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On a Fractional Operator of Adjoint Hybrid Fractional Derivative Operator

Mine Aylin Bayrak and Ali Demir

Abstract. The achievement of this paper is to propose a new kind of fractional derivative which is called New Constant Proportional Caputo (NCPC) operator and to construct the solution of time-fractional initial value problem (TFIVPs) with NCPC derivative by taking the combination of Laplace transform (LT) and Homotopy Analysis method (HAM) into account. Later, the obtained solution is compared with the solutions of TFIVPs with Caputo and Constant Proportional Caputo (CPC) derivatives. The gained results reveal that the combination of LT and HAM together form an efficient method to build the approximate results of TFIVPs in NCPC sense.

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Key words and phrases. Time fractional initial value problem, Caputo Fractional derivative, Homotopy analysis method.

1. Introduction

Last couple of decades, utilizing fractional derivatives in mathematical models of processes draws remarkable attention from many scientists, since fractional derivative plays a critical role for modeling non-local dynamical behaviors of processes arising in various important phenomena in many branches of physics, economics, engineering, and biology, by fractional differential equations [1, 2, 3, 4, 5, 6, 7]. Since modelling many systems with the memory and hereditary by using fractional derivatives and integrals is better, there is a growing need to construct new kinds of fractional operators such as Atangana-Baleanu fractional and Caputo-Fabrizio operators, to offer much more better mathematical models of systems as compared to the other fractional operators. As a result, significant developments in the definition of new fractional derivatives have been done and based on their structure and properties, various fractional derivatives such as Riemann-Liouville, Caputo, Marchaud, tempered, Hilfer, and Atangana-Baleanu etc. have been defined [8, 9, 10, 11, 12, 13, 14, 15] which allow us to model real data from diverse processes.
The main aim of the present research involves proposition of new kind of fractional operator NCPC and the construction of the truncated solution of TFIVPs including NCPC operator proposed by applying the combination of LT and HAM. Hybrid fractional operator [16, 17], utilizing the Caputo derivative, the Riemann-Liouville integral and CPC proposed by Baleanu et all. [16] together. New proportional Caputo

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operator is defined as

$$ {}_0^{NPC}D_t^{\beta}f(t)=\frac{1}{\Gamma(1-\beta)}\int_0^t\left(\begin{array}{c} M_0(\beta,\zeta)f(\zeta)+M_1(\beta,\zeta){}_0^{C}D_t^{\frac{1}{n}}f(\zeta)\\ +M_2(\beta,\zeta)f'(\zeta), \end{array}\right)(t-\zeta)^{-\beta}d\zeta \quad (1) $$

where the functions M_0, M_1 and M_2 depend on fractional order $\beta\in[0,1]$ and fulfill the following properties

$$ \lim_{\beta\to 0^+}M_0(\beta,t)=1,\ \lim_{\beta\to(\frac{1}{n})^-}M_0(\beta,t)=0; M_0(\beta,t)\neq 0, \beta\in[0,\frac{1}{n}), \quad (2) $$

$$ \lim_{\beta\to(\frac{1}{n})^+}M_0(\beta,t)=\lim_{\beta\to 1^-}M_0(\beta,t)=0; M_0(\beta,t)\neq 0, \beta\in(\frac{1}{n},1), \quad (3) $$

$$ \lim_{\beta\to 0^+}M_1(\beta,t)=0,\ \lim_{\beta\to(\frac{1}{n})^-}M_1(\beta,t)=1; M_1(\beta,t)\neq 0, \beta\in(0,\frac{1}{n}], \quad (4) $$

$$ \lim_{\beta\to(\frac{1}{n})^+}M_1(\beta,t)=1,\ \lim_{\beta\to 1^-}M_1(\beta,t)=0; M_1(\beta,t)\neq 0, \beta\in[1/n,1), \quad (5) $$

$$ \lim_{\beta\to 0^+}M_2(\beta,t)=\lim_{\beta\to(\frac{1}{n})^-}M_2(\beta,t)=0; M_2(\beta,t)\neq 0, \beta\in(0,\frac{1}{n}), \quad (6) $$

$$ \lim_{\beta\to(\frac{1}{n})^+}M_2(\beta,t)=0,\ \lim_{\beta\to 1^-}M_2(\beta,t)=1; M_2(\beta,t)\neq 0, \beta\in(\frac{1}{n},1], \quad (7) $$

and f is a Caputo differentiable function of $t\in R$.
Especially, by taking the functions M_0, M_1 and M_2 independent of t, we have the definition of NCPC operator which is formalised as follows:

$$ \begin{aligned} {}_0^{NCPC}D_t^{\beta}f(t) &= \frac{1}{\Gamma(1-\beta)}\int_0^t\left(\begin{array}{c} M_0(\beta)f(\zeta)+M_1(\beta){}_0^{C}D_t^{\frac{1}{n}}f(\zeta)\\ +M_2(\beta)f'(\zeta) \end{array}\right)(t-\zeta)^{-\beta}d\zeta \\ &= M_0(\beta){}_0^{RL}J_t^{1-\beta}f(t)+M_1(\beta){}_0^{C}D_t^{\beta-\frac{(n-1)}{n}}f(t)+M_2(\beta){}_0^{C}D_t^{\beta}f(t) \end{aligned} \quad (8) $$

where

$$ {}_0^{C}D_t^{\beta}f(t)={}_0^{RL}J_t^{1-\beta}f'(t). $$

Remark 1.1. We originally wrote this paper using the specific case

$$ \begin{aligned} M_0(\beta) &= n(\beta-1)(\beta-\frac{1}{n}),\\ M_1(\beta) &= -\frac{n^2}{n-1}(\beta-1)(\beta-\frac{1}{n}),\\ M_2(\beta) &= \frac{n}{n-1}\beta(\beta-\frac{1}{n}) \end{aligned} \quad (9) $$

which is afforded special attention in [16].
This operator may arise in mathematical modelling of various processes such as control theory, medicine, etc., since it can be considered as expansion of different fractional derivatives.
In order to establish the numerical solution of TFIVPs with NCPC derivative, first LT is applied to reduced the time fractional diffusion equation into a simpler equation and then HAM is utilized to establish the numerical solution.
This paper is organised as: In Section 2, Fundamental concepts in fractional calculus

are presented. Section 3 covers application of LT and HAM to the (TFIVPs). Illustrative examples of the proposed fractional derivative with mathematical problems are given in Section 4. The last section includes the concluded results.

2. Preliminaries

Definition 2.1. [18]

$$\begin{aligned} {}_0^{RL}J_t^{\beta} f(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} f(\zeta) d\zeta, \beta > 0, t > 0, \\ J^0 f(t) &= f(t). \end{aligned} \tag{10}$$

where $f \in C_\mu, \mu \geq -1$ and β denotes to order of the operator, is called the Riemann-Liouville fractional integral operator.

Definition 2.2. [18] The fractional derivative in Caputo sense is defined as

$${}_0^{C}D_t^{\beta} f(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^{(n)}(\zeta)}{(t-\zeta)^{\beta+1-n}} d\zeta, \ n-1 < \beta \leq n, \\ \\ \frac{d^n}{dt^n} f(t) \ , \beta = n \in N \end{cases}$$

where $\beta > 0, t \in R$.

Theorem 2.3. [16] The operator NCPC for a differentiable function $f(t)$ fulfilling the conditions $f, f' \in L_1$ locally has the LT as follows:

$$\begin{aligned} \mathcal{L}\Big\{ {}_0^{NCPC}D_t^{\beta} f(t) \Big\} =& \Big(\frac{M_0(\beta)}{s} + \frac{M_1(\beta)}{s^{\frac{1}{n}}} + M_2(\beta) \Big) s^{\beta} F(s) \\ & - \Big(\frac{M_1(\beta)}{s^{\frac{(n+1)}{n}}} + \frac{M_2(\beta)}{s} \Big) s^{\beta} f(0) \end{aligned} \tag{11}$$

where $F(s)$ denotes the LT of $f(t)$.

3. Analysis of LT and HAM

In this section, main steps of the method presented in this study are established on the following FPDEs.

$$(D^{\beta}w)(\xi,\tau) + Lw(\xi,\tau) + Nw(\xi,\tau) = f(\xi,\tau), 0 < \beta \leq 1, \tag{12}$$

where N, L and f represent the nonlinear operator, linear operator and source function, respectively and $(D^{\beta}w)$ is the fractional differential operator.
Employing the LT to (12) with the CPC fractional derivative we have

$$\begin{aligned} ({}^{C}D^{\beta}w)(\xi,\tau) + \frac{M_0(\beta)}{M_2(\beta)} \mathcal{L}^{-1}\{ \frac{1}{s^{1-\beta}} \mathcal{L}\{w(\xi,\tau)\}\} + \frac{M_1(\beta)}{M_2(\beta)} ({}^{C}D^{\beta - \frac{(n-1)}{n}} w)(\xi,\tau) \\ - \frac{1}{M_2(\beta)} [L(w(\xi,\tau)) + N(w(\xi,\tau)) - f(\xi,\tau)] = 0, \end{aligned} \tag{13}$$

where $({}^C D^\beta w)\xi,\tau$ is the β^{th}-order fractional derivative in Caputo sense. The deformation equation of order 0 is constructed by HAM [19, 20, 21],

$$\begin{aligned}(1-p)L[\phi(\xi,\tau;p)-w_0(\xi,\tau)] &= p\hbar H(\xi,\tau)\Big[{}^C D^\beta \phi(\xi,\tau;p)\\ &+ \frac{M_0(\beta)}{M_2(\beta)}\mathcal{L}^{-1}\{\frac{1}{s^{1-\beta}}\mathcal{L}\{\phi(\xi,\tau;p)\}\} + \frac{M_1(\beta)}{M_2(\beta)}{}^C D^{\beta-\frac{(n-1)}{n}}\phi(\xi,\tau;p)\\ &- \frac{1}{M_2(\beta)}[L(\phi(\xi,\tau;p)) + N(\phi(\xi,\tau;p)) - f(\xi,\tau)]\Big],\end{aligned} \tag{14}$$

where $\phi(\xi,\tau;p)$ with parameter $p\in[0,1]$ is defined on R, $\hbar\neq 0$ and $w_0(\xi,\tau)$ represent an parameter and an starting point, respectively. Furthermore, we take $H(\xi,\tau)=1$ in the computations. It follows from Eq. (15) that

$$\phi(\xi,\tau;0)=w_0(\xi,\tau), \phi(\xi,\tau;1)=w(\xi,\tau). \tag{15}$$

Thus as p tempts to 1, the initial guess $w_0(\xi,\tau)$ tempts to $w(\xi,\tau)$. The function $\phi(\xi,\tau;p)$ can be rewritten in the series form as

$$\phi(\xi,\tau;p)=w_0(\xi,\tau)+\sum_{m=1}^{\infty} w_m(\xi,\tau)p^m, \tag{16}$$

where

$$w_m(\xi,\tau)=\frac{1}{m!}\frac{\partial^m\phi(\xi,\tau;p)}{\partial p^m}\Big|_{p=0}. \tag{17}$$

The correct choices lead to the following convergent series,

$$w(\xi,\tau)=w_0(\xi,\tau)+\sum_{m=1}^{\infty} w_m(\xi,\tau). \tag{18}$$

Hence for $H(\xi,\tau)=1$,

$$L[w_m(\xi,\tau)-\kappa_m w_{m-1}(\xi,\tau)]=\hbar R_m(\vec{w}_{m-1}) \tag{19}$$

with the requirements

$$w_m^{(k)}(\xi,0)=0, k=0,1,2,...,m-1, \tag{20}$$

where

$$\begin{aligned}R_m(\vec{w}_{m-1}) =& \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}\Big[{}^C D^\beta \phi(\xi,\tau;p)\\ &+ \frac{M_0(\beta)}{M_2(\beta)}\mathcal{L}^{-1}\{\frac{1}{s^{1-\beta}}\mathcal{L}\{\phi(\xi,\tau;p)\}\} + \frac{M_1(\beta)}{M_2(\beta)}{}^C D^{\beta-\frac{(n-1)}{n}}\phi(\xi,\tau;p)\\ &- \frac{1}{M_2(\beta)}[L(\phi(\xi,\tau;p)) + N(\phi(\xi,\tau;p)) - f(\xi,\tau)]\Big]\Big|_{p=0},\end{aligned} \tag{21}$$

and

$$\kappa_m = \begin{cases} 0, m\leq 1,\\ 1, m>1.\end{cases} \tag{22}$$

Employing the operator J^β in Eq. (20) leads to

$$w_m(\xi,\tau)=\kappa_m w_{m-1}(\xi,\tau)-\kappa_m\sum_{i=0}^{n-1} w_{m-1}^{(i)}(\xi,0^+)\frac{\tau^i}{i!}+\hbar J^\beta[R_m(\vec{w}_{m-1})]. \tag{23}$$

In this manner, we establish the approximate solution $w(\xi,\tau)$ in terms of $w_m(\xi,\tau)$ as follows

$$w(\xi,\tau)=\sum_{m=0}^{M} w_m(\xi,\tau). \tag{24}$$

Note that as M tends to infinity, the exact analytical solution of Eq.(14) is obtained.

4. Numerical examples

In this work, we consider the following time fractional diffusion equation

$${}_0^{NCPC}D_\tau^\beta w(\xi,\tau)=\lambda w_{\xi\xi}(\xi,\tau)-(F(\xi)w(\xi,\tau))_\xi, 0<\beta\le 1, \xi>0, \tau>0 \tag{25}$$

with initial condition

$$w(\xi,0)=\varphi(\xi). \tag{26}$$

Here, λ is a positive constant, $F(\xi)$ is the external force, $w(\xi,\tau)$ represents the probability density function of finding a particle at the point ξ in the time τ.

Example 1. Taking $F(\xi)=-\xi, \lambda=1$ and choosing $\varphi(\xi)=1$, we get the following initial value problem:

$${}_0^{NCPC}D_t^\beta w(\xi,\tau)=w_{\xi\xi}(\xi,\tau)+w(\xi,\tau)+\xi(w(\xi,\tau))_\xi \tag{27}$$

$$w(\xi,0)=1. \tag{28}$$

Thus, the exact solution of the above problem with Caputo derivative is given by

$$w(\xi,\tau)=E_\beta(t^\beta). \tag{29}$$

The solution for the deformation equations of the order m is constructed for $n=2$ as follows:

$$w_m(\xi,\tau)=(\kappa_m+\hbar)(w_{m-1}(\xi,\tau)-w_{m-1}(\xi,0))+\hbar J^\beta[R_m(\overrightarrow{w_{m-1}})], \tag{30}$$

where

$$\begin{aligned}R_m(\overrightarrow{w_{m-1}}) =& {}^C D_\tau^\beta w_{m-1}(\xi,\tau)+\frac{M_0(\beta)}{M_2(\beta)}\mathcal{L}^{-1}\{\frac{1}{s^{1-\beta}}\mathcal{L}\{w_{m-1}(\xi,\tau)\}\}\\ &+\frac{M_1(\beta)}{M_2(\beta)}{}^C D_\tau^{\beta-\frac{1}{2}} w_{m-1}(\xi,\tau)\\ &-\frac{1}{M_2(\beta)}[(w_{m-1})_{\xi\xi}(\xi,\tau)-w_{m-1}(\xi,\tau)-\xi(w_{m-1})_\xi(\xi,\tau)].\end{aligned} \tag{31}$$

As a result, we obtain:

$w_0(\xi,\tau)=1,$

$w_1(\xi,\tau)=\hbar(\frac{M_0(\beta)}{M_2(\beta)}\tau+\frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}-\frac{1}{M_2(\beta)}\frac{\tau^\beta}{\Gamma(1+\beta)}),$

$w_2(\xi,\tau)=(\hbar+\hbar^2)(\frac{M_0(\beta)}{M_2(\beta)}\tau+\frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}-\frac{1}{M_2(\beta)}\frac{\tau^\beta}{\Gamma(1+\beta)})+\frac{M_0(\beta)}{M_2(\beta)}\hbar^2(\frac{M_0(\beta)}{M_2(\beta)}\frac{\tau^2}{\Gamma(3)}$
$+\frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}-\frac{1}{M_2(\beta)}\frac{\tau^{\beta+1}}{\Gamma(2+\beta)})+\frac{M_1(\beta)}{M_2(\beta)}\hbar^2(\frac{M_0(\beta)}{M_2(\beta)}\frac{\tau^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}+\frac{M_1(\beta)}{M_2(\beta)}\tau-\frac{1}{M_2(\beta)}\frac{\tau^{\beta+\frac{1}{2}}}{\Gamma(\beta+\frac{3}{2})})$
$-\frac{1}{M_2(\beta)}\hbar^2(\frac{M_0(\beta)}{M_2(\beta)}\frac{\tau^{\beta+1}}{\Gamma(\beta+2)}+\frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\beta+\frac{1}{2}}}{\Gamma(\beta+\frac{3}{2})}-\frac{1}{M_2(\beta)}\frac{\tau^{2\beta}}{\Gamma(1+2\beta)}),$

and so on.

The figures of numerical solution with 3-terms are given in Figs. 1-6. Table 1 presents

TABLE 1. HAM solutions ($\hbar = -1$) with exact solution at $x = 0.8$ of Example 1 with different fractional derivatives and for various α.

		$\alpha = 0.95$			$\alpha = 0.9$			$\alpha = 0.85$		
t	Exact	Caputo	CPC	NCPC	Caputo	CPC	NCPC	Caputo	CPC	NCPC
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.25	1.28125	1.31273	1.31441	1.22167	1.34778	1.35541	1.15930	1.38682	1.40640	1.09454
0.5	1.62500	1.67489	1.67756	1.55477	1.72849	1.74036	1.47948	1.78595	1.81583	1.39855
0.75	2.03125	2.09329	2.09660	1.96030	2.15797	2.17242	1.88201	2.22509	2.26070	1.79486
1	2.50000	2.56777	2.57135	2.43690	2.63624	2.65162	2.36502	2.70490	2.74191	2.28217

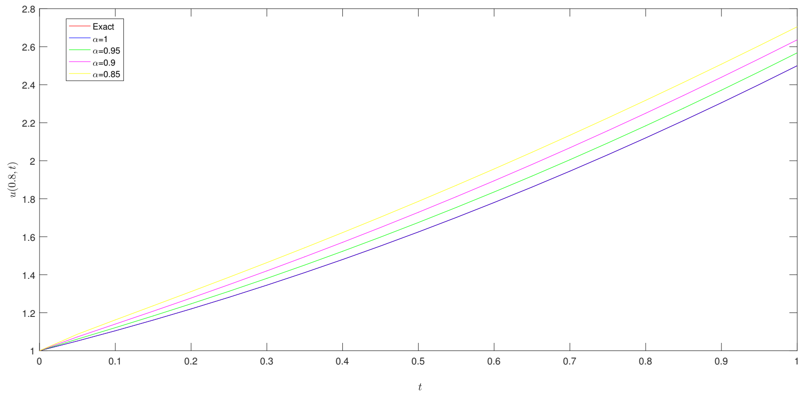

FIGURE 1. The graphics of approximate solutions for Ex.1 with Caputo derivative for various α.

the values of approximate solutions.

Example 2. Taking $F(\xi) = -\xi, \lambda = 1$ and choosing $\varphi(\xi) = \xi$, we get the following initial value problem:

$$ {}_{0}^{NCPC}D_t^{\beta} w(\xi,\tau) = w_{\xi\xi}(\xi,\tau) + w(\xi,\tau) + \xi(w(\xi,\tau))_{\xi} \tag{32} $$

$$ w(\xi, 0) = \xi. \tag{33} $$

Thus, the exact solution of the above problem with Caputo derivative is given by

$$ w(\xi,\tau) = \xi E_{\beta}(2\tau^{\beta}). \tag{34} $$

The solution for the deformation equations of the order m is constructed as follows:

$$ w_m(\xi,\tau) = (\kappa_m + \hbar)(w_{m-1}(\xi,\tau) - w_{m-1}(\xi,0)) + \hbar J^{\beta}[R_m(\vec{w_{m-1}})], \tag{35} $$

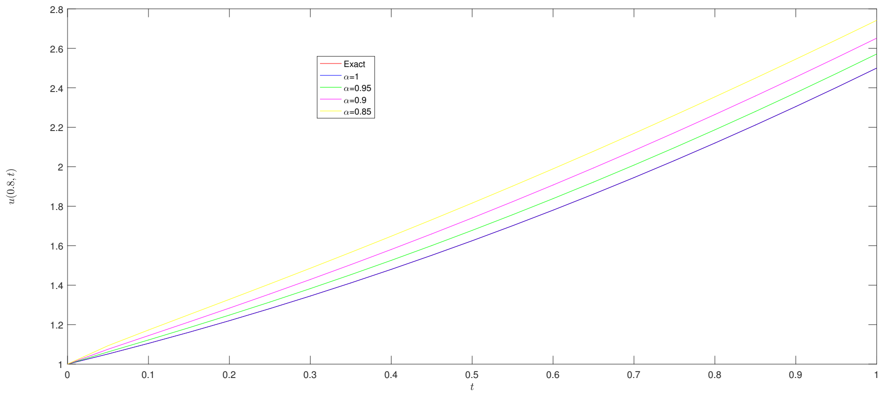

FIGURE 2. The graphics of approximate solutions for Ex.1 with CPC derivative for various α .

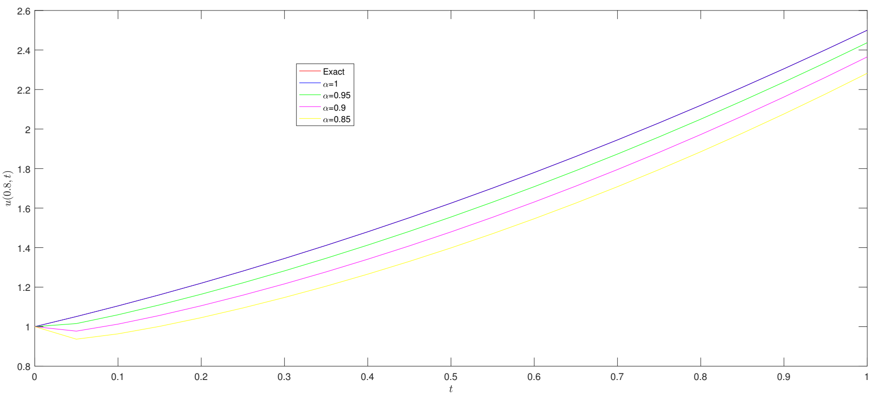

FIGURE 3. The graphics of approximate solutions for Ex.1 with NCPC derivative for various α and $n = 2$.

where

$$\begin{aligned} R_m(\overrightarrow{w_{m-1}}) = {}^C D_\tau^\beta w_{m-1}(\xi,\tau) + \frac{M_0(\beta)}{M_2(\beta)} \mathcal{L}^{-1}\{\frac{1}{s^{1-\beta}} \mathcal{L}\{w_{m-1}(\xi,\tau)\}\} \\ + \frac{M_1(\beta)}{M_2(\beta)} {}^C D_\tau^{\beta-\frac{1}{2}} w_{m-1}(\xi,\tau) \\ - \frac{1}{M_2(\beta)} [(w_{m-1})_{\xi\xi}(\xi,\tau) - w_{m-1}(\xi,\tau) - \xi(w_{m-1})_\xi(\xi,\tau)]. \end{aligned} \tag{36}$$

The first three deformation equations are established for $n = 2$ as follows:
$w_0(\xi,\tau) = \xi$,
$w_1(\xi,\tau) = \hbar\xi(\frac{M_0(\beta)}{M_2(\beta)}\tau + \frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} - \frac{1}{M_2(\beta)}\frac{2\tau^\beta}{\Gamma(1+\beta)})$,

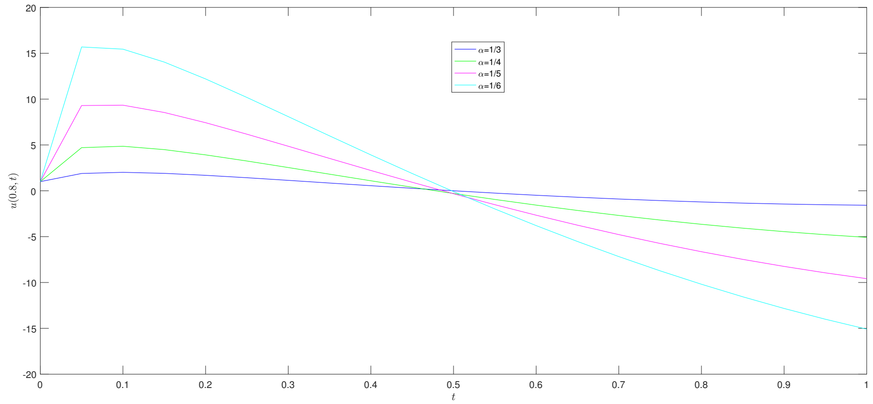

FIGURE 4. The graphics of approximate solutions for Ex.1 with NCPC derivative for various $\alpha < \frac{1}{2}$.

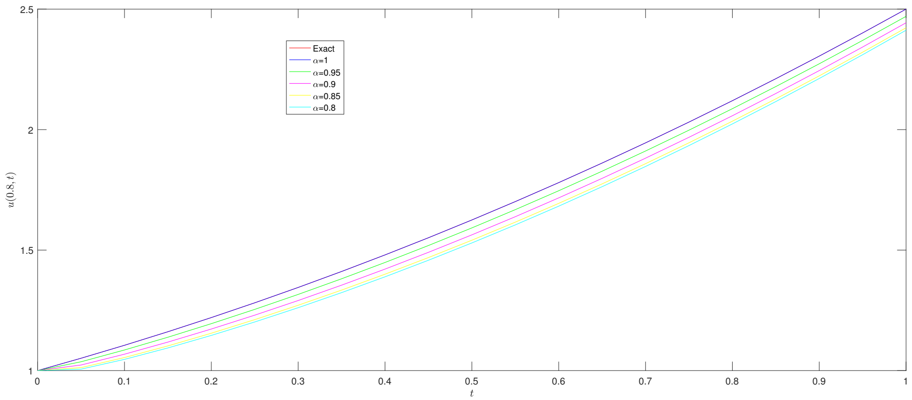

FIGURE 5. The graphics of approximate solutions for Ex.1 with NCPC derivative for various α and $n = 3$.

$$\begin{aligned}
w_2(\xi,\tau) &= (\hbar+\hbar^2)\xi\Big(\frac{M_0(\beta)}{M_2(\beta)}\tau + \frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} - \frac{1}{M_2(\beta)}\frac{2\tau^{\beta}}{\Gamma(1+\beta)}\Big) + \frac{M_0(\beta)}{M_2(\beta)}\hbar^2\xi\Big(\frac{M_0(\beta)}{M_2(\beta)}\frac{\tau^2}{\Gamma(3)} \\
&+ \frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} - \frac{1}{M_2(\beta)}\frac{2\tau^{\beta+1}}{\Gamma(2+\beta)}\Big) + \frac{M_1(\beta)}{M_2(\beta)}\hbar^2\xi\Big(\frac{M_0(\beta)}{M_2(\beta)}\frac{\tau^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{M_1(\beta)}{M_2(\beta)}\tau - \frac{1}{M_2(\beta)}\frac{2\tau^{\beta+\frac{1}{2}}}{\Gamma(\beta+\frac{3}{2})}\Big) \\
&- \frac{2}{M_2(\beta)}\hbar^2\xi\Big(\frac{M_0(\beta)}{M_2(\beta)}\frac{\tau^{\beta+1}}{\Gamma(\beta+2)} + \frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\beta+\frac{1}{2}}}{\Gamma(\beta+\frac{3}{2})} - \frac{1}{M_2(\beta)}\frac{2\tau^{2\beta}}{\Gamma(1+2\beta)}\Big),
\end{aligned}$$

and so on.

The figures of numerical solution with 3-terms are given in Figs. 7-12. Table 2 presents the values of approximate solutions.

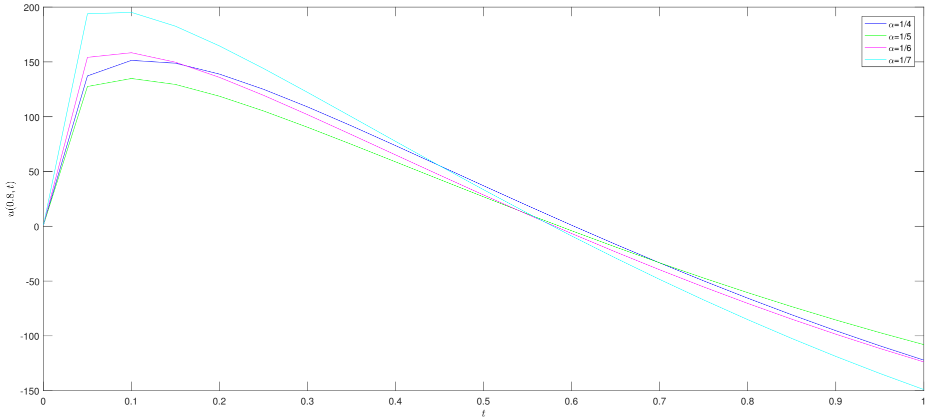

FIGURE 6. The graphics of approximate solutions for Ex.1 with NCPC derivative for various $\alpha < \frac{1}{3}$.

TABLE 2. HAM solutions ($\hbar = -1$) with exact solution at $x = 0.8$ of Example 2 for various α.

		$\alpha = 0.95$			$\alpha = 0.9$			$\alpha = 0.85$		
t	Exact	Caputo	CPC	NCPC	Caputo	CPC	NCPC	Caputo	CPC	NCPC
0	0.80000	0.80000	0.80000	0.80000	0.80000	0.80000	0.80000	0.80000	0.80000	0.80000
0.25	1.30000	1.36323	1.38317	1.32224	1.43516	1.48773	1.36821	1.51703	1.62057	1.45504
0.5	2.00000	2.11443	2.16482	2.11430	2.23965	2.36614	2.29297	2.37633	2.61406	2.58212
0.75	2.90000	3.05616	3.14717	3.15091	3.22138	3.44178	3.52297	3.39530	3.79512	4.10052
1	4.00000	4.18402	4.32536	4.42875	4.37236	4.70479	5.04943	4.56365	5.14942	5.99169

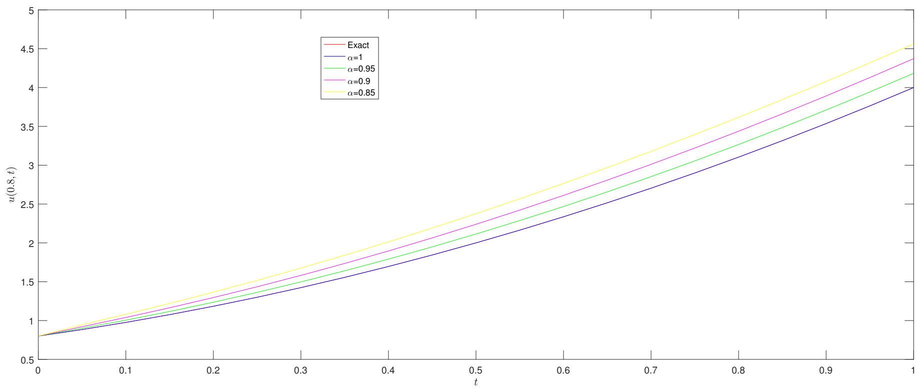

FIGURE 7. The graphics of approximate solutions for Ex.2 with Caputo derivative for various α.

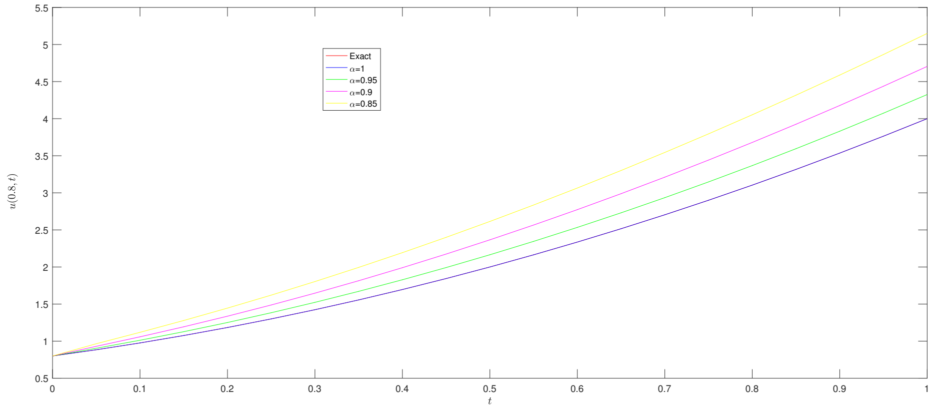

FIGURE 8. The graphics of approximate solutions for Ex.2 with CPC derivative for various α .

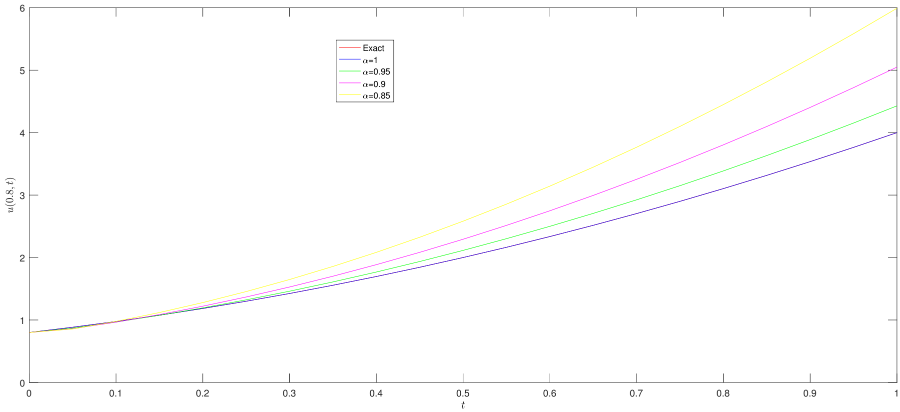

FIGURE 9. The graphics of approximate solutions for Ex.2 with NCPC derivative for various α and $n = 2$.

Example 3. Taking $F(\xi) = -\xi, \lambda = 1$ and choosing $\varphi(\xi) = \xi^2$, we get the following initial value problem:

$$ {}_0^{NCPC}D_t^{\beta} w(\xi,\tau) = w_{\xi\xi}(\xi,\tau) + w(\xi,\tau) + \xi(w(\xi,\tau))_\xi \tag{37} $$

$$ w(\xi,0) = \xi^2. \tag{38} $$

Thus, the exact solution of the above problem with Caputo derivative is given by

$$ w(\xi,\tau) = E_\beta(r\tau^\beta) \tag{39} $$

where $r = 3^k(1+\xi^2) - 1, k = 0,1,2,....$ The solution for the deformation equations of the order m is constructed as follows:

$$ w_m(\xi,\tau) = (\kappa_m + \hbar)(w_{m-1}(\xi,\tau) - w_{m-1}(\xi,0)) + \hbar J^\beta[R_m(\overrightarrow{w_{m-1}})], \tag{40} $$

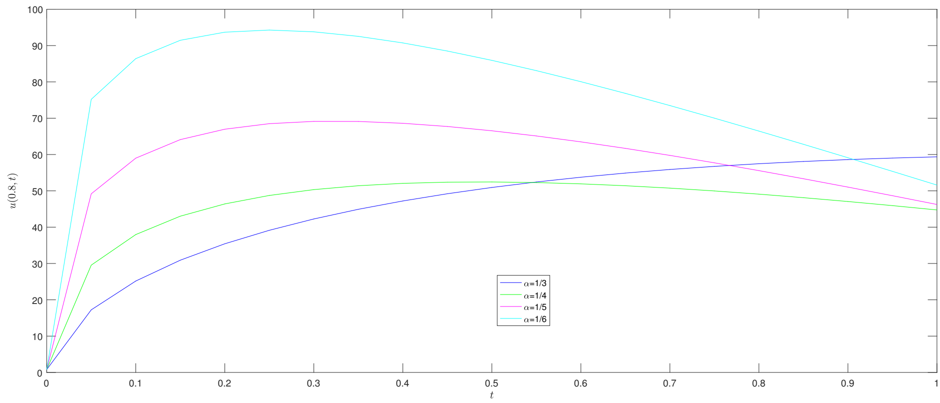

FIGURE 10. The graphics of approximate solutions for Ex.2 with NCPC derivative for various $\alpha < \frac{1}{2}$.

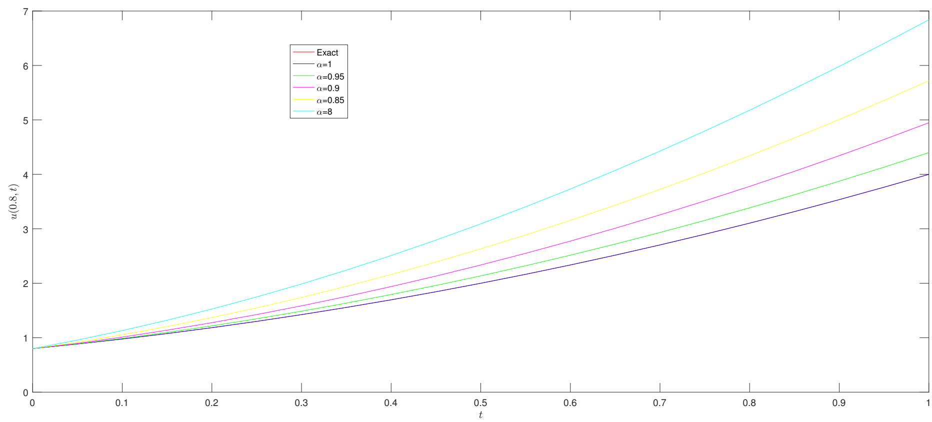

FIGURE 11. The graphics of approximate solutions for Ex.2 with NCPC derivative for various α and $n = 3$.

where

$$\begin{aligned} R_m(\overrightarrow{w_{m-1}}) = {}^C D_\tau^\beta w_{m-1}(\xi,\tau) &+ \frac{M_0(\beta)}{M_2(\beta)} \mathcal{L}^{-1}\{\frac{1}{s^{1-\beta}} \mathcal{L}\{w_{m-1}(\xi,\tau)\}\} \\ &+ \frac{M_1(\beta)}{M_2(\beta)} {}^C D_\tau^{\beta-\frac{1}{2}} w_{m-1}(\xi,\tau) \\ &- \frac{1}{M_2(\beta)}[(w_{m-1})_{\xi\xi}(\xi,\tau) - w_{m-1}(\xi,\tau) - \xi(w_{m-1})_\xi(\xi,\tau)]. \end{aligned} \tag{41}$$

The first three deformation equations are constructed for $n = 2$ as follows:
$w_0(\xi,\tau) = \xi^2$,
$w_1(\xi,\tau) = \hbar(\frac{M_0(\beta)}{M_2(\beta)}\xi^2\tau + \frac{M_1(\beta)}{M_2(\beta)}\xi^2\frac{\tau^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} - \frac{1}{M_2(\beta)}(2+3\xi^2)\frac{\tau^\beta}{\Gamma(1+\beta)})$,

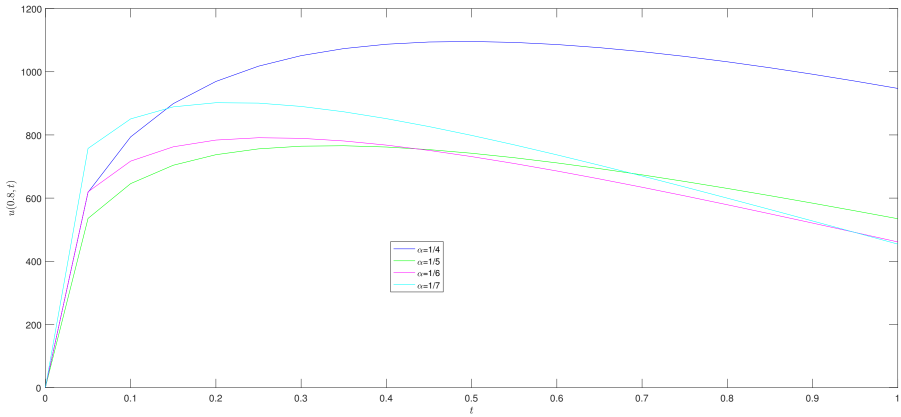

FIGURE 12. The graphics of approximate solutions for Ex.2 with NCPC derivative for various $\alpha < \frac{1}{3}$.

TABLE 3. HAM solutions ($\hbar = -1$) with exact solution at $x = 0.8$ of Example 3 for various α.

		$\alpha = 0.95$			$\alpha = 0.9$			$\alpha = 0.85$		
t	Exact	Caputo	CPC	NCPC	Caputo	CPC	NCPC	Caputo	CPC	NCPC
0	0.64000	0.64000	0.64000	0.64000	0.64000	0.64000	0.64000	0.64000	0.64000	0.64000
0.25	2.05000	2.25251	2.34374	2.37228	2.48735	2.72062	2.86035	2.75979	3.20986	3.64430
0.5	4.32000	4.72840	4.97990	5.22195	5.18121	5.80028	6.57917	5.68159	6.82843	8.74057
0.75	7.45000	8.04309	8.51975	9.16501	8.67631	9.81521	11.72432	9.34859	11.39391	15.75163
1	11.44000	12.17050	12.93442	14.17983	12.92346	14.70474	18.23305	13.69343	16.81297	24.53891

$$w_2(\xi,\tau) = (\hbar+\hbar^2)\left(\frac{M_0(\beta)}{M_2(\beta)}\xi^2\tau + \frac{M_1(\beta)}{M_2(\beta)}\xi^2\frac{\tau^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} - \frac{1}{M_2(\beta)}(2+3\xi^2)\frac{\tau^{\beta}}{\Gamma(1+\beta)}\right)$$
$$+\frac{M_0(\beta)}{M_2(\beta)}\hbar^2\left(\frac{M_0(\beta)}{M_2(\beta)}\xi^2\frac{\tau^2}{\Gamma(3)} + \frac{M_1(\beta)}{M_2(\beta)}\xi^2\frac{\tau^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} - \frac{1}{M_2(\beta)}(2+3\xi^2)\frac{\tau^{\beta+1}}{\Gamma(2+\beta)}\right) + \frac{M_1(\beta)}{M_2(\beta)}\hbar^2\left(\frac{M_0(\beta)}{M_2(\beta)}\xi^2\frac{\tau^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{M_1(\beta)}{M_2(\beta)}\xi^2\tau - \frac{1}{M_2(\beta)}(2+3\xi^2)\frac{\tau^{\beta+\frac{1}{2}}}{\Gamma(\beta+\frac{3}{2})}\right) - \frac{2}{M_2(\beta)}\hbar^2\left(\frac{M_0(\beta)}{M_2(\beta)}\xi^2\frac{\tau^{\beta+1}}{\Gamma(\beta+2)} + \frac{M_1(\beta)}{M_2(\beta)}\xi^2\frac{\tau^{\beta+\frac{1}{2}}}{\Gamma(\beta+\frac{3}{2})} - \frac{1}{M_2(\beta)}(2+3\xi^2)\frac{\tau^{2\beta}}{\Gamma(1+2\beta)} + 2\frac{M_0(\beta)}{M_2(\beta)}(1+\xi^2)\frac{\tau^{\beta+1}}{\Gamma(2+\beta)} + 2\frac{M_2(\beta)}{M_2(\beta)}(1+\xi^2)\frac{\tau^{\beta+\frac{1}{2}}}{\Gamma(\beta+\frac{3}{2})}\right.$$
$$\left. + \frac{6}{M_2(\beta)}(1+\xi^2)\frac{\tau^{2\beta}}{\Gamma(1+2\beta)}\right),$$

and so on.
The figures of numerical solution with 3-terms are given in Figs. 13-18. Table 3 presents the values of approximate solutions.

Example 4. Taking $F(\xi) = e^{-\xi}, \lambda = 1$ and choosing $\varphi(\xi) = e^{\xi}$, we get the following initial value problem:

$$ {}_{0}^{NCPC}D_t^{\beta}w(\xi,\tau) = w_{\xi\xi}(\xi,\tau) + e^{-\xi}w(\xi,\tau) - e^{-\xi}(w(\xi,\tau))_{\xi} \tag{42}$$
$$w(\xi,0) = e^{\xi}. \tag{43}$$

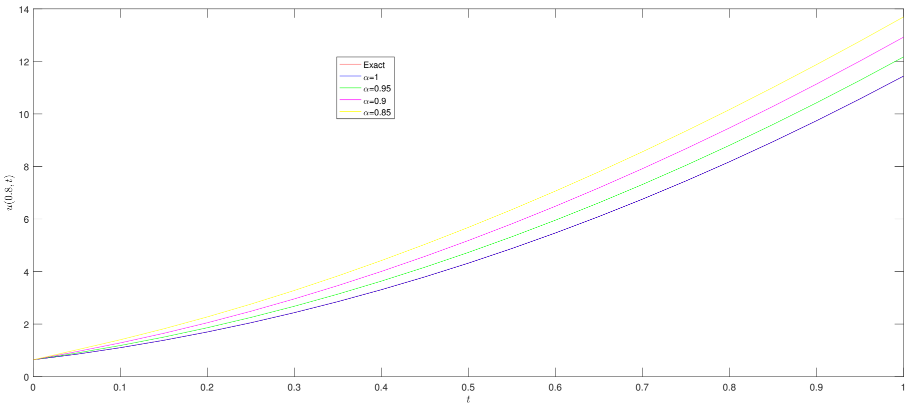

FIGURE 13. The graphics of approximate solutions for Ex.3 with Caputo derivative for various α.

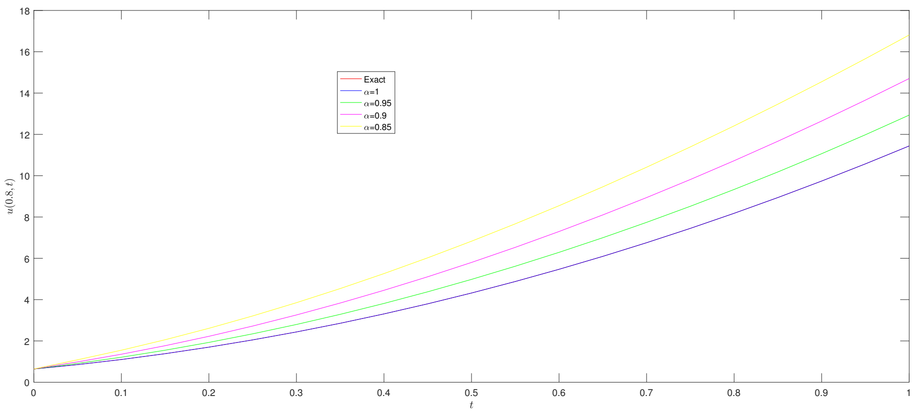

FIGURE 14. The graphics of approximate solutions for Ex.3 with CPC derivative for various α.

Thus, the exact solution of the above problem with Caputo derivative is given by

$$w(\xi,\tau) = e^{\xi} E_{\beta}(\tau^{\beta}). \tag{44}$$

The solution for the deformation equations of the order m is constructed as follows:

$$w_m(\xi,\tau) = (\kappa_m + \hbar)(w_{m-1}(\xi,\tau) - w_{m-1}(\xi,0)) + \hbar J^{\beta}[R_m(\overrightarrow{w_{m-1}})], \tag{45}$$

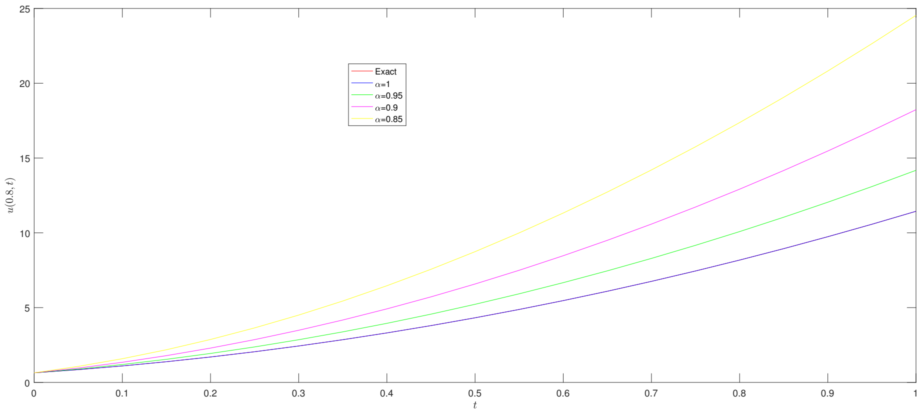

FIGURE 15. The graphics of approximate solutions for Ex.3 with NCPC derivative for various α and $n = 2$.

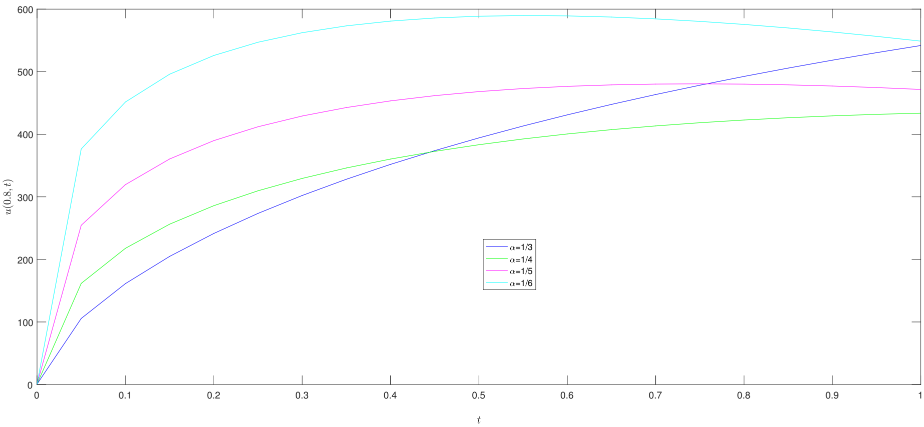

FIGURE 16. The graphics of approximate solutions for Ex.3 with NCPC derivative for various $\alpha < \frac{1}{2}$.

where

$$\begin{aligned} R_m(\vec{w_{m-1}}) =& {}^C D_\tau^\beta w_{m-1}(\xi,\tau) + \frac{M_0(\beta)}{M_2(\beta)}\mathcal{L}^{-1}\{\frac{1}{s^{1-\beta}}\mathcal{L}\{w_{m-1}(\xi,\tau)\}\} \\ &+ \frac{M_1(\beta)}{M_2(\beta)} {}^C D_\tau^{\beta-\frac{1}{2}} w_{m-1}(\xi,\tau) \\ &- \frac{1}{M_2(\beta)}[(w_{m-1})_{\xi\xi}(\xi,\tau) + e^{-\xi}w_{m-1}(\xi,\tau) - e^{-\xi}(w_{m-1}(\xi,\tau))_\xi]. \end{aligned} \tag{46}$$

The first three deformation equations are constructed for $n = 2$ as follows:
$w_0(\xi,\tau) = e^\xi$,
$w_1(\xi,\tau) = \hbar e^\xi(\frac{M_0(\beta)}{M_2(\beta)}\tau + \frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} - \frac{1}{M_2(\beta)}\frac{\tau^\beta}{\Gamma(1+\beta)})$,

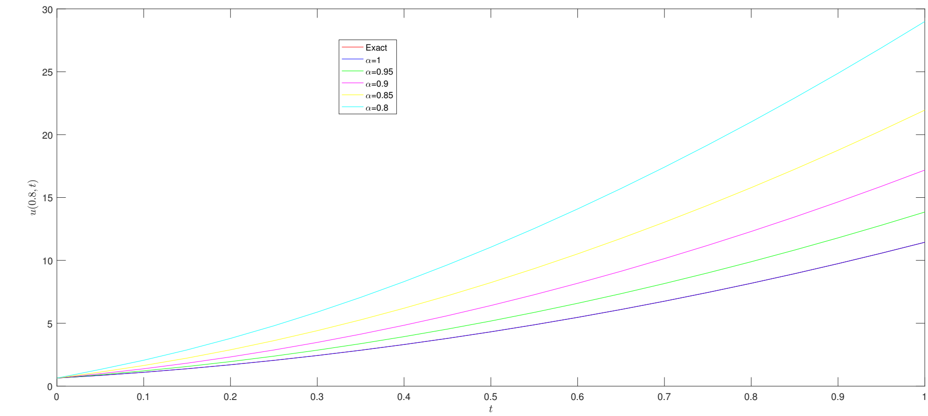

FIGURE 17. The graphics of approximate solutions for Ex.3 with NCPC derivative for various α and $n = 3$.

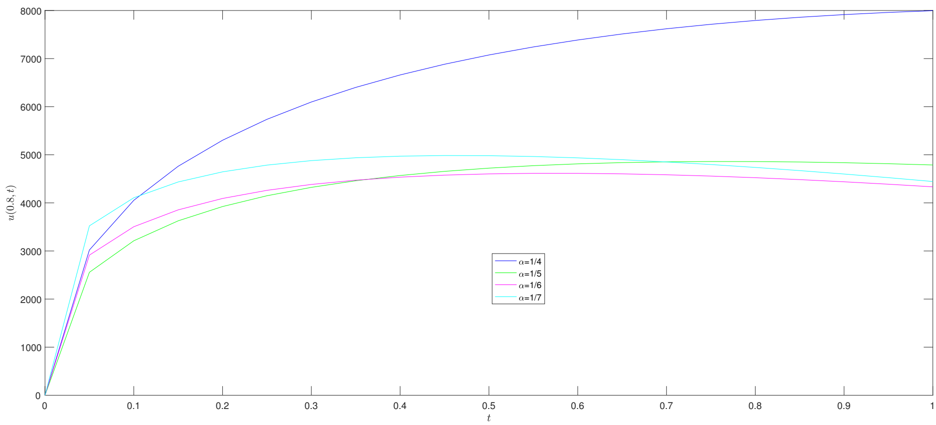

FIGURE 18. The graphics of approximate solutions for Ex.3 with NCPC derivative for various $\alpha < \frac{1}{3}$.

$$w_2(\xi,\tau) = (\hbar+\hbar^2)e^{\xi}\big(\frac{M_0(\beta)}{M_2(\beta)}\tau + \frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} - \frac{1}{M_2(\beta)}\frac{\tau^{\beta}}{\Gamma(1+\beta)}\big) + \frac{M_0(\beta)}{M_2(\beta)}\hbar^2 e^{\xi}\big(\frac{M_0(\beta)}{M_2(\beta)}\frac{\tau^2}{\Gamma(3)}$$
$$+ \frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} - \frac{1}{M_2(\beta)}\frac{\tau^{\beta+1}}{\Gamma(2+\beta)}\big) + \frac{M_1(\beta)}{M_2(\beta)}\hbar^2 e^{\xi}\big(\frac{M_0(\beta)}{M_2(\beta)}\frac{\tau^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{M_1(\beta)}{M_2(\beta)}\tau - \frac{1}{M_2(\beta)}\frac{\tau^{\beta+\frac{1}{2}}}{\Gamma(\beta+\frac{3}{2})}\big)$$
$$- \frac{1}{M_2(\beta)}\hbar^2 e^{\xi}\big(\frac{M_0(\beta)}{M_2(\beta)}\frac{\tau^{\beta+1}}{\Gamma(\beta+2)} + \frac{M_1(\beta)}{M_2(\beta)}\frac{\tau^{\beta+\frac{1}{2}}}{\Gamma(\beta+\frac{3}{2})} - \frac{1}{M_2(\beta)}\frac{\tau^{2\beta}}{\Gamma(1+2\beta)}\big),$$

and so on.
The figures of numerical solution with 3-terms are given in Figs. 19-24. Table 4 presents the values of approximate solutions.

TABLE 4. HAM solutions ($\hbar = -1$) with exact solution at $x = 0.8$ of Example 4 for various α .

		$\alpha = 0.95$			$\alpha = 0.9$			$\alpha = 0.85$		
t	Exact	Caputo	CPC	NCPC	Caputo	CPC	NCPC	Caputo	CPC	NCPC
0	2.22554	2.22554	2.22554	2.22554	2.22554	2.22554	2.22554	2.22554	2.22554	2.22554
0.25	2.85147	2.92154	2.92489	2.71887	2.99955	3.01554	2.58007	3.08642	3.12818	2.43593
0.5	3.61650	3.72754	3.73197	3.46021	3.84682	3.86962	3.29264	3.97470	4.03462	3.11254
0.75	4.52063	4.65871	4.51975	4.36273	4.80265	4.82699	4.18848	4.95203	5.01735	3.99454
1	5.56385	5.71468	5.93442	5.42343	5.86706	5.88778	5.26346	6.01986	6.07850	5.07905

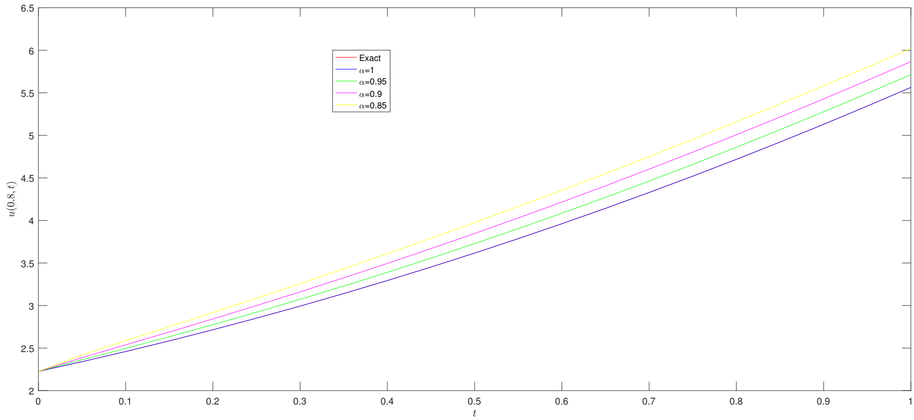

FIGURE 19. The graphics of approximate solutions for Ex.4 with Caputo derivative for various α.

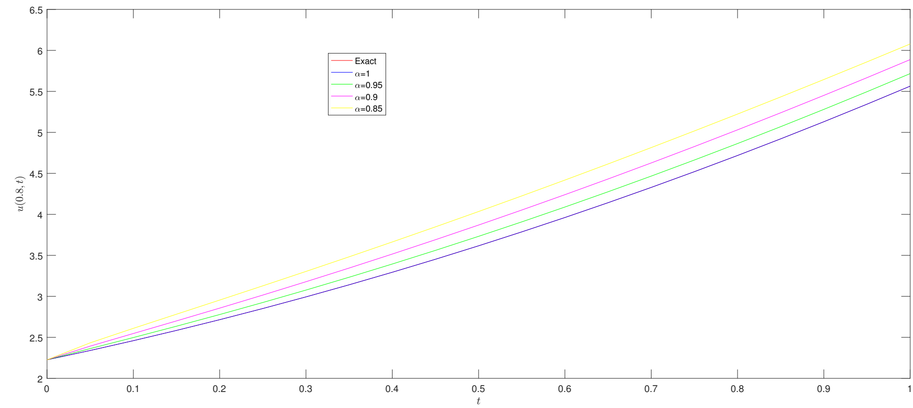

FIGURE 20. The graphics of approximate solutions for Ex.4 with CPC derivative for various α.

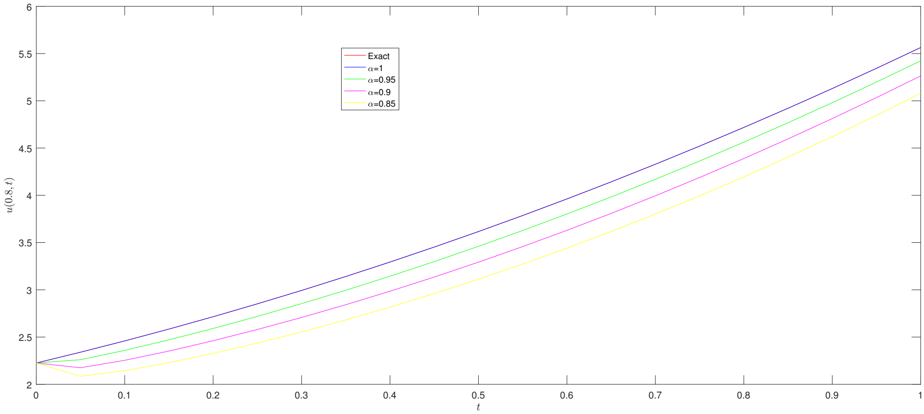

FIGURE 21. The graphics of approximate solutions for Ex.4 with NCPC derivative for various α and $n = 2$..

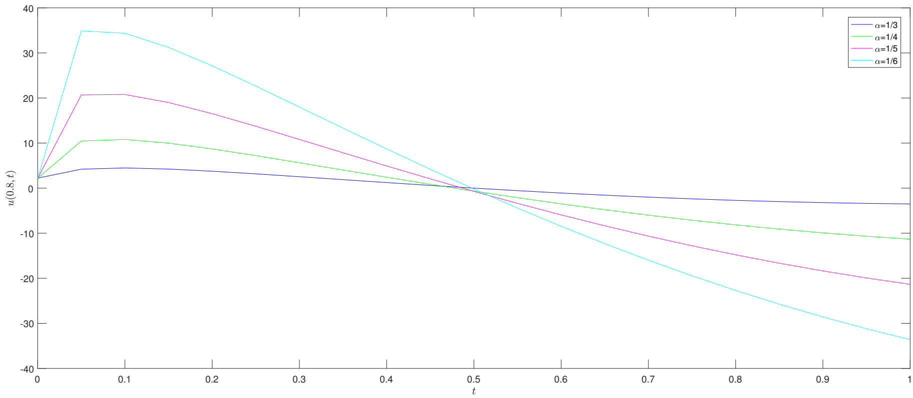

FIGURE 22. The graphics of approximate solutions for Ex.4 with NCPC derivative for various $\alpha < \frac{1}{2}$.

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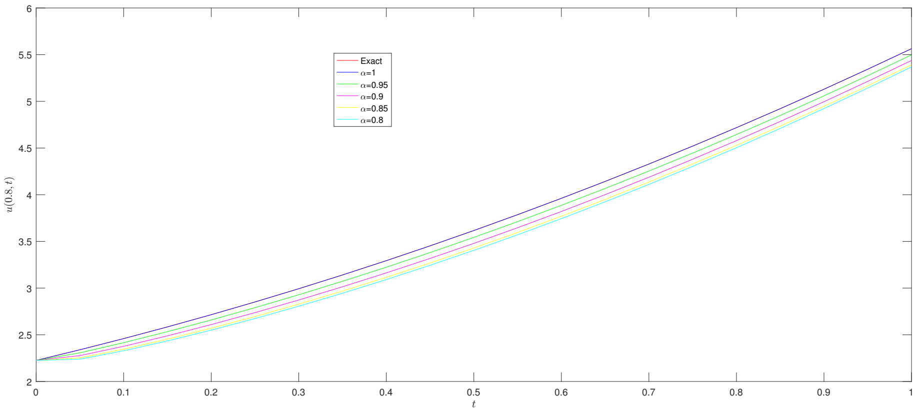

FIGURE 23. The graphics of approximate solutions for Ex.4 with NCPC derivative for various α and $n = 3$.

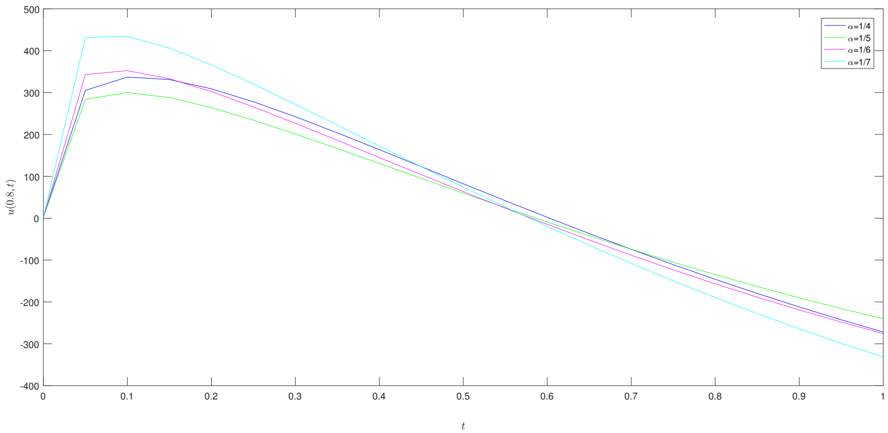

FIGURE 24. The graphics of approximate solutions for Ex.4 with NCPC derivative for various $\alpha < \frac{1}{3}$.

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(Mine Aylin Bayrak, Ali Demir) Department of Mathematics, Kocaeli University, 41380, Izmit, Kocaeli, Turkey
E-mail address: aylin@kocaeli.edu.tr, ademir@kocaeli.edu.tr