu)$. Define

$$\beta : r(N \cap M^{''}) \to K^{'} \oplus K_\mu \quad \text{by} \quad \beta(n) = \beta(m_1 + m_2) = \alpha^{'}(m_1) + \alpha^{''}(m_2)$$

where $n \in r(N \cap M^{''})$, $m_1 \in r(N \cap M^{'})$, $m_2 \in r(N \cap M_\mu)$. It is easy to check that β is a monomorphism. Furthermore, $\beta(N \cap M^{''}) = \alpha^{'}(N \cap M^{'}) \oplus \alpha^{''}(N \cap M_\mu)$ is essential submodule of $K^{'} \oplus K_\mu$. Repeating this argument, there exists a direct summand K of M and a monomorphism $\gamma : r(N) \to K$ such that $\gamma(N)$ is essential in K. Thus M satisfies rC_{12}. $\square$

Our next objective is to give a condition when the rC_{12} property is inherited by essential extensions.

Theorem 3.3. *Let M and T be right R-modules such that M_R is essential in T_R and M_R satisfies rC_{12}. If M_R is T-injective then T_R satisfies rC_{12}.*

Proof. Let X be any submodule of T_R. Let $\overline{X} = X \cap M$. Since M_R is an rC_{12}-module, there exist $e^2 = e \in End(M_R)$ and a monomorphism $\varphi : r(\overline{X}) \to eM$ such that $\varphi(r(\overline{X}))$ is essential in eM. By relative injectivity assumption, there exists $\theta : X \to T$ which lifts φ to X. Let $\pi : T \to eT$ be the canonical projection with kernel $(1-e)T$. So, define $\alpha : r(X) \to eT$ by $\alpha(x) = \pi(\theta(x))$ where $x \in r(X)$. Then it is easy to see that α is a monomorphism. Now, we show that $\alpha(r(X))$ is essential in eT. For, let $0 \neq et \in eT$. Hence there exists $r \in R$ such that $0 \neq etr \in M$. Thus $0 \neq etr \in eM$. It follows that there exists $s \in R$ such that $0 \neq etrs \in \varphi(r(\overline{X}))$. Hence $0 \neq etrs = \varphi(a)$ for some $a \in r(\overline{X})$. Therefore $\alpha(a) = \pi(\theta(a)) = \pi(\varphi(a)) = \varphi(a) = etrs$. Thus $\alpha(r(X))$ is essential in eT which yields that T_R satisfies rC_{12}. □

It is natural to think of whether we can remove the relative injectivity assumption in the previous theorem. However the following example illustrates that the relative injectivity condition is not superfluous.

Example 3.1. Let $T = F[x]/ < x^4 >= \{a\overline{1} + b\overline{x} + c\overline{x}^2 + d\overline{x}^3 \mid a, b, c, d \in F$ and $\overline{x} = x+ < x^4 >\}$ where F is a field. Let $R = F + F\overline{x}^2 + F\overline{x}^3 = \{a\overline{1} + c\overline{x}^2 + d\overline{x}^3 \mid a, c, d \in F\} \leq T$ a subring of T . Then $Z(R) = \{a + b\overline{x}^2 + c\overline{x}^3 \mid (a + b\overline{x}^2 + c\overline{x}^3)(F\overline{x}^2 + F\overline{x}^3) = \overline{0}\} = F\overline{x}^2 + F\overline{x}^3 = SocR_R$ is essential in R_R. Also, $SocR_R$ is not R-injective. In fact, note that $F\overline{x}^2$ is an ideal of R. Let $f : F\overline{x}^2 \to SocR$ defined by $f(a\overline{x}^2) = a\overline{x}^3$, where $a \in F$. Hence f is an R-homomorphism. But there is no $\alpha \in R$ such that $\alpha a\overline{x}^2 = a\overline{x}^3$ for all $a \in F$. So, $SocR$ is not R-injective by Baers Criterion. The ideals of R are $0, R, F\overline{x}^2, F\overline{x}^3$ and $F\overline{x}^2 + F\overline{x}^3$. So, the ideals of $SocR$ are $0, SocR, F\overline{x}^2$ and $F\overline{x}^3$. Also, $Z(F\overline{x}^2) = F\overline{x}^2$ and $Z(F\overline{x}^3) = F\overline{x}^3$. Now, it is easy to see that $SocR_R$ satisfies rC_{12} for $r = Z$. Let us show that R_R does not satisfy rC_{12} for $r = Z$. Note that R is an indecomposable R-module. Let $N = F\overline{x}^2$. Assume that R_R satisfy rC_{12}. Then there exist a monomorphism $\alpha : Z(F\overline{x}^2) \to R$ such that $\alpha(Z(F\overline{x}^2)) = \alpha(F\overline{x}^2)$ is essential in R. Thus, $\alpha(Soc(F\overline{x}^2)) = SocR = F\overline{x}^2 \oplus F\overline{x}^3$. Since α is a monomorphism, $F\overline{x}^2 \cong \alpha(F\overline{x}^2) = \alpha(Soc(F\overline{x}^2)) = F\overline{x}^2 \oplus F\overline{x}^3$, a contradiction. Hence, R_R does not satisfy rC_{12}.

The following proposition is useful in the sense of determining when a direct summand of an rC_{12}-module satisfies the rC_{12} property.

Proposition 3.4. *Let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 of M. Then M_1 is an rC_{12}-module if and only if for every submodule N of M_1, there exists a direct summand K of M and φ monomorphism on N such that $M_2 \subseteq K$, $\varphi(r(N)) \cap K = 0$ and $\varphi(r(N)) \oplus K$ is an essential submodule of M.*

Proof. Assume M_1 satisfies rC_{12}. Let N be any submodule of M_1. Then there exists a direct summand H of M_1 and a monomorphism $\alpha : r(N) \to H$ such that $\varphi(r(N))$ is an essential submodule of H. Thus $M_1 = H \oplus L$ for some submodule L of M. Now, it is clear that $L \oplus M_2$ is a direct summand of M, $(L \oplus M_2) \cap \varphi(r(N)) = 0$ and $(L \oplus M_2) \oplus \varphi(r(N))$ is an essential submodule of M. Conversely, suppose that M_1 has the stated property. Let N be a submodule of M_1. By hypothesis, there exists a direct summand K of M and a monomorphism α on $r(N)$ such that $M_2 \subseteq K$, $\alpha(r(N)) \cap K = 0$ and $\alpha(r(N)) \oplus K$ is an essential submodule of M. Since $K = K \cap (M_1 \oplus M_2) = (K \cap M_1) \oplus M_2$, $K \cap M_1$ is a direct summand of M, and hence also of M_1. Let $M_1 = (K \cap M_1) \oplus X$ for some submodule X of M_1 and $\pi : M \to X$ be the canonical projection with kernel K. Now, let us define $\varphi : r(N) \to X$ by $\varphi(n) = \pi(\alpha(n))$ where $n \in r(N)$. It is easy check that φ is a monomorphism. Now, let us show that $\varphi(r(N))$ is essential in X. For, let $0 \neq x \in X$. Then there exists $r \in R$ such that $0 \neq xr \in \alpha(r(N)) \oplus K$. Thus $xr = \pi(xr) = \pi(n + k) = \varphi(n) + \pi(k)$ for some $n \in r(N)$, $k \in K$. Hence $0 \neq xr = \varphi(n) \in \varphi(r(N))$. It follows that $\varphi(r(N))$ is an essential submodule of X. So, M_1 is an rC_{12}-module. □

Proposition 3.5. *If M is an rC_{12}-module, then $M = M_1 \oplus M_2$ for some submodules M_1, M_2 such that $r(M_1)$ is essential in M_1.*

Proof. Let $N = r(M)$. Then there exist a direct summand M_1 of M and a monomorphism $\alpha : r(N) \to M_1$ such that $\alpha(N)$ is essential in M_1. So $M = M_1 \oplus M_2$ for some submodule M_2 of M. Since $\alpha(r(N)) = \alpha(r(r(M))) = \alpha(r(M)) \leq r(M_1) \leq M_1$, $r(M_1)$ is essential in M_1 as required. □

The following theorem shows that the rC_{12} property on distributive modules is inherited by direct summands.

Theorem 3.6. *Let M be an rC_{12}-module. If M is distributive then any direct summand is also an rC_{12}-module.*

Proof. Let N be a direct summand of M and X be any submodule of N. By hypothesis, there exist a direct summand K of M and a monomorphism $f : r(X) \to K$ such that $f(r(X))$ is essential in K. Now, $M = K \oplus K' = N \oplus N'$ for some submodules N', K' of M. So, we have $N = N \cap (K \oplus K') = (N \cap K) \oplus (N \cap K')$. Let $\pi_1 : M \to N$ and $\pi_2 : N \to N \cap K$ be the canonical projections with $\ker(\pi_1) = N'$, $\ker(\pi_2) = N \cap K'$, respectively. Also, let $i : K \to M$ be the inclusion mapping. Define $\alpha : r(X) \to N \cap K$ by $\alpha(x) = (\pi_2 \circ \pi_1 \circ i)(f(x))$ where $x \in r(X)$. It is easy to check that α is a monomorphism. Let us show that $\alpha(r(X))$ is an essential submodule of $N \cap K$. For, let $0 \neq a \in N \cap K$. Then there exists $r \in R$ such that $0 \neq ar \in f(r(X))$. Now $(\pi_2 \circ \pi_1 \circ i)(ar) = (\pi_2 \circ \pi_1)(ar) = \pi_2(ar) = ar \neq 0$. Hence $0 \neq ar \in \alpha(r(X))$. So, $\alpha(r(X))$ is essential in $N \cap K$. It follows that N is an rC_{12}-module. □

We can not drop the module is being distributive in the previous result as the following example illustrates.

Example 3.2. Let $\mathbb{R}$ be the real field and S the polynomial ring $\mathbb{R}[x, y, z]$. Then the ring $R = S/Ss$, where $s = x^2 + y^2 + z^2 - 1$, is a commutative Noetherian domain. Let $r = id$. The free R-module $M = R \oplus R \oplus R$ satisfies rC_{12}, but M contains a direct summand K which does not satisfy rC_{12} (see [13]). It can be seen that M is not a distributive module. For this, if we choose three submodules I, J, and K in M such that neither J nor K is contained in I, then the distributivity fails.

Open Problem Whether being distributive and duo are superfluous in Theorem 3.2?

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