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Inverse Coefficient Problem by Fractional Taylor Series Method

Mine Aylin Bayrak and Ali Demir

Abstract. This study focus on determining the unknown function of time or space in space-time fractional differential equation by fractional Taylor series method. A significant advantage of this method is that over-measured data is not used unlike most inverse problems. This advantage allows us to determine the unknown function with less error. The presented examples illustrate that the obtained solutions are in a high agreement with the exact solutions of the corresponding inverse problems.

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1. Introduction

Since fractional derivatives have non-locality properties, fractional differential equations provide a significant tool for modelling of many processes. As a result, this subject draws interest of many scientists in various research areas [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Therefore, inverse problems including fractional differential equations becomes an essential part of diverse processes in science [13, 14, 15, 16].

In this study, we focus on establishing unknown coefficient in space-time fractional differential equations by means of fractional Taylor series, which is a Taylor series including fractional powers. This series also allows us to solve direct problems including fractional differential equations efficiently. Unlike, many methods in inverse problems, this method does not require any over-measured data which makes solution of inverse problem more precise. Furthermore, a few condition is taken into account in determination of unknown coefficient. The main goal in this article is to reveal the unknown coefficient of the following inverse space-time fractional diffusion problem:

$$D_t^{\alpha} u(x,t) = D_x^{2\beta} u(x,t) + r f(x,t), 0 < \alpha, \beta \leq 1, \tag{1}$$

$$u(x,0) = \varphi(x), \tag{2}$$

$$u(0,t) = \mu_1(t), \tag{3}$$

$$u(1,t) = \mu_2(t) \tag{4}$$

where $r = r(t)$ or $r = r(x)$ is the unknown function.

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2. Preliminaries

Essential concepts and features of fractional derivatives are presented in this section [1, 2, 3, 4].

Definition 2.1. The Riemann-Liouville fractional integral of order α ($\alpha \geqslant 0$) is given as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \ \alpha > 0, \ x > 0, \tag{5}$$

$$J^0 f(x) = f(x). \tag{6}$$

Definition 2.2. The Liouville-Caputo fractional derivative of order α is given as

$$D^{\alpha} f(x) = J^{n-\alpha} D^n f(x) = \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} f(t)dt, \ n-1 < \alpha < n, \ x > 0, \tag{7}$$

where D^n denotes the ordinary derivative of order n.

Definition 2.3. The α^{th} order derivative of $u(x,t)$ in Liouville-Caputo sense is given as

$$D_t^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} \frac{\partial^n u(x,\xi)}{\partial t^n} d\xi, \ n-1 < \alpha < n, \\ \frac{\partial^n u(x,t)}{\partial t^n} \ , \alpha = n \in N. \end{cases} \tag{8}$$

Definition 2.4. An (α, β)-fractional Taylor series is defined as follows [17]:

$$\sum_{i+j=0}^{\infty} g_{i,j} t^{i\alpha} x^{j\beta} = \underbrace{g_{0,0}}_{i+j=0} + \underbrace{g_{1,0} t^{\alpha} + g_{0,1} x^{\beta}}_{i+j=1} + ... + \underbrace{\sum_{k=0}^{n} g_{n-k,k} t^{(n-k)\alpha} x^{k\beta}}_{i+j=n} + ... \tag{9}$$

where $g_{i,j}, i, j \epsilon N$ are the coefficients of the series.

Based on definition 2.4 fractional Taylor series of $u(x,t)$ can be written in the following form:

$$u(x,t) = \sum_{i+j=0}^{\infty} \frac{D_t^{i\alpha} D_x^{j\beta} (u(x,t))|_{(x,t)=(0,0)}}{\Gamma(i\alpha+1)\Gamma(j\beta+1)} t^{i\alpha} x^{j\beta}. \tag{10}$$

Lemma 2.1. *Let $u(x,t)$ has a fractional Taylor series representation as (9) for $(x,t)\epsilon[0,R_x) \times [0,R_t)$. If $D_t^{r\alpha} D_x^{s\beta} u(x,t) \epsilon((0,R_x) \times (0,R_t))$ for $r, s \epsilon N$, then*

$$D_t^{r\alpha} u(x,t) = \sum_{i+j=0}^{\infty} g_{i+r,j} \frac{\Gamma((i+r)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)} t^{i\alpha} x^{j\beta} \tag{11}$$

$$D_x^{s\beta} u(x,t) = \sum_{i+j=0}^{\infty} g_{i,j+s} \frac{\Gamma((j+s)\beta+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)} t^{i\alpha} x^{j\beta} \tag{12}$$

3. Fractional Taylor series method

In order to determine the unknown coefficient $r(t)$ of time in the space-time fractional diffusion problem (1)-(4), in the series form we plug the fractional Taylor series of $u=u(x,t)$ and $r=r(t)$ into (1)-(4) which leads to:

$$\sum_{i+j=0}^{\infty} g_{i+1,j}\frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}t^{i\alpha}x^{j\beta}=\sum_{i+j=0}^{\infty} g_{i,j+2}\frac{\Gamma((j+2)\beta+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}t^{i\alpha}x^{j\beta} + \sum_{k=0}^{\infty} r_k\Big\{\sum_{i+j=0}^{\infty}\frac{D_t^{i\alpha}D_x^{j\beta}(f(x,t))|_{(x,t)=(0,0)}}{\Gamma(i\alpha+1)\Gamma(j\beta+1)\Gamma(k\alpha+1)}t^{(i+k)\alpha}x^{j\beta}\Big\} \quad (13)$$

Making two series on both sides of above equation equal to each other, the unknown coefficients in the fractional Taylor series of $r(t)$ are acquired.

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$$\sum_{i+j=0}^{\infty} g_{i+1,j}\frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}t^{i\alpha}x^{j\beta}=\sum_{i+j=0}^{\infty} g_{i,j+2}\frac{\Gamma((j+2)\beta+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}t^{i\alpha}x^{j\beta} + \sum_{k=0}^{\infty} r_k\Big\{\sum_{i+j=0}^{\infty}\frac{D_t^{i\alpha}D_x^{j\beta}(f(x,t))|_{(x,t)=(0,0)}}{\Gamma(i\alpha+1)\Gamma(j\beta+1)\Gamma(k\beta+1)}t^{i\alpha}x^{(j+k)\beta}\Big\} \quad (14)$$

Making two series on both sides of above equation equal to each other, the unknown coefficients in the fractional Taylor series of $r(x)$ are acquired.

4. Illustrative examples

The following example is about unknown coefficient depending on t.

Example 1. Consider the inverse coefficient problem involving space-time fractional differential equations:

$$D_t^{\alpha}u(x,t)=D_x^{2\beta}u(x,t)+r(t)E_{\alpha}(t^{\alpha})sin_{\beta}(x^{\beta}), \quad (15)$$

$$u(x,0)=sin_{\beta}(x^{\beta}), \quad (16)$$

$$u(0,t)=0, \quad (17)$$

$$u(1,t)=E_{\alpha}(2t^{\alpha})sin_{\beta}(1), \quad (18)$$

where $sin_{\beta}(x^{\beta})=\sum_{j=1}^{\infty}\frac{(-1)^j x^{(2j+1)\beta}}{\Gamma((2j+1)\beta+1)}$ is the fractional generalization of the function $sin(x)$. We determine the unknown function $r(t)$ in fractional Taylor series form as follows:

$$r(t)=\sum_{k=0}^{\infty} r_k\frac{t^{k\alpha}}{\Gamma(1+k\alpha)}, 0<\alpha\leq 1. \quad (19)$$

$$\sum_{i+j=0}^{\infty} g_{i+1,j}\frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}t^{i\alpha}x^{j\beta} = \sum_{i+j=0}^{\infty} g_{i,j+2}\frac{\Gamma((j+2)\beta+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}t^{i\alpha}x^{j\beta}$$
$$+\sum_{k=0}^{\infty} r_k\Big\{\sum_{i+j=0}^{\infty}(-1)^j\frac{t^{(i+k)\alpha}x^{(2j+1)\beta}}{\Gamma(i\alpha+1)\Gamma(k\alpha+1)\Gamma((2j+1)\beta+1)}\Big\} \quad (20)$$

with the initial coefficients

$$g_{0,2j+1} = \frac{(-1)^j}{\Gamma((2j+1)\beta+1)}, \quad (21)$$

$$g_{0,0} = 0 \quad (22)$$

The coefficients $g_{i,j}$ are obtained by equating two series in Eq. (20), which enable us to form the solution of Eq.(15) as follows:

$$u(x,t) = \frac{x^{\beta}}{\Gamma(1+\beta)} + (r_0-1)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\frac{x^{\beta}}{\Gamma(1+\beta)} + (r_1+1)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\frac{x^{\beta}}{\Gamma(1+\beta)}$$
$$-\frac{x^{3\beta}}{\Gamma(1+3\beta)} + \Big(-1+r_0-r_1+r_2+\frac{r_1\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\Big)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\frac{x^{\beta}}{\Gamma(1+\beta)}$$
$$+(1-r_0)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\frac{x^{3\beta}}{\Gamma(1+3\beta)} + \ldots \quad (23)$$

In order to determine the unknown coefficient $r(t)$ the boundary condition at $x = 1$ into account in (23) produce the coefficients r_k as follows:
$r_0 = 3,$
$r_1 = 3,$
$r_2 = 9 - 3\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2},$
$\vdots$
As a result, the unknown coefficient $r(t)$ determine in the series of as follows:

$$r(t) = 3 + 3\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \Big(9 - 3\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\Big)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \ldots \quad (24)$$

The following example is about unknown coefficient depending on x.

Example 2. Consider the inverse coefficient problem involving space-time fractional differential equations:

$$D_t^{\alpha}u(x,t) = D_x^{2\beta}u(x,t) + r(x)E_{\alpha}(-t^{\alpha})E_{\beta}(x^{\beta}), \quad (25)$$

$$u(x,0) = E_{\beta}(2x^{\beta}), \quad (26)$$

$$u(0,t) = E_{\alpha}(-t^{\alpha}), \quad (27)$$

$$u(1,t) = E_{\alpha}(-t^{\alpha})E_{\beta}(2), \quad (28)$$

TABLE 1. Comparison of absolute errors at $t = 0.5$ with $E(\alpha, \beta)$ of Example 1.

x	$Exact$	$E(1,1)$	$E(1,0.9)$	$E(1,0.8)$	$E(0.9,1)$	$E(0.9,0.9)$	$E(0.9,0.8)$	$E(0.8,1)$	$E(0.8,0.9)$	$E(0.8,0.8)$
0.1	0.26622	1.11e-04	2.71e-03	8.07e-03	2.72e-03	4.69e-04	7.73e-03	9.53e-03	6.57e-03	1.91e-03
0.2	0.52978	8.89e-04	6.54e-03	1.72e-02	4.46e-03	3.05e-03	1.81e-02	1.76e-02	9.08e-03	1.01e-02
0.3	0.78800	3.00e-03	1.26e-02	2.97e-02	4.24e-03	9.11e-03	3.37e-02	2.29e-02	6.19e-03	2.67e-02
0.4	1.03822	7.11e-03	2.18e-02	4.66e-02	1.08e-03	1.98e-02	5.59e-02	2.38e-02	3.67e-03	5.32e-02
0.5	1.27778	1.39e-02	3.48e-02	6.84e-02	6.01e-03	3.62e-02	8.56e-02	1.90e-02	2.20e-02	9.10e-02
0.6	1.50400	2.40e-02	5.22e-02	9.57e-02	1.80e-02	5.92e-02	1.23e-01	7.10e-03	5.00e-02	1.41e-01
0.7	1.71422	3.81e-02	7.48e-02	1.29e-01	3.59e-02	8.97e-02	1.70e-01	1.34e-02	8.92e-02	2.04e-01
0.8	1.90578	5.69e-02	1.03e-01	1.69e-01	6.06e-02	1.29e-01	2.26e-01	4.40e-02	1.41e-01	2.81e-01
0.9	2.07600	8.10e-02	1.38e-01	2.15e-01	9.32e-02	1.77e-01	2.93e-01	8.60e-02	2.06e-01	3.73e-01
1	2.22222	1.11e-01	1.79e-01	2.69e-01	1.35e-01	2.35e-01	3.69e-01	1.41e-01	2.86e-01	4.80e-01

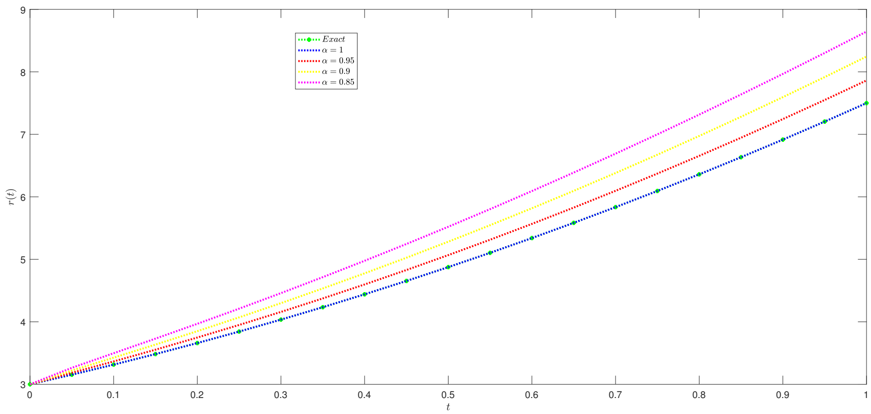

FIGURE 1. The graphics of approximate solutions of $r(t)$.

where $E_\beta(2x^\beta) = \sum_{j=1}^{\infty} \frac{2^j x^{j\beta}}{\Gamma(j\beta+1)}$ is the fractional generalization of the function $exp(2x)$. We determine the unknown coefficient $r(x)$ in fractional Taylor series form as follows:

$$r(x) = \sum_{k=0}^{\infty} r_k \frac{x^{k\beta}}{\Gamma(1+k\beta)}, 0 < \beta \leq 1. \tag{29}$$

$$\sum_{i+j=0}^{\infty} g_{i+1,j} \frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)} t^{i\alpha} x^{j\beta} = \sum_{i+j=0}^{\infty} g_{i,j+2} \frac{\Gamma((j+2)\beta+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)} t^{i\alpha} x^{j\beta} + \sum_{k=0}^{\infty} r_k \Big\{ \sum_{i+j=0}^{\infty} 2^j \frac{t^{i\alpha} x^{(j+k)\beta}}{\Gamma(i\alpha+1)\Gamma(j\beta+1)\Gamma(k\beta+1)} \Big\} \tag{30}$$

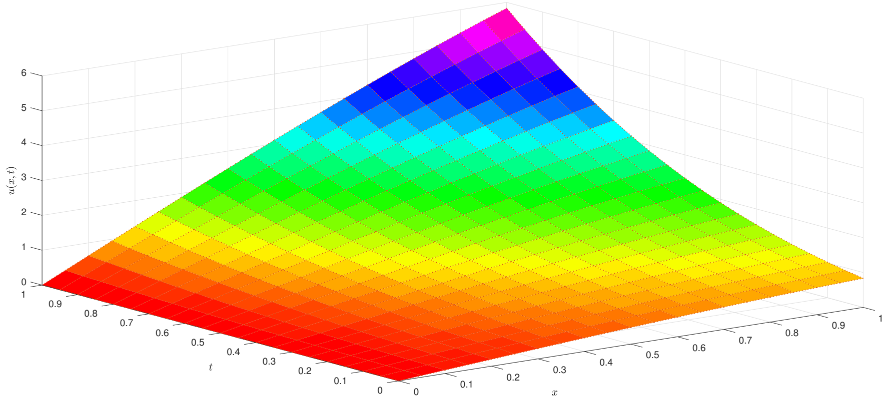

FIGURE 2. The graphics of approximate solutions for Example 1 when $\alpha = 1, \beta = 1$.

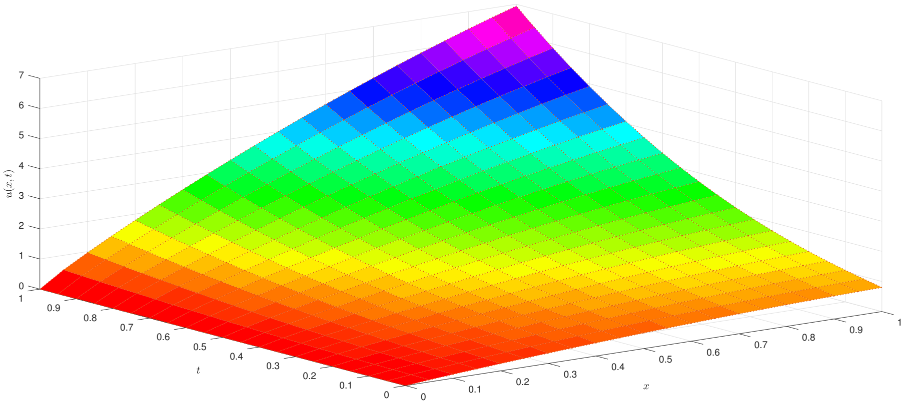

FIGURE 3. The graphics of approximate solutions for Example 1 when $\alpha = 0.9, \beta = 0.9$.

with the initial coefficients

$$g_{0,j} = \frac{2^j}{\Gamma(j\beta + 1)}, g_{0,0} = 1 \tag{31}$$

The coefficients $g_{i,j}$ are obtained by equating two series in Eq. (30), which enable us to form the solution of Eq.(25) as follows:

$$\begin{aligned} u(x,t) &= 1 + (4+r_0)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + 2\frac{x^{\beta}}{\Gamma(1+\beta)} + \left(16 + r_2 + r_1\frac{\Gamma(1+2\beta)}{(\Gamma(1+\beta))^2}\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ (8+r_0+r_1)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\frac{x^{\beta}}{\Gamma(1+\beta)} + 4\frac{x^{2\beta}}{\Gamma(1+2\beta)}\Big(64 + r_0 + r_4 - r_2 \\ &+ \frac{(r_1+r_3)\Gamma(1+4\beta)}{\Gamma(1+\beta)\Gamma(1+3\beta)} + \frac{f_2\Gamma(1+4\beta)}{(\Gamma(1+2\beta))^2} - \frac{r_1\Gamma(1+2\beta)}{(\Gamma(1+\beta))^2}\Big)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &+ \left(32 - r_1 + r_3 + \frac{(r_1+r_2)\Gamma(1+3\beta)}{\Gamma(1+2\beta)\Gamma(1+\beta)}\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\frac{x^{\beta}}{\Gamma(1+\beta)} \\ &+ \left(16 + r_0 + r_2 + \frac{r_1\Gamma(1+2\beta)}{\Gamma(1+2\beta)\Gamma(1+\beta)}\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\frac{x^{2\beta}}{\Gamma(1+2\beta)} + 8\frac{x^{3\beta}}{\Gamma(1+3\beta)} + \ldots \end{aligned} \tag{32}$$

In order to determine the unknown coefficient $r(x)$ the boundary condition at $x = 1$ into account in (31) produce the coefficients r_k as follows:

$r_0 = -5,$

$r_1 = -5,$

$r_2 = -15 + \frac{5\Gamma(1+2\beta)}{(\Gamma(1+\beta))^2},$

$r_3 = -35 - \frac{\Gamma(1+3\beta)(-20(\Gamma(1+\beta))^2+5\Gamma(1+2\beta)))}{\Gamma(1+2\beta)(\Gamma(1+\beta))^3},$

$r_4 = -75 - \frac{\Gamma(1+4\beta)(-40\Gamma(1+2\beta)(\Gamma(1+\beta))^3-\Gamma(1+3\beta)(-20(\Gamma(1+\beta))^2+5\Gamma(1+2\beta)))}{\Gamma(1+3\beta)\Gamma(1+2\beta)(\Gamma(1+\beta))^4} + \frac{10\Gamma(1+2\beta)}{(\Gamma(1+\beta))^2}$
$- \frac{\Gamma(1+2\alpha)\Gamma(1+4\beta)(-15(\Gamma(1+\beta))^2+5\Gamma(1+2\beta)}{\Gamma(1+3\alpha)\Gamma(1+\alpha)(\Gamma(1+\beta))^2(\Gamma(1+2\beta))^2},$

$\vdots$

As a result, the unknown coefficient $r(x)$ determine in the series of as follows:

$$\begin{aligned} r(x) &= -5 - 5\frac{x^{\beta}}{\Gamma(1+\beta)} + \left(-15 + \frac{5\Gamma(1+2\beta)}{(\Gamma(1+\beta))^2}\right)\frac{x^{2\beta}}{\Gamma(1+2\beta)} \\ &+ \left(-35 - \frac{\Gamma(1+3\beta)(-20(\Gamma(1+\beta))^2+5\Gamma(1+2\beta))}{\Gamma(1+2\beta)(\Gamma(1+\beta))^3}\right)\frac{x^{3\beta}}{\Gamma(1+3\beta)} \\ &+ \Bigg(-75 + 40\frac{\Gamma(1+4\beta)}{\Gamma(1+3\beta)\Gamma(1+\beta)} + \frac{\Gamma(1+4\beta)(-20(\Gamma(1+\beta))^2+5\Gamma(1+2\beta))}{\Gamma(1+2\beta)(\Gamma(1+\beta))^4} \\ &- \frac{\Gamma(1+2\alpha)\Gamma(1+4\beta)(-15(\Gamma(1+\beta))^2+5\Gamma(1+2\beta)}{\Gamma(1+3\alpha)(\Gamma(1+2\beta))^2(\Gamma(1+\beta))^3\Gamma(1+\alpha)} + \frac{10\Gamma(1+2\beta)}{(\Gamma(1+\beta))^2}\Bigg)\frac{x^{4\beta}}{\Gamma(1+4\beta)} + \ldots \end{aligned} \tag{33}$$

5. Conclusion

In this research, we take the inverse coefficient problem including space-time fractional differential equation in hand. Fractional Taylor series method is employed for this inverse problem since we don't need any over-measured data for the determination of unknown function of space or time which is a substantial advantage to establish the unknown function more precise. We first obtain the solution of the direct problem in

TABLE 2. Comparison of absolute errors at $t = 0.5$ with $E(\alpha, \beta)$ of Example 2.

x	$Exact$	$E(1,1)$	$E(1,0.9)$	$E(1,0.8)$	$E(0.9,1)$	$E(0.9,0.9)$	$E(0.9,0.8)$	$E(0.8,1)$	$E(0.8,0.9)$	$E(0.8,0.8)$
0	0.60417	0.00	1.50e-02	1.30e-02	9.37e-02	6.16e-02	5.71e-02	3.46e-01	2.80e-01	2.56e-01
0.1	0.73789	2.61e-03	1.20e-02	8.98e-03	9.89e-02	6.78e-02	6.47e-02	3.56e-01	2.92e-01	2.71e-01
0.2	0.90061	4.22e-03	8.74e-03	5.07e-04	1.02e-01	7.25e-02	7.42e-02	3.63e-01	3.00e-01	2.84e-01
0.3	1.09717	8.00e-03	5.75e-04	2.07e-02	1.07e-01	8.23e-02	9.57e-02	3.70e-01	3.13e-01	3.07e-01
0.4	1.33239	1.71e-02	2.02e-02	5.90e-02	1.16e-01	1.02e-01	1.34e-01	3.82e-01	3.34e-01	3.48e-01
0.5	1.61111	3.47e-02	5.36e-02	1.18e-01	1.34e-01	1.35e-01	1.94e-01	4.01e-01	3.69e-01	4.09e-01
0.6	1.93817	6.40e-02	1.04e-01	2.00e-01	1.63e-01	1.86e-01	2.77e-01	4.31e-01	4.22e-01	4.94e-01
0.7	2.31839	1.08e-01	1.75e-01	3.07e-01	2.07e-01	2.58e-01	3.86e-01	4.77e-01	4.95e-01	6.07e-01
0.8	2.75661	1.70e-01	2.70e-01	4.42e-01	2.69e-01	3.54e-01	5.24e-01	5.41e-01	5.93e-01	7.49e-01
0.9	3.25767	2.53e-01	3.90e-01	6.07e-01	3.53e-01	4.76e-01	6.93e-01	6.27e-01	7.19e-01	9.23e-01
1	3.82639	3.61e-01	5.39e-01	8.03e-01	4.62e-01	6.29e-01	8.95e-01	7.38e-01	8.76e-01	1.13e+00

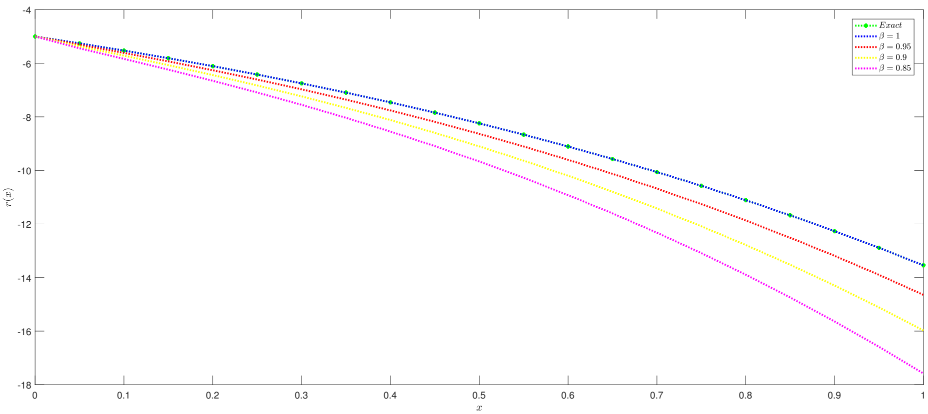

FIGURE 4. The graphics of approximate solutions of $r(x)$.

series for in terms of the coefficients of unknown function. Later, taking the boundary condition into account the coefficients of the unknown functions are determined which allows us to construct the unknown function in fractional Taylor series form. In the future research, we apply this method or modification of this method to various problem in science.

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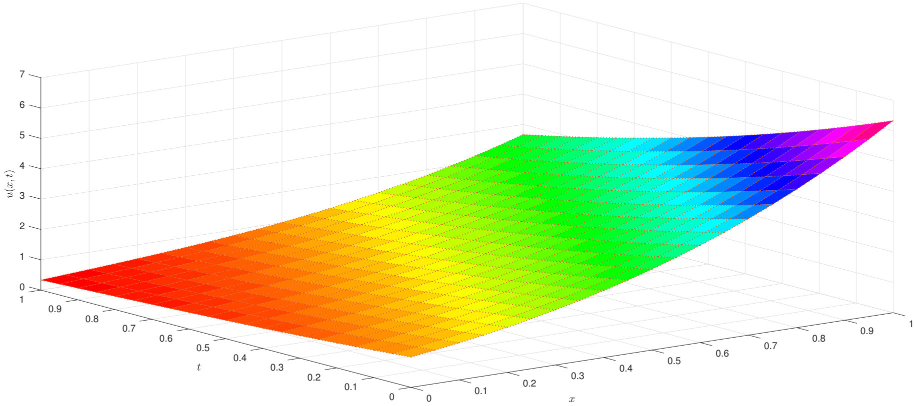

FIGURE 5. The graphics of approximate solutions for Ex. 2 when $\alpha = 1, \beta = 1$.

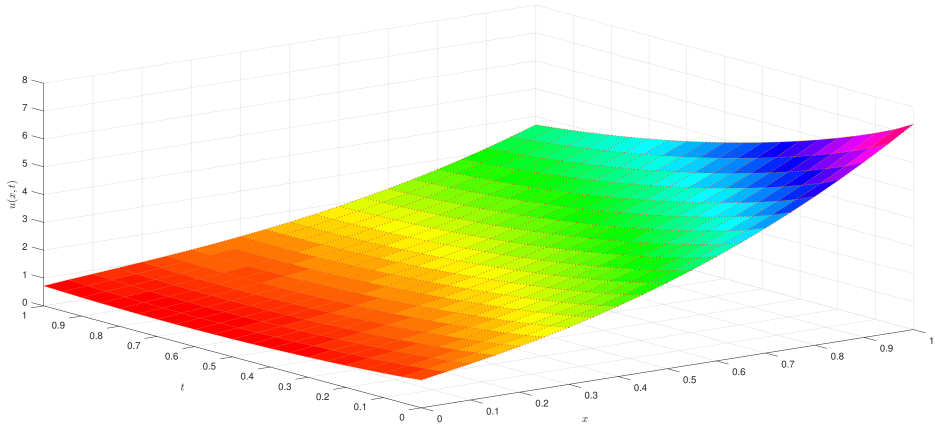

FIGURE 6. The graphics of approximate solutions for Ex. 2 when $\alpha = 0.9, \beta = 0.9$.

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(Mine Aylin Bayrak) Department of Mathematics, Kocaeli University, 41380, Izmit, Kocaeli, Turkey
E-mail address: aylin@kocaeli.edu.tr

(Ali Demir) Department of Mathematics, Kocaeli University, 41380, Izmit, Kocaeli, Turkey
E-mail address: ademir@kocaeli.edu.tr