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# Continuous Dependence of Renormalized Solution for Convection-diffusion Problems Involving a Nonlocal Operator

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Abstract. In Ouédraogo A. et al (cf. [30]), it is provided existence and uniqueness results of $L^1-$renormalized entropy solution for the Cauchy problem associated to the following vast class of nonlinear anisotropic degenerate parabolic-hyperbolic equations involving a nonlocal diffusion term:

$$\partial_t u+\nabla.F(u)-\sum_{i,j=1}^{N}\partial^2_{x_ix_j}A_{ij}(u)-\mathcal{L}_\mu[u]=f(u) \quad \text{in} \quad Q=(0,T)\times\mathbb{R}^N \text{ with } T>0 \text{ and } N\geq 1.$$

Our goal is to complement this previous work with a continuous dependence result of the $L^1-$solution with respect to the data set $(F,a,\mu,f,u_0)$. The strategy is to follow the approach developed by Karlsen and Ulusoy in [28]. However, we must manage the difficulties due to the fact that we are working in the whole space $\mathbb{R}^N$ with an only integrable initial datum $u_0$ and the term source $f$ depends on the unknown function $u$.

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## 1. Introduction

This paper deals with continuous dependence of renormalized entropy solution of anisotropic diffusion-convection problems involving a nonlocal diffusion term. More precisely, we consider initial-value problems of the form:

$$(CP)\begin{cases} \partial_t u+\nabla.F(u)-\displaystyle\sum_{i,j=1}^{N}\partial^2_{x_ix_j}A_{ij}(u)-\mathcal{L}_\mu[u]=f(u) & \text{in} \quad Q=(0,T)\times\mathbb{R}^N, \\ u(0,.)=u_0 & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1}$$

where $u=u(t,x)$ is the scalar unknown function, $\nabla$ denotes the gradient operator with respect to $x$; $\mathcal{L}_\mu$ is a nonlocal operator properly defined on the Schwartz class $\mathcal{S}(\mathbb{R}^N)$ via the Lévy-Khintchine formula by

$$\mathcal{L}_\mu[u](t,x)=\int_{\mathbb{R}^N}\left(u(t,x+z)-u(t,x)-z.\nabla u\mathbb{1}_{\{|z|<1\}}\right)\,d\mu(z) \quad \text{for} \quad a.e.\ (t,x)\in Q, \tag{2}$$

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where $d\mu$ is a measure on $\mathbb{R}^N$ satisfying $d\mu(z) = g(z)\ dz$ with $g \geq 0, g(-z) = g(z)$ for all $z \in \mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} \min(1, |z|^2) g(z)\ dz < +\infty; \tag{3}$$

$f : \mathbb{R} \longrightarrow \mathbb{R}$ is a term source depending on the unknown function $u$ and satisfies

$$f \in Lip(\mathbb{R}) \text{ with } f(0) = 0, \tag{4}$$

where $Lip(\mathbb{R})$ is the set of Lipschitz functions on $\mathbb{R}$; the flux function $F : \mathbb{R} \longrightarrow \mathbb{R}^N$ is assumed to satisfy

$$F \in Lip_{loc}(\mathbb{R}; \mathbb{R}^N); \tag{5}$$

$a = (a_{ij})_{1\leq i,j\leq N}$ is a nonnegative symmetric matrix with nonnegative and locally integrable coefficients such that

$$a_{ij}(u) = \sum_{k=1}^{K} \sigma_{ik}^a(u)\sigma_{jk}^a(u),\ 0 \leq \sigma_{ik}^a(u) \in L^\infty_{loc}(\mathbb{R}),\ i = 1, ..., N \text{ and } k = 1, ..., K \tag{6}$$

with $1 \leq K \leq N$;
we define the diffusion matrix $A = (A_{ij})_{1\leq i,j\leq N}$ with respect to $a_{ij}$ by nonlinearities

$$A_{ij}(r) = \int_0^r a_{ij}(\xi) d\xi \quad i, j = 1, ..., N; \tag{7}$$

the initial datum $u_0$ is assumed to satisfy

$$u_0 \in L^1(\mathbb{R}^N). \tag{8}$$

The problem (CP) on which we would like to investigate and its variants appear in many different areas of research and in a wide variety of important physical problems, including overdriven detonations of gases (cf. [22]), anomalous diffusion in semiconductor growth (cf. [38]), flows in porous media (cf. [24]), radiation hydrodynamics (cf. [33, 34, 36]), mathematical models in finance (cf. [11, 12, 23, 26, 35]), molecular biology [1, 25], . . . So the mathematical theory received much attention from several authors. Let us give some representative examples of $(CP)$ :

- dropping the nonlocal term $\mathcal{L}_\mu[u]$, we fall into the pure local non isotropic diffusion-convection problem which seems to be well understood with the pioneer's work of Chen and Perthame [20] from which other extensions have also been derived (see. [10, 19, 31, 32]);
- if we remove the diffusion term $\sum_{i,j=1}^{N} \partial^2_{x_i x_j} A_{ij}(u)$, we recover the fractal conservation laws of which the well-posedness issue is solved by Alibaud (see [1]) in the sense of entropy solutions when initial data are bounded functions. From other extensions, we refer to [2, 6, 7, 8, 13, 15, 16, 17].

  More recently the concept of kinetic solution is extended to nonlocal conservation laws problems by Alibaud N. et al in [4]. For the same notion of solution, we refer to [29, 37]. For other important contributions on this topic, we refer the reader to [3, 5, 14, 18, 21, 27, 25, 26] and the references cited therein.

The purpose of the present paper is to complement the study of the Cauchy problem (CP) provided in [30] with a continuous dependence result of the $L^1-$ renormalized solution. So, we will get the well-posedness of this problem in the sense of Hadamard and this will predispose us in a future work to develop numerical schemes. Our approach in this investigation lies essentially on ideas and techniques developed in [19, 28] where authors deal only with bounded variation space on $\mathbb{R}^N$ denoted by $BV(\mathbb{R}^N)$.

The remaining part of this work is organized as follows: in Section 2, we present some preliminary results and recall the definitions of the type of solutions that we use (entropy solution and renormalized solution) as well as the useful results presented in [30]; in Section 3, we will restrict our attention to state and prove our main result.

## 2. Preliminary results and definitions

We recall some useful notations and results already used in [30] and [31].

- For any $\varepsilon > 0$, we define the operators $H, H_0$ and $H_\varepsilon$ respectively by:

$$H(s) := \begin{cases} 1 & \text{if } \ s > 0 \\ [0,1] & \text{if } \ s = 0 \\ 0 & \text{if } \ s < 0 \end{cases}, \quad H_0(s) := \begin{cases} -1 & \text{if } \ s < 0 \\ 0 & \text{if } \ s = 0 \\ 1 & \text{if } \ s > 0 \end{cases}, \tag{9}$$

$$H_\varepsilon(s) := \begin{cases} -1 & \text{if } s < -\varepsilon \\ \sin\left(\dfrac{\pi}{2\varepsilon} s\right) & \text{if } |s| < \varepsilon \\ 1 & \text{if } s > \varepsilon \end{cases} ; \tag{10}$$

  for any fixed real $k$ and for $i, j = 1, ..., N$, we introduce the corresponding entropy functions :

$$r \mapsto \gamma_\varepsilon(r-k) := \int_k^r H_\varepsilon(s-k)ds, \tag{11}$$

$$r \mapsto \theta_{\varepsilon,i}(r-k) := \int_k^r H_\varepsilon(s-k)F_i'(s)ds, \tag{12}$$

$$r \mapsto \nu_{\varepsilon,ij}(r-k) := \int_k^r H_\varepsilon(s-k)a_{ij}(s)ds; \tag{13}$$

- for $i, j = 1, ..., N$, we set $\rho := (\rho_{ij})$ and $\rho^\eta := (\rho^\eta_{ij})$ with

$$\rho_{ij}(r) := \int_0^r \sigma^a_{ij}(\tau)d\tau, \qquad \rho^\eta_{ij}(r) := \int_0^r \eta(\tau)\sigma^a_{ij}(\tau)d\tau, \ \ \forall \eta \in \mathcal{C}(\mathbb{R});$$

- for any $\mathcal{C}^2$ convex entropy function $\gamma : \mathbb{R} \to \mathbb{R}$, we define the entropy flux $\theta := (\theta_i) : \mathbb{R} \to \mathbb{R}^N$, $\nu := (\nu_i) : \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}^N$ by $\theta'(r) := \gamma'(r)F'(r)$, $\nu'(r) := \gamma'(r)a(r)$.

In addition, we complete the above notations by setting that:

- for a function $\pi : Q \to \mathbb{R}$, $(t, x, z) \in (0, T) \times \mathbb{R}^2$ and $\tau \in (0, 1)$,

$$\delta_\pi(t,x,z) := \pi(t, x+z) - \pi(t,x), \tag{14}$$

$$\pi^\tau(t,x,z) := (1-\tau)\pi(t,x) + \tau\pi(t,x+z), \tag{15}$$

$$\Gamma^{\gamma''}_\pi(t,x,z) := \int_0^1 (1-\tau)\gamma''\left(\pi^\tau(t,x,z)\right)\, d\tau; \tag{16}$$

- for any test function $\phi \in D(Q)$,

$$\mathcal{L}_\mu[\phi]_x(t,x) := \int_{\mathbb{R}^N} \left(\phi(t,x+z) - \phi(t,x) - z.\nabla_x\phi \mathbb{1}_{\{|z|<1\}}\right) \, d\mu(z). \tag{17}$$

Now, we can state the following elementary lemma

**Lemma 2.1.** *For any function* $\gamma \in \mathcal{C}^2(\mathbb{R})$, *one has:*

$$\gamma(\pi(t,x+z)) - \gamma(\pi(t,x)) - \gamma'(\pi(t,x))\delta_\pi(t,x,z) = \Gamma_\pi^{\gamma''}(t,x,z)\delta_\pi^2(t,x,z). \tag{18}$$

*Proof.* The proof trivially follows from the Taylor's Formula with integral reminder applied to $\gamma$ in the neighborhood of $\pi(t,x)$. □

**Remark 2.1.** For any $r$, $k \in \mathbb{R}$ and $i = 1, ..., N$, we have:

$$H_\varepsilon(r) = -H_\varepsilon(-r), \quad \gamma_\varepsilon(r-k) = \gamma_\varepsilon(k-r), \tag{19}$$

$$\theta_{\varepsilon,i}(r-k) = \theta_{\varepsilon,i}(k-r), \quad \nu_{\varepsilon,ij}(r-k) = \nu_{\varepsilon,ij}(k-r). \tag{20}$$

Besides, when $\varepsilon$ goes to zero, then

$$H_\varepsilon(r) \to H_0(r) \text{ a.e.}, \tag{21}$$

$$\gamma_\varepsilon(r-k) \to \gamma(r-k) = H_0(r-k)(r-k) \text{ a.e.}, \tag{22}$$

$$\theta_{\varepsilon,i}(r-k) \to \theta_i(r-k) = H_0(r-k)(F_i(r) - F_i(k)) \text{ a.e}, \tag{23}$$

$$\nu_{\varepsilon,ij}(r-k) \to \nu_{ij}(r-k) = H_0(r-k)(A_{ij}(r) - A_{ij}(k)) \text{ a.e..} \tag{24}$$

**Remark 2.2.** If the density function g is not defined in $0_{\mathbb{R}^N}$, under the assumptions of the introduction, the integral

$$\mathcal{L}_\mu[\varphi]_x \quad := \quad p.v. \int_{\mathbb{R}^N} (\varphi(x+z) - \varphi(x)) \, d\mu(z) \tag{25}$$

is always well-defined in the principal value sense whenever $\varphi \in \mathcal{S}(\mathbb{R}^N)$.
This principal value is defined as the limit

$$\mathcal{L}_\mu[\varphi]_x \quad := \quad \lim_{r \downarrow 0} \int_{|z|>r} (\varphi(x+z) - \varphi(x)) \, d\mu(z).$$

Now, in order to recall the notion of entropy solution as given in [28, 30], we suppose that the initial datum satisfies the following assumption

$$u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N). \tag{26}$$

**Definition 2.1.** (Entropy solution of (CP))
Let $u_0$ satisfying (26). An entropy solution of $(CP)$ is a measurable function $u : Q \to \mathbb{R}$ such that:

$(i)$ $u \in L^\infty\left(0,T;L^1(\mathbb{R}^N)\right) \cap L^\infty(Q)$, $\displaystyle\sum_{i=1}^N \partial_{x_i}\rho_{ij}(u) \in L^2(Q)$, for any $j = 1, ..., N$ and

$$\int_Q \int_{\mathbb{R}^N} \delta_u^2(t,x,z) \quad d\mu(z) < +\infty. \tag{27}$$

$(ii)$ For $j = 1, ..., N$ and for any $\psi \in \mathcal{C}(\mathbb{R})$,

$$\sum_{i=1}^N \partial_{x_i}\rho_{ij}^\psi(u) = \psi(u) \sum_{i=1}^N \partial_{x_i}\rho_{ij}(u), \; a.e \text{ and in } \mathrm{L}^2(\mathrm{Q}). \tag{28}$$

$(iii)$ For any entropy flux triple $(\gamma,\psi,\nu)$ and for any nonnegative function $\phi\in\mathcal{D}(Q)$,

$$\begin{cases} \displaystyle\int_Q \left(\gamma(u)\partial_t\phi+\psi(u).\nabla\phi+\sum_{i,j=1}^N \nu_{ij}(u)\partial^2_{x_ix_j}\phi+\gamma(u)\mathcal{L}_\mu[\phi]+\gamma'(u)f(u)\phi\right)\,dxdt \\ \displaystyle +\int_{\mathbb{R}^N}\gamma(u_0(x))\phi(0,x)\,dx\geq\int_Q\left(n^{\gamma''}+m_\mu^{\gamma''}\right)\phi\,dxdt, \end{cases} \tag{29}$$

where

$$\begin{aligned} n^{\gamma''}(t,x) &= \gamma''(u(t,x))\sum_{j=1}^N\left(\sum_{i=1}^N\partial_{x_i}\rho_{ij}(u(t,x))\right)^2 \\ m_\mu^{\gamma''}(t,x) &= \int_{\mathbb{R}^N}\Gamma_u^{\gamma''}(t,x,z)\delta_u^2(t,x,z)\,d\mu(z). \end{aligned}$$

Let us introduce the Lipschitz continuous truncation function $T_k:\mathbb{R}\to\mathbb{R}$ at height $k>0$:

$$T_k(s):=\operatorname{sgn}(s)\min(k,|s|)=\begin{cases} k & \text{if } s>k \\ s & \text{if } |s|\leq k \\ -k & \text{if } s<-k. \end{cases} \tag{30}$$

**Definition 2.2.** (Renormalized solution of $(CP)$)
Let $u_0$ satisfying (8). A renormalized entropy solution of $(CP)$ is a measurable function $u:Q\to\mathbb{R}$ such that:

$(i)$ $u\in L^\infty\left(0,T;L^1(\mathbb{R}^N)\right)$, $\displaystyle\sum_{i=1}^N\partial_{x_i}\rho_{ij}(T_k(u))\in L^2(Q)$, for any $k>0$ and $j=1,...,N$

and

$$\int_Q\int_{\mathbb{R}^N}\delta^2_{T_k(u)}(t,x,z)\,d\mu(z)<+\infty. \tag{31}$$

$(ii)$ For $j=1,...,N$, for any $\psi\in\mathcal{C}(\mathbb{R})$ and for any $k>0$,

$$\sum_{i=1}^N\partial_{x_i}\rho_{ij}^\psi(T_k(u))=\psi(T_k(u))\sum_{i=1}^N\partial_{x_i}\rho_{ij}(T_k(u)),\ a.e \text{ in Q and in } \mathrm{L}^2(\mathrm{Q}). \tag{32}$$

$(iii)$ For any $k>0$ and any entropy flux triple $(\gamma,\theta,\nu)$ with $|\gamma'|\leq K$ (for some given positive constant $K$), there exists a nonnegative bounded Radon measure $m_k^{u,K}$ on $Q$ such that

$$\begin{aligned} &\int_Q\left(\gamma(T_k(u))\partial_t\phi+\theta(T_k(u)).\nabla\phi+\sum_{i,j=1}^N\nu_{ij}(T_k(u))\partial^2_{x_ix_j}\phi+\gamma(T_k(u))\mathcal{L}_\mu[\phi]\right)\,dxdt \\ &\qquad+\int_Q\gamma'(T_k(u))f(T_k(u))\phi\,dxdt+\int_{\mathbb{R}^N}\gamma(T_k(u_0)(x))\phi(0,x)\,dx \\ &\qquad\geq\int_Q\left(n_k^{\gamma''}+m_{\mu,k}^{\gamma''}-m_k^{u,K}\right)\phi\,dxdt, \end{aligned} \tag{33}$$

for any nonnegative function $\phi \in \mathcal{D}(Q)$ where

$$n_k^{\gamma''}(t,x) = \gamma''(T_k(u(t,x)))\sum_{j=1}^{N}\Big(\sum_{i=1}^{N}\partial_{x_i}\rho_{ij}(T_k(u(t,x)))\Big)^2$$

$$m_{\mu,k}^{\gamma''}(t,x) = \int_{\mathbb{R}^N}\Gamma_{T_k(u)}^{\gamma''}(t,x,z)\delta^2_{T_k(u)}(t,x,z)\,d\mu(z). \quad (34)$$

$(iv)$ The total mass of the renormalized measure $m_k^{u,K}$ vanishes as $k\uparrow\infty$:

$$\lim_{k\uparrow\infty} m_k^{u,K}(Q) = 0.$$

**Remark 2.3.** For any $k > 0$, $T_k(u)$ belongs to $L^\infty(Q)$ and (33) is well defined. Moreover, if $u$ is bounded on $Q$, taking $k$ larger than $\|u\|_{L^\infty(0,T;L^1(\mathbb{R}^N))}$, one can see that $T_k(u) = u$ and $u$ satisfies Definition 2.1. This means that any bounded renormalized solution of $(CP)$ is also an entropy solution of the same problem although the concept of renormalized solution is more general. Therefore, in $L^\infty$ framework, the two notions of solutions are equivalent.

**Theorem 2.2.** *(Comparison principle) Let $u, v$ be two renormalized solution of $(CP)(A,F,f,u_0,\mathcal{L}_\mu)$ and $(CP)(A,F,g,v_0,\mathcal{L}_\mu)$ respectively with $u_0$ and $v_0$ satisfying (26). For a.e. $t\in(0,T)$, we have*

$$\int_{\mathbb{R}^N}(u(t,x)-v(t,x))^+\,dx \le \int_{\mathbb{R}^N}(u_0(x)-v_0(x))^+\,dx + \int_0^t\int_{\mathbb{R}^N}\kappa\left(f(u(\tau,x))-g(v(\tau,x))\right)\,dxd\tau \quad (35)$$

*with $\kappa\in H(u-v)$ a.e. so that:*

$$\|u(t)-v(t)\|_{L^1(\mathbb{R}^N)} \le \|u_0-v_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|f(u)-g(v)\|_{L^1(\mathbb{R}^N)}ds. \quad (36)$$

After this preliminary part, we are now in the position to state and prove our continuous dependence result.

## 3. The main result

**Theorem 3.1.** *Let $u$ and $\widehat{u}$ be two renormalized solutions of (CP) respectively with data sets $(F,a,\mu,f,u_0)$ and $(\widehat{F},\widehat{a},\widehat{\mu},\widehat{f},\widehat{u}_0)$ satisfying (3)–(8). Then, for all $r>0$ and for any $t\in(0,T)$, we have:*

$$\begin{aligned}\|u(t,\cdot)-\widehat{u}(t,\cdot)\|_{L^1(\mathbb{R}^N)} \le\ & \|u_0-\widehat{u}_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|f(u)-\widehat{f}(\widehat{u})\|_{L^1(\mathbb{R}^N)}ds \\ & + C_1 t\sqrt{\|F-\widehat{F}\|_{Lip(I)}} + C_2\sqrt{t}\|\sigma^a-\sigma^{\widehat{a}}\|_{L^\infty(I;\mathbb{R}^{N\times K})} \quad (37)\\ & + C_3\sqrt{t}\sqrt{\left(\int_{|z|<r}|z|^2|g(z)-\widehat{g}(z)|\,dz\right)} + C_4 t\int_{|z|\ge r}|z||g(z)-\widehat{g}(z)|\,dz,\end{aligned}$$

*where $I$ is some bounded subset of $\mathbb{R}$ and the constants $C_i$, $i=1,\cdots,4$ depend on the data and the norms $\|u\|_{L^1(\mathbb{R}^N)}$ and $\|\widehat{u}\|_{L^1(\mathbb{R}^N)}$.*

*Proof.* Let $u$ and $\widehat{u}$ be renormalized entropy solutions of $(CP)$ with respect to $(F, a, \mu, f, u_0)$ and $(\widehat{F}, \widehat{a}, \widehat{\mu}, \widehat{f}, \widehat{u}_0)$ respectively. From the definition of renormalized solution for $u = u(t,x)$ and $\gamma = \gamma_\varepsilon(r-c)$ for any $c \in \mathbb{R}$, let us take $\phi = \phi(t,x) \in \mathcal{D}(Q)$ as a nonnegative test function in (33) , we obtain for any $k > 0$

$$\begin{aligned}
\int_Q \gamma_\varepsilon(T_k(u) - c)\partial_t\phi \, dxdt &+ \int_Q \Big(\theta_\varepsilon(T_k(u) - c).\nabla\phi + \sum_{i,j=1}^N \nu_{\varepsilon,ij}(T_k(u) - c)\partial^2_{x_ix_j}\phi\Big) dxdt \\
&+ \int_Q \Big(\gamma_\varepsilon(T_k(u) - c)\mathcal{L}_\mu[\phi]_x + H_\varepsilon(T_k(u) - c)f(T_k(u))\phi\Big) \, dxdt \\
&+ \int_{\mathbb{R}^N} \gamma_\varepsilon(T_k(u_0) - c)\phi(0,x) \, dx \\
&\geq \int_Q H_\varepsilon^{'}(T_k(u) - c)\sum_{j=1}^N \Big(\sum_{i=1}^N \partial_{x_i}\rho_{ij}(T_k(u))\Big)^2 \phi \, dxdt \\
&+ \int_Q \Big(\int_{\mathbb{R}^N} \Gamma^{H_\varepsilon^{'}}_{T_k(u)}(t,x,z)\delta^2_{T_k(u)}(t,x,z) \, d\mu(z) \Big)\phi \, dxdt \\
&- \int_Q m_k^{u,K}(t,x)\phi \, dxdt. \qquad (38)
\end{aligned}$$

Similarly, specifying $\widehat{u} = \widehat{u}(s,y)$ and $\gamma = \gamma_\varepsilon(r-c)$ for any $c \in \mathbb{R}$, we take $\phi = \phi(s,y) \in \mathcal{D}(Q)$ as a nonnegative test function in (33) and obtain for any $k > 0$

$$\begin{aligned}
\int_Q \gamma_\varepsilon(T_k(\widehat{u}) - c)\partial_s\phi \, dyds &+ \int_Q \Big(\widehat{\theta}_\varepsilon(T_k(\widehat{u}) - c).\nabla\phi + \sum_{i,j=1}^N \widehat{\nu}_{\varepsilon,ij}(T_k(\widehat{u}) - c)\partial^2_{y_iy_j}\phi\Big) dyds \\
&+ \int_Q \Big(\gamma_\varepsilon(T_k(\widehat{u}) - c)\mathcal{L}_{\widehat{\mu}}[\phi]_y + H_\varepsilon(T_k(\widehat{u}) - c)\widehat{f}(T_k(\widehat{u}))\phi\Big) \, dyds \\
&+ \int_{\mathbb{R}^N} \gamma_\varepsilon(T_k(\widehat{u}_0) - c)\phi(0,y) \, dy \\
&\geq \int_Q H_\varepsilon^{'}(T_k(\widehat{u}) - c)\sum_{j=1}^N \Big(\sum_{i=1}^N \partial_{y_i}\widehat{\rho}_{ij}(T_k(\widehat{u}))\Big)^2 \phi \, dyds \\
&+ \int_Q \Big(\int_{\mathbb{R}^N} \Gamma^{H_\varepsilon^{'}}_{T_k(\widehat{u})}(s,y,z)\delta^2_{T_k(\widehat{u})}(s,y,z) \, d\widehat{\mu}(z) \Big)\phi \, dyds \\
&- \int_Q \widehat{m}_k^{\widehat{u},K}(s,y)\phi \, dyds. \qquad (39)
\end{aligned}$$

Now we take $c = T_k(\widehat{u}(s,y))$ in (38) and integrate with respect to $(s,y)$, then we take $c = T_k(u(t,x))$ in (39) and integrate with respect to $(t,x)$. Thanks to (19) and (20), by summing up the results we obtain

$$\begin{aligned}
I^k_{time}(\varepsilon) \quad &+ \quad I^k_{conv}(\varepsilon) + I^k_{diffus}(\varepsilon) + I^k_{fdiff}(\varepsilon) + I^k_{source}(\varepsilon) \\
&+ \quad I^k_{init}(\varepsilon) \geq I^k_{diss}(\varepsilon) + I^k_{fdiss}(\varepsilon) - I^k_{renormeas}(\varepsilon), \qquad (40)
\end{aligned}$$

where

$$I^k_{time}(\varepsilon) = \int_{Q\times Q} \gamma_\varepsilon(T_k(u)-T_k(\widehat{u}))(\partial_t+\partial_s)\phi \; dxdydtds$$

$$I^k_{conv}(\varepsilon) = \int_{Q\times Q} \theta_\varepsilon(T_k(u)-T_k(\widehat{u})).\nabla_x\phi + \widehat{\theta}_\varepsilon(T_k(\widehat{u})-T_k(u)).\nabla_y\phi \; dxdydtds$$

$$I^k_{diffus}(\varepsilon) = \int_{Q\times Q} \sum_{i,j=1}^{N} \int_{T_k(\widehat{u})}^{T_k(u)} \left(\sum_{k=1}^{K} H_\varepsilon(\xi-T_k(\widehat{u}))\sigma^a_{ik}(\xi)\sigma^a_{jk}(\xi)\; d\xi\right) \; \partial^2_{x_ix_j}\phi \; dxdydtds$$
$$+ \int_{Q\times Q} \sum_{i,j=1}^{N} \int_{T_k(\widehat{u})}^{T_k(u)} \left(\sum_{k=1}^{K} H_\varepsilon(\xi-T_k(\widehat{u}))\sigma^{\widehat{a}}_{ik}(\xi)\sigma^{\widehat{a}}_{jk}(\xi)\; d\xi\right) \; \partial^2_{y_iy_j}\phi \; dxdydtds$$

$$I^k_{fdiff}(\varepsilon) = \int_{Q\times Q} \gamma_\varepsilon(T_k(u)-T_k(\widehat{u}))\left(\mathcal{L}_\mu[\phi]_x + \mathcal{L}_{\widehat{\mu}}[\phi]_y\right) \; dxdydtds$$

$$I^k_{source}(\varepsilon) = \int_{Q\times Q} H_\varepsilon(T_k(u)-T_k(\widehat{u}))\left(f(T_k(u)) - \widehat{f}(T_k(\widehat{u}))\right)\phi \; dxdydtds$$

$$I^k_{init}(\varepsilon) = \int_Q\int_{\mathbb{R}^N} \gamma_\varepsilon(T_k(u_0(x))-T_k(\widehat{u}(s,y)))\phi(0,x,s,y)\; dxdyds$$
$$+ \int_Q\int_{\mathbb{R}^N} \gamma_\varepsilon(T_k(\widehat{u}_0(y))-T_k(u(t,x)))\phi(t,x,0,y)\; dxdydt$$

$$I^k_{diss}(\varepsilon) = \int_{Q\times Q} H^{'}_\varepsilon(T_k(u)-T_k(\widehat{u}))\sum_{j=1}^{N}\left(\sum_{i=1}^{N}\partial_{x_i}\rho_{ij}(T_k(u))\right)^2 \phi \; dxdydtds$$
$$+ \int_{Q\times Q} H^{'}_\varepsilon(T_k(u)-T_k(\widehat{u}))\sum_{j=1}^{N}\left(\sum_{i=1}^{N}\partial_{y_i}\widehat{\rho}_{ij}(T_k(\widehat{u}))\right)^2 \phi \; dxdydtds$$

$$I^k_{fdiss}(\varepsilon) = \int_{Q\times Q}\left(\int_{\mathbb{R}^N}\Gamma^{H^{'}_\varepsilon}_{T_k(u)}(t,x,z)\delta^2_{T_k(u)}(t,x,z)\; d\mu(z)\right)\phi \; dxdydtds$$
$$+ \int_{Q\times Q}\left(\int_{\mathbb{R}^N}\Gamma^{H^{'}_\varepsilon}_{T_k(\widehat{u})}(s,y,z)\delta^2_{T_k(\widehat{u})}(s,y,z)\; d\widehat{\mu}(z)\right)\phi \; dxdydtds$$

$$I^k_{renormeas}(\varepsilon) = \int_{Q\times Q}\left(m^{u,K}_k(t,x) + \widehat{m}^{\widehat{u},K}_k(s,y)\right)\phi \; dxdydtds.$$

Otherwise, one can see that for any $k>0$ and for any nonnegative $\phi=\phi(t,x,s,y)\in \mathcal{D}(Q\times Q)$, thank to the inequality "$a^2+b^2\geq 2ab$" we have $I^k_{diss}(\varepsilon)\geq \bar{I}^k_{diss}(\varepsilon)$ with

$$\bar{I}^k_{diss}(\varepsilon) = 2\int_{Q\times Q} H^{'}_\varepsilon(T_k(u)-T_k(\widehat{u}))$$
$$\times \sum_{k=1}^{N}\left(\sum_{i,j=1}^{N}\partial_{x_i}\rho_{ik}(T_k(u))\partial_{y_j}\widehat{\rho}_{jk}(T_k(\widehat{u}))\right)\phi \; dxdydtds. \tag{41}$$

Next, for positive real parameters $\sigma$, $\lambda$ and $\alpha$, we define mollifier sequences $\psi_\sigma$ and $\delta_\lambda$ such that $\psi_\sigma\in\mathcal{D}((-\sigma,\sigma))$ and $\delta_\lambda\in\mathcal{D}(B(0,\lambda))$ of symmetric approximate delta functions. Herein, $B(0,\lambda)$ is the open ball centered at $0_{\mathbb{R}^N}$ and of radius $\lambda$. We take the test function $0 \leq \phi = \phi(t,x,s,y) \in \mathcal{D}(Q \times Q)$ to be of the form

$$\phi(t,x,s,y) = \psi_\sigma(s-t)\delta_\lambda(y-x)\chi_\alpha(t), \tag{42}$$

where, for a fixed time $\tau \in (0,T)$, we define for any $\alpha > 0$ with $0 < \alpha < \min(\tau_0, T-\tau)$.

$$\chi_\alpha(t) = H_\alpha(t) - H_\alpha(t-\tau), \quad H_\alpha(t) = \int_{-\infty}^{t} \psi_\alpha(\sigma)\, d\sigma$$

so that $\chi'_\alpha(t) = \psi_\alpha(t) - \psi_\alpha(t-\tau)$.
Simple calculations reveal that:

$$(\partial_t + \partial_s)\phi = \psi_\sigma(s-t)\delta_\mu(y-x)\chi'_\alpha(t), \quad (\nabla_x + \nabla_y)\phi = 0 \quad \text{and} \quad \partial^2_{x_i x_j}\phi = \partial^2_{y_i y_j}\phi.$$

In the rest of this proof, putting the above test function in previous calculations, we are looking for the expected estimates by analyzing the differents terms.

**Lemma 3.2.** *From Inequality (3.16) in* [19], *we deduce*

$$\lim_{\varepsilon\to 0} \bar{I}^k_{diss}(\varepsilon) = -2\int_{Q\times Q}\sum_{i,j=1}^{N}\sum_{k=1}^{K} \times \int_{T_k(\widehat{u})}^{T_k(u)} \Big(H_0(\xi - T_k(\widehat{u}))\sigma^a_{ik}(\xi)\sigma^{\widehat{a}}_{jk}(\xi)d\xi\Big)\partial^2_{x_i y_j}\phi\ dxdydtds. \tag{43}$$

If we use the test function given in (42) and sending $\varepsilon \to 0$ in (40), we find:

$$-\lim_{\varepsilon\to 0} I^k_{time}(\varepsilon,\sigma,\alpha,\lambda) \leq \lim_{\varepsilon\to 0} I^k_{conv}(\varepsilon,\sigma,\alpha,\lambda) + \lim_{\varepsilon\to 0} I^k_{diffus}(\varepsilon,\sigma,\alpha,\lambda) - \lim_{\varepsilon\to 0} \bar{I}^k_{diss}(\varepsilon,\sigma,\alpha,\lambda) + \lim_{\varepsilon\to 0} I^k_{fdiff}(\varepsilon,\sigma,\alpha,\lambda) + \lim_{\varepsilon\to 0} I^k_{source}(\varepsilon,\sigma,\alpha,\lambda). \tag{44}$$

- **Estimate of** $-\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0} I^k_{time}(\varepsilon,\sigma,\alpha,\lambda)$.

This test function allows to rewrite our first term as follow:

$$-I^k_{time}(\varepsilon,\sigma,\alpha,\lambda) = -\int_{Q\times Q} \gamma_\varepsilon(T_k(u) - T_k(\widehat{u}))\psi_\sigma(s-t)\delta_\lambda(y-x)\chi'_\alpha(t)\ dxdydtds.$$

Sending $\varepsilon$ to 0, one has

$$-\lim_{\varepsilon\to 0} I^k_{time}(\varepsilon,\sigma,\alpha,\lambda) = -\int_{Q\times Q} |T_k(u) - T_k(\widehat{u})|\psi_\sigma(s-t)\delta_\lambda(y-x)\chi'_\alpha(t)\ dxdydtds;$$

so, using triangle's inequality, we observe that:

$$\begin{aligned}
-\int_{Q\times Q} &|T_k(u(t,x)) - T_k(\widehat{u}(s,y))|\psi_\sigma(s-t)\delta_\lambda(y-x)\chi'_\alpha(t)\ dxdydtds \\
&\geq -\int_{Q\times Q} |T_k(u(t,y)) - T_k(\widehat{u}(t,y))|\psi_\sigma(s-t)\delta_\lambda(y-x)|\chi'_\alpha(t)|\ dxdydtds \\
&\quad -\int_{Q\times Q} |T_k(\widehat{u}(t,y)) - T_k(\widehat{u}(s,y))|\psi_\sigma(s-t)\delta_\lambda(y-x)|\chi'_\alpha(t)|\ dxdydtds \\
&\quad -\int_{Q\times Q} |T_k(u(t,x)) - T_k(u(t,y))|\psi_\sigma(s-t)\delta_\lambda(y-x)|\chi'_\alpha(t)|\ dxdydtds \\
&= L(\sigma,\alpha,\lambda) + R^t(\sigma,\alpha,\lambda) + R^x(\sigma,\alpha,\lambda).
\end{aligned}$$

Next, proceeding as [9] and [19], we show that:

$$\lim_{\alpha\to 0} L(\sigma,\alpha,\lambda) = \|u(\tau,\cdot)-\widehat{u}(\tau,\cdot)\|_{L^1(\mathbb{R}^N)} - \|u_0-\widehat{u}_0\|_{L^1(\mathbb{R}^N)} \text{ as } k\to+\infty,$$

$$\lim_{\sigma\to 0} R^t(\sigma,\alpha,\lambda)=0 \quad\text{and}\quad \limsup_{\alpha\to 0}|R^x(\sigma,\alpha,\lambda)| \le 2\lambda\|u(\tau,.)\|_{L^1(\mathbb{R}^N)} \text{ as } k\to+\infty.$$

Thus, by passing to the limit, we obtain the following estimate:

$$\begin{aligned}&-\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0} I^k_{time}(\varepsilon,\sigma,\alpha,\lambda)\\ &\qquad\ge \|u(\tau,\cdot)-\widehat{u}(\tau,\cdot)\|_{L^1(\mathbb{R}^N)} - \|u_0-\widehat{u}_0\|_{L^1(\mathbb{R}^N)} + 2\lambda\|u(\tau,.)\|_{L^1(\mathbb{R}^N)}, \qquad (45)\end{aligned}$$

as $k \longrightarrow +\infty$.

- **Estimate of** $\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0} I^k_{conv}(\varepsilon,\sigma,\alpha,\lambda)$.

The second term that we analyze is defined by

$$\begin{aligned}&I^k_{conv}(\varepsilon,\sigma,\alpha,\lambda)\\ &\quad=\int_{Q\times Q} H_\varepsilon(T_k(u(t,x))-T_k(\widehat{u}(s,y)))\big[(F(T_k(u(t,x)))-F(T_k(\widehat{u}(s,y))))\\ &\quad -(\widehat{F}(T_k(u(t,x)))-\widehat{F}(T_k(\widehat{u}(s,y))))\big]\nabla_x\delta_\mu(y-x)\psi_\sigma(s-t)\chi_\alpha(t)\ dxdydtds.\end{aligned}$$

When $\varepsilon$ tends to 0, exploiting also that $\displaystyle\int |\partial_{x_i}\delta_\lambda| \le C/\lambda$, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}|I^k_{conv}(\varepsilon,\sigma,\alpha,\lambda)|\\ &= \Big|\int_{Q\times Q}\big[(F(T_k(u(t,x)))-\widehat{F}(T_k(u(t,x))))-(F(T_k(\widehat{u}(s,y))-\widehat{F}(T_k(\widehat{u}(s,y))))\big]\\ &\qquad\qquad \times\nabla_x\delta_\lambda(y-x)\psi_\sigma(s-t)\chi_\alpha(t)\ dxdydtds\Big|\\ &\le \int_{Q\times Q}\Big|\big[(F-\widehat{F})(T_k(u(t,x)))-(F-\widehat{F})(T_k(\widehat{u}(s,y)))\big]\nabla_x\delta_\lambda(y-x)\Big|\\ &\qquad\qquad \times\psi_\sigma(s-t)\chi_\alpha(t)\ dxdydtds\\ &\le \int_{Q\times Q}\sum_{i=1}^N\Big|(F_i-\widehat{F}_i)(T_k(u(t,x)))-(F_i-\widehat{F}_i)(T_k(\widehat{u}(s,y)))\Big|\Big|\partial_{x_i}\delta_\lambda(y-x)\Big|\\ &\qquad\qquad \times\psi_\sigma(s-t)\chi_\alpha(t)\ dxdydtds\\ &\le \|F-\widehat{F}\|_{Lip(I)}\int_{Q\times Q}\Big|T_k(u(t,x))\partial_{y_i}\delta_\lambda(y-x)+T_k(\widehat{u}(s,y))\partial_{x_i}\delta_\lambda(y-x)\Big|\\ &\qquad\qquad \times\psi_\sigma(s-t)\chi_\alpha(t)\ dxdydtds\end{aligned}$$

where

$$I=\vartheta(\iota)\subset I_k=[-k,k] \text{ such as } \|F-\widehat{F}\|_{Lip(I)} = \sup_{\vartheta(\iota)\subset I_k}\|F-\widehat{F}\|_{Lip(\vartheta(\iota))}$$

with $\vartheta(\iota)$ a neighborhood of $\iota\in I_k$.

A passage to the limit when $\sigma$ and $\alpha$ tend successively to 0 leads to

$$\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0}|I^k_{conv}(\varepsilon,\sigma,\alpha,\lambda)| \le 2\frac{C}{\lambda}\tau\|F-\widehat{F}\|_{Lip(I)}\|u(\tau,.)\|_{L^1(\mathbb{R}^N)} \quad \text{as } k\longrightarrow+\infty.$$

Choosing $\lambda = \sqrt{\|F - \widehat{F}\|_{Lip(I)}}$, we obtain:

$$\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0}|I^k_{conv}(\varepsilon,\sigma,\alpha,\lambda)| \le C\tau\sqrt{\|F-\widehat{F}\|_{Lip(I)}}. \tag{46}$$

- **Estimate of** $\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0}|I_{diffus}(\sigma,\alpha,\lambda) - \bar{I}^k_{diss}(\sigma,\alpha,\lambda)|$.

From the test function, we get

$$I^k_{diffus}(\varepsilon,\sigma,\alpha,\lambda) - \bar{I}^k_{diss}(\varepsilon,\sigma,\alpha,\lambda) = \int_{Q\times Q}\sum_{i,j=1}^{N}\psi_\sigma(s-t)\chi_\alpha(t)\partial^2_{x_ix_j}\delta_\lambda(y-x) \times \int_{T_k(\widehat{u})}^{T_k(u)} H_\varepsilon(\xi - T_k(\widehat{u}))\varepsilon^{a,\widehat{a}}_{ij}(\xi)\, d\xi\ dxdydtds,$$

with $\varepsilon^{a,\widehat{a}}_{ij}(\xi) = \sum_{k=1}^{N}\Big(\sigma^a_{ik}(\xi)\sigma^a_{jk}(\xi) - 2\sigma^a_{ik}(\xi)\sigma^{\widehat{a}}_{jk}(\xi) + \sigma^{\widehat{a}}_{ik}(\xi)\sigma^{\widehat{a}}_{jk}(\xi)\Big)$.

Letting $\varepsilon$ to 0, we have

$$\lim_{\varepsilon\to 0}|I^k_{diffus}(\varepsilon,\sigma,\alpha,\lambda) - \bar{I}^k_{diss}(\varepsilon,\sigma,\alpha,\lambda)| = \int_{Q\times Q}\sum_{i,j=1}^{N}\psi_\sigma(s-t)\chi_\alpha(t)\partial^2_{x_ix_j}\delta_\lambda(y-x)\int_{T_k(\widehat{u})}^{T_k(u)} H_0(\xi - T_k(\widehat{u}))\varepsilon^{a,\widehat{a}}_{ij}(\xi)\, d\xi\ dxdydtds,$$

Sending $\alpha$ and $\sigma$ to 0 and exploiting also that $\int |\partial_{x_i}\delta_\lambda| \le C/\lambda$, we get the following estimate

$$\lim_{\alpha\to 0}\lim_{\sigma\to 0}|I_{diffus}(\sigma,\alpha,\lambda) - \bar{I}^k_{diss}(\sigma,\alpha,\lambda)| \le \frac{C}{\lambda}\tau\|(\sigma^a - \sigma^{\widehat{a}})(\sigma^a - \sigma^{\widehat{a}})^{tr}\|_{L^\infty(I,\mathbb{R}^{N\times K})},$$

then we choose $\lambda = \sqrt{\tau}\|\sigma^a - \sigma^{\widehat{a}}\|_{L^\infty(I,\mathbb{R}^{N\times K})}$ to obtain that

$$\lim_{\alpha\to 0}\lim_{\sigma\to 0}|I_{diffus}(\sigma,\alpha,\lambda) - \bar{I}^k_{diss}(\sigma,\alpha,\lambda)| \le C\sqrt{\tau}\|\sigma^a - \sigma^{\widehat{a}}\|_{L^\infty(I,\mathbb{R}^{N\times K})}. \tag{47}$$

- **Estimate of** $\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0} I^k_{fdiff}(\varepsilon,\sigma,\alpha,\lambda)$.

Using the above test function, we have

$$\begin{aligned} I^k_{fdiff}&(\varepsilon,\sigma,\alpha,\lambda)\\ =\ & \int_{Q\times Q}\int_{\{z\le r\}}\gamma_\varepsilon(T_k(u)-T_k(\widehat{u}))\psi_\sigma(s-t)\chi_\alpha(t)\\ & \times\left(\delta_\lambda(y-x-z) - \delta_\lambda(y-x) - z\cdot\nabla\delta_\lambda(y-x)\right)(g(z)-\tilde{g}(z))\ dzdxdydtds\\ & + \int_{Q\times Q}\int_{\{z>r\}}\gamma_\varepsilon(T_k(u)-T_k(\widehat{u}))\psi_\sigma(s-t)\chi_\alpha(t)\left(\delta_\lambda(y-x-z)-\delta_\lambda(y-x)\right)\\ & \times(g(z)-\tilde{g}(z))\ dzdxdydtds = I^k_{fdiff,r}(\varepsilon,\sigma,\alpha,\lambda) + I^{k,r}_{fdiff}(\varepsilon,\sigma,\alpha,\lambda). \end{aligned}$$

When $\varepsilon$ goes to 0, using the Taylor's and Fubini's theorems, we show that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} I^k_{fdiff,r}(\varepsilon,\sigma,\alpha,\lambda)\\
&\quad= \int_{Q\times Q}\int_{\{z\le r\}} H_0(u-\widehat{u})(T_k(u)-T_k(\widehat{u}))\psi_\sigma(s-t)\chi_\alpha(t)\\
&\quad\times\Big(\delta_\lambda(y-x-z)-\delta_\lambda(y-x)-z\cdot\nabla\delta_\lambda(y-x)\Big)(g(z)-\tilde{g}(z))\ dzdxdydtds\\
&\quad= \int_{(0,\tau)\times(0,\tau)}\int_{\{z\le r\}}\int_0^1 |z|^2(1-\xi)\psi_\sigma(s-t)\chi_\alpha(t)(g(z)-\tilde{g}(z))\\
&\quad\times\Big(\int_{\mathbb{R}^N\times\mathbb{R}^N} H_0(u-\widehat{u})\big(T_k(u(t,x))-T_k(\widehat{u}(s,y))\big)D^2\delta_\lambda(y-x-\xi z)\ dxdy\Big)\ d\xi dzdtds\\
&\quad\le \int_{(0,\tau)\times(0,\tau)}\int_{\{z\le r\}}\int_0^1 |z|^2(1-\xi)\psi_\sigma(s-t)\chi_\alpha(t)(g(z)-\tilde{g}(z))\frac{C}{\lambda}\\
&\qquad\qquad\times\Big(\|u(t,.)\|_{L^1(\mathbb{R}^N)}+\|\widehat{u}(s,.)\|_{L^1(\mathbb{R}^N)}\Big)\ d\xi dzdtds\\
&\quad\le \frac{1}{2}\Big(\int_{\{z\le r\}}|z|^2(g(z)-\tilde{g}(z))\ dz\Big)\\
&\qquad\times\int_{(0,\tau)\times(0,\tau)}\psi_\sigma(s-t)\chi_\alpha(t)\frac{C}{\lambda}\Big(\|u(t,.)\|_{L^1(\mathbb{R}^N)}+\|\widehat{u}(s,.)\|_{L^1(\mathbb{R}^N)}\Big)\ dtds.
\end{aligned}$$

Passing to the limit, we reach to the following estimate

$$\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0}|I^k_{fdiff,r}(\varepsilon,\sigma,\alpha,\lambda)|\le\frac{C}{\lambda}\tau\int_{\{|z|<r\}}|z|^2|g(z)-\tilde{g}(z)|\ dz;$$

and choosing $\lambda=\sqrt{\tau}\sqrt{\int_{\{|z|<r\}}|z|^2|g(z)-\tilde{g}(z)|\ dz}$, we get:

$$\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0}|I^k_{fdiff,r}(\varepsilon,\sigma,\alpha,\lambda)|\le C\sqrt{\tau}\sqrt{\int_{\{|z|<r\}}|z|^2|g(z)-\tilde{g}(z)|\ dz}. \tag{48}$$

Similarly, we get:

$$\lim_{\sigma\to 0}\lim_{\alpha\to 0}\lim_{\varepsilon\to 0}|I^{k,r}_{fdiff}(\varepsilon,\sigma,\alpha,\lambda)|\le C\tau\int_{|z|\ge r}|z||g(z)-\tilde{g}(z)|\ dz. \tag{49}$$

- **Estimate of** $\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0} I^k_{source}(\varepsilon,\sigma,\alpha,\lambda)$.

We have

$$\begin{aligned}
I^k_{source}(\varepsilon,\sigma,\alpha,\lambda) &= \int_{Q\times Q} H_\varepsilon(T_k(u(t,x))-T_k(\widehat{u}(s,y)))\\
&\times\Big(f(T_k(u(t,x)))-\widehat{f}(T_k(\widehat{u}(s,y)))\Big)\psi_\sigma(s-t)\delta_\lambda(y-x)\chi_\alpha(t)\ dxdydtds.
\end{aligned}$$

When $\varepsilon$ goes to 0, by using triangle's inequality, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}|I^k_{source}(\varepsilon,\sigma,\alpha,\lambda)|\\
&= \int_{Q\times Q}|f(T_k(u(t,x)))-\widehat{f}(T_k(\widehat{u}(s,y)))|\psi_\sigma(s-t)\delta_\lambda(y-x)\chi_\alpha(t)dxdydtds\\
&\leq \int_{Q\times Q}|f(T_k(u(t,x)))-\widehat{f}(T_k(\widehat{u}(t,x)))|\psi_\sigma(s-t)\delta_\lambda(y-x)\chi_\alpha(t)\ dxdydtds\\
&+ \int_{Q\times Q}|\widehat{f}(T_k(\widehat{u}(t,x)))-\widehat{f}(T_k(\widehat{u}(s,y)))|\psi_\sigma(s-t)\delta_\lambda(y-x)\chi_\alpha(t)\ dxdydtds\\
&\leq \int_{Q\times Q}|f(T_k(u(t,x)))-\widehat{f}(T_k(\widehat{u}(t,x)))|\psi_\sigma(s-t)\delta_\lambda(y-x)\chi_\alpha(t)dxdydtds\\
&+ \|\widehat{f}\|_{Lip(I)}\int_{Q\times Q}|T_k(\widehat{u}(t,x))-T_k(\widehat{u}(s,y))|\psi_\sigma(s-t)\delta_\lambda(y-x)\chi_\alpha(t)\ dxdydtds
\end{aligned}$$

and then

$$\begin{aligned}
&\lim_{\sigma\to 0}\lim_{\varepsilon\to 0}|I^k_{source}(\varepsilon,\sigma,\alpha,\lambda)|\\
&\qquad\leq \int_0^\tau \|f(u)-\widehat{f}(\widehat{u})\|_{L^1(\mathbb{R}^N)}dt+\|\widehat{f}\|_{Lip(I)}\lambda\Big(\|u\|_{L^1(\mathbb{R}^N)}+\|\widehat{u}\|_{L^1(\mathbb{R}^N)}\Big),
\end{aligned}$$

as $k\to+\infty$.
We conclude

$$\lim_{\sigma\to 0}\lim_{\varepsilon\to 0}|I^k_{source}(\varepsilon,\sigma,\alpha,\lambda)|\leq\int_0^\tau \|f(u)-\widehat{f}(\widehat{u})\|_{L^1(\mathbb{R}^N)}dt,\qquad as\ \lambda\to 0. \tag{50}$$

- **Estimate of** $\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0} I^k_{init}(\varepsilon,\sigma,\alpha,\lambda)$.

We have

$$\begin{aligned}
&I^k_{init}(\varepsilon,\sigma,\alpha,\lambda)\\
&\qquad=\int_Q\int_{\mathbb{R}^N}\gamma_\varepsilon(T_k(u_0(x))-T_k(\widehat{u}(s,y)))\psi_\sigma(s)\delta_\lambda(y-x)\chi_\alpha(0)\ dxdyds\\
&\qquad\ +\int_Q\int_{\mathbb{R}^N}\gamma_\varepsilon(T_k(\widehat{u}_0(y))-T_k(u(t,x)))\psi_\sigma(-t)\delta_\lambda(y-x)\chi_\alpha(t)\ dxdydt.
\end{aligned}$$

Letting $\varepsilon$ to 0, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}I^k_{init}(\varepsilon,\sigma,\alpha,\lambda)\\
&\qquad=\int_Q\int_{\mathbb{R}^N}|T_k(u_0(x))-T_k(\widehat{u}(s,y))|\psi_\sigma(s)\delta_\lambda(y-x)\chi_\alpha(0)\ dxdyds\\
&\qquad\ +\int_Q\int_{\mathbb{R}^N}|T_k(\widehat{u}_0(y))-T_k(u(t,x))|\psi_\sigma(-t)\delta_\lambda(y-x)\chi_\alpha(t)\ dxdydt
\end{aligned}$$

$$\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0}I^k_{init}(\varepsilon,\sigma,\alpha,\lambda)=2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|T_k(u_0(x))-T_k(\widehat{u}_0(y))|\delta_\lambda(y-x)\ dxdy.$$

Then,

$$\lim_{\alpha\to 0}\lim_{\sigma\to 0}\lim_{\varepsilon\to 0}|I^k_{init}(\varepsilon,\sigma,\alpha,\lambda)|\leq 2\lambda\Big(\|u_0\|_{L^1(\mathbb{R}^N)}+\|\widehat{u}_0\|_{L^1(\mathbb{R}^N)}\Big)\quad\text{as}\quad k\longrightarrow+\infty.$$

Finally,

$$\lim_{\alpha \to 0} \lim_{\sigma \to 0} \lim_{\varepsilon \to 0} |I^k_{init}(\varepsilon, \sigma, \alpha, \lambda)| = 0 \quad \text{as} \quad \lambda \longrightarrow 0. \tag{51}$$

To end this proof, we add the estimates (45)-(51) and reach to the desired continuous dependence result (37). □

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