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A New Approach to Korovkin-Type Theorems based on Deferred Nörlund Summability Mean

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Abstract. This paper aims to introduce the notions of deferred Nörlund statistical Riemann integrability and statistical deferred Nörlund Riemann summability for sequence of real-valued functions and to apply them in Korovkin-type new approximations. First, we present an inclusion theorem to understand the connection between these new notions. Then, based on these potential notions we establish new versions of Korovkin-type theorems with three algebraic test functions. Finally, we compute an example, under the consideration of a positive linear operator in association with the Bernstein polynomials to exhibit the effectiveness of our findings.

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1. Introduction and motivation

The study of *convergence* on sequence space is one of the most important and fascinating aspects in the domain of real and functional analysis. The gradual improvement on this study leads to the development of *statistical convergence* which is more general than the usual convergence. The credit for independently defining this beautiful concept goes to both Fast [8] and Steinhaus [25]. Now a days, this potential notion of statistical convergence has been a field of interest of many researchers and becoming an active research area in various fields of pure and applied Mathematics. In particular, it is very much useful in the study of Machine Learning, Soft Computing, Number Theory, Measure theory, Probability Theory etc. For some recent research works in this direction, see [1], [3], [4], [7], [15], [16], [20] and [22] .

Suppose $\mathfrak{J} \subseteq \mathbb{N}$, and let $\mathfrak{J}_k = \{\xi : \xi \leqq k \quad \text{and} \quad \xi \in \mathfrak{J}\}$. Then the natural density $d(\mathfrak{J})$ of $\mathfrak{J}$ is defined by

$$d(\mathfrak{J}) = \lim_{k\to\infty} \frac{|\mathfrak{J}_k|}{k} = \rho,$$

where the number ρ is real and finite, and $|\mathfrak{J}_k|$ is the cardinality of $\mathfrak{J}_k$.

A given sequence (η_k) is statistically convergent to α if, for each $\epsilon > 0$,

$$\mathfrak{J}_\epsilon = \{\xi : \xi \in \mathbb{N} \quad \text{and} \quad |\eta_\xi - \alpha| \geqq \epsilon\}$$

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has zero natural density (see [8] and [25]). Thus, for each $\epsilon > 0$, we have

$$d(\mathfrak{J}_\epsilon) = \lim_{k\to\infty} \frac{|\mathfrak{J}_\epsilon|}{k} = 0.$$

We write

$$\text{stat} \lim_{k\to\infty} \eta_k = \alpha.$$

Let $[a, b] \subset \mathbb{R}$, and for all $k \in \mathbb{N}$ there is a function $g_k : [a, b] \to \mathbb{R}$ and it is called a sequence (g_k) of functions on $[a, b]$.

We now define the Riemann sum of a sequence (g_i) of functions associated with a tagged partition $\dot{\mathcal{P}}$ which is given by

$$\delta(g_i; \dot{\mathcal{P}}) := \sum_{i=1}^{k} g(\gamma_i)(r_i - r_{i-1}).$$

Next, we recall the definition of Riemann integrability of a sequence of functions over an interval $[a, b]$.

A sequence $(g_k)_{k\in\mathbb{N}}$ of functions is Riemann integrable to a function g on $[a, b]$ if, for all $\epsilon > 0$ there exists $\sigma_\epsilon > 0$ and let $\dot{\mathcal{P}}$ be any tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \sigma_\epsilon$ such that

$$|\delta(g_k; \dot{\mathcal{P}}) - g| < \epsilon.$$

We now present the definition of statistical convergence of Riemann integrable functions.

A sequence $(g_k)_{k\in\mathbb{N}}$ of functions is *statistically Riemann integrable* to a function g on $[a, b]$ if, for all $\epsilon > 0$ and for each $x \in [a, b]$, there exists $\sigma_\epsilon > 0$, and for $\dot{\mathcal{P}}$ be any tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \sigma_\epsilon$, the set

$$\mathfrak{J}_\epsilon = \{\xi : \xi \in \mathbb{N} \quad \text{and} \quad |\delta(g_\xi; \dot{\mathcal{P}}) - g| \geqq \epsilon\}$$

has zero natural density. That is, for every $\epsilon > 0$,

$$d(\mathfrak{J}_\epsilon) = \lim_{k\to\infty} \frac{|\mathfrak{J}_\epsilon|}{k} = 0.$$

We write

$$\text{stat}_{\text{Rie}} \lim_{k\to\infty} \delta(g_k; \dot{\mathcal{P}}) = g.$$

The following example demonstrates that every Riemann integrable function is statistically Riemann integrable; however the converse is not true.

Example 1.1. Let $g_k : [0, 1] \to \mathbb{R}$ be a sequence of functions defined by

$$g_k(x) = \begin{cases} \frac{1}{2} & (x \in \mathbb{Q} \cap [0, 1];\ k = j^2,\ j \in \mathbb{N}) \\ \\ \frac{n}{n+1} & (\text{otherwise}). \end{cases} \tag{1}$$

It is easy to see that the sequence (g_k) of functions is statistically Riemann integrable to 1 over $[0, 1]$, but not Riemann integrable (in the ordinary sense) over $[0, 1]$.

Motivated essentially by the above-mentioned investigations and studies, we introduce here the notions of deferred Nörlund statistical Riemann integrability and statistical deferred Nörlund Riemann summability for sequence of real-valued functions. We first present an inclusion theorem connecting these new notions. Moreover, as an application point of view, we state and prove new versions of Korovkin-type theorems with three algebraic test functions by using these potential notions. Finally, we compute an example under the consideration of a positive linear operator in association with the Bernstein polynomials to exhibit the effectiveness of our findings.

2. Deferred Nörlund statistical Riemann integrability

Let (ϕ_k) and (φ_k) be sequences of non-negative integers satisfying the regularity conditions, $\phi_k < \varphi_k$ and $\lim_{k\to\infty} \varphi_k = +\infty$, and let (p_i) be a sequence of non-negative real numbers such that $P_k = \sum_{i=\phi_k+1}^{\varphi_k} p_{\varphi_k - i}$.

Then, we define the deferred Nörlund summability mean for the Riemann sum of a sequence of functions $\delta(g_k; \dot{\mathcal{P}})$ with tagged partition $\dot{\mathcal{P}}$ of the form,

$$\mathcal{N}(\delta(g_k; \dot{\mathcal{P}})) = \frac{1}{P_k} \sum_{\varrho=\phi_k+1}^{\varphi_k} p_{\varphi_k - \varrho}\, \delta(g_\varrho; \dot{\mathcal{P}}). \tag{2}$$

We now present the notions of statistical Riemann integrability and statistical Riemann summability of a sequence of functions via deferred Nörlund mean.

Definition 2.1. Let (ϕ_k) and (φ_k) be sequences of non-negative integers, and let (p_k) be a sequence of non-negative real numbers. A sequence $(g_k)_{k\in\mathbb{N}}$ of functions is deferred Nörlund statistically Riemann integrable to a function g on $[a, b]$ if, for all $\epsilon > 0$ there exists $\sigma_\epsilon > 0$, and for $\dot{\mathcal{P}}$ be any tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \sigma_\epsilon$, the set

$$\{\xi : \xi \leqq P_k \quad \text{and} \quad p_\xi |\delta(g_\xi; \dot{\mathcal{P}}) - g| \geqq \epsilon\}$$

has zero natural density. This implies that for each $\epsilon > 0$,

$$\lim_{k\to\infty} \frac{|\{\xi : \xi \leqq P_k \quad \text{and} \quad p_\xi |\delta(g_\xi; \dot{\mathcal{P}}) - g| \geqq \epsilon\}|}{P_k} = 0.$$

We write

$$\mathrm{DNR}_{\mathrm{stat}} \lim_{k\to\infty} \delta(g_k; \dot{\mathcal{P}}) = g.$$

Definition 2.2. Let (ϕ_k) and (φ_k) be sequences of non-negative integers and let (p_k) be a sequence of non-negative real numbers. A sequence $(g_k)_{k\in\mathbb{N}}$ of functions is statistically deferred Nörlund Riemann summable to a function g on $[a, b]$ if, for all $\epsilon > 0$ there exists $\sigma_\epsilon > 0$, and for $\dot{\mathcal{P}}$ be any tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \sigma_\epsilon$, the set

$$\{\xi : \xi \leqq k \quad \text{and} \quad |\mathcal{N}(\delta(g_\xi; \dot{\mathcal{P}})) - g| \geqq \epsilon\}$$

has zero natural density. This implies that for all $\epsilon > 0$,

$$\lim_{k\to\infty} \frac{|\{\xi : \xi \leqq k \quad \text{and} \quad |\mathcal{N}(\delta(g_\xi; \dot{\mathcal{P}})) - g| \geqq \epsilon\}|}{k} = 0.$$

We write

$$\mathrm{stat}_{\mathrm{DNR}} \lim_{k\to\infty} \delta(g_k; \dot{\mathcal{P}}) = g.$$

Now, we establish an inclusion theorem between these two new potentially useful notions that every deferred Nörlund statistically Riemann integrable sequence of functions is statistically deferred Nörlund Riemann summable, but the converse is not true.

Theorem 2.1. *Let* (ϕ_k) *and* (φ_k) *be sequences of non-negative integers and let* (p_k) *be a sequence of non-negative real numbers. If a sequence* $(g_k)_{k\in\mathbb{N}}$ *of functions is deferred Nörlund statistically Riemann integrable to a function* g *on* $[a,b]$*, then it is statistically deferred Nörlund Riemann summable to the same function* g *on* $[a,b]$*, but not conversely.*

Proof. Suppose $(g_k)_{k\in\mathbb{N}}$ is deferred Nörlund statistically Riemann integrable to a function g on $[a,b]$, then by Definition 2.1, we have

$$\lim_{k\to\infty}\frac{|\{\xi:\phi_k<\xi\leqq\varphi_k \quad \text{and} \quad p_\xi|\delta(g_\xi;\dot{\mathcal{P}})-g|\geqq\epsilon\}|}{P_k}=0.$$

Now assuming two sets as follows:

$$\mathcal{L}_\epsilon=\{\xi:\phi_k<\xi\leqq\varphi_k \quad \text{and} \quad p_\xi|\delta(g_\xi;\dot{\mathcal{P}})-g|\geqq\epsilon\}$$

and

$$\mathcal{L}_\epsilon^c=\{\xi:\phi_k<\xi\leqq\varphi_k \quad \text{and} \quad p_\xi|\delta(g_\xi;\dot{\mathcal{P}})-g|<\epsilon\},$$

we have

$$\begin{aligned}
\left|\mathcal{N}(\delta(g_k;\dot{\mathcal{P}}))-g\right| &= \left|\frac{1}{P_k}\sum_{\varrho=\phi_k+1}^{\varphi_k}p_{\varphi_k-\varrho}\delta(g_\varrho;\dot{\mathcal{P}})-g\right| \\
&\leqq \left|\frac{1}{P_k}\sum_{\varrho=\phi_k+1}^{\varphi_k}p_{\varphi_k-\varrho}\left[\delta(g_\varrho;\dot{\mathcal{P}})-g\right]\right|+\left|\frac{1}{P_k}\sum_{\varrho=\phi_k+1}^{\varphi_k}p_{\varphi_k-\varrho}g-g\right| \\
&\leqq \frac{1}{P_k}\sum_{\substack{\varrho=\phi_k+1\\(\xi\in\mathcal{L}_\epsilon)}}^{\varphi_k}p_{\varphi_k-\varrho}\left|\delta(g_\varrho;\dot{\mathcal{P}})-g\right|+\frac{1}{P_k}\sum_{\substack{\varrho=\phi_k+1\\(\xi\in\mathcal{L}_\epsilon^c)}}^{\varphi_k}p_{\varphi_k-\varrho}\left|\delta(g_\varrho;\dot{\mathcal{P}})-g\right| \\
&\quad +|g|\left|\frac{1}{P_k}\sum_{\varrho=\phi_k+1}^{\varphi_k}p_{\varphi_k-\varrho}-1\right| \\
&\leqq \frac{1}{P_k}\left|\mathcal{L}_\epsilon\right|+\frac{1}{P_k}|\mathcal{L}_\epsilon^c|=0.
\end{aligned}$$

This implies that

$$|\mathcal{N}(\delta(g_k;\dot{\mathcal{P}}))-g|<\epsilon.$$

Thus, the sequence of functions (g_k) is statistically deferred Nörlund Riemann summable to the function g on $[a,b]$.

Next, in view of the non-validity of the converse statement, the following example illustrates that a statistically deferred Nörlund Riemann summable sequence of functions is not deferred Nörlund statistically Riemann integrable.

Example 2.1. Let $\phi_k = 2k-1$, $\varphi_k = 4k-1$ and $p_k = 1$ and let $g_k : [0,1] \to \mathbb{R}$ be a sequence of functions of the form given by

$$g_k(x) = \begin{cases} 0 & (x \in \mathbb{Q} \cap [0,1];\ k \text{ is even}) \\ \\ 1 & (x \in \mathbb{R} - \mathbb{Q} \cap [0,1];\ k \text{ is odd}). \end{cases} \tag{3}$$

The given sequence (g_k) of functions trivially indicates that it is neither Riemann integrable nor deferred Nörlund statistically Riemann integrable. However, as per our proposed mean (2), it is easy to see that

$$\begin{aligned} \mathcal{N}(\delta(g_k;\dot{\mathcal{P}})) &= \frac{1}{\varphi_k - \phi_k} \sum_{\varrho=\phi_k+1}^{\varphi_k} \delta(g_\varrho;\dot{\mathcal{P}}) \\ &= \frac{1}{2k} \sum_{m=2k+1}^{4k} \delta(g_\varrho;\dot{\mathcal{P}}) = \frac{1}{2}. \end{aligned}$$

Thus, the sequence (g_k) of functions has deferred Nörlund Riemann sum $\frac{1}{2}$ under the tagged partition $\dot{\mathcal{P}}$. Therefore, the sequence (g_k) of functions is statistically deferred Nörlund Riemann summable to $\frac{1}{2}$ over $[0,1]$ but it is not deferred Nörlund statistically Riemann integrable.

□

3. Korovkin-type theorems via the $\mathcal{N}(\delta(g_k;\dot{\mathcal{P}}))$-mean

Very recently, a number of researchers worked toward extending (or generalizing) various aspects of the Korovkin-type approximation theorems with several settings in different fields of mathematics such as (for example) sequence spaces, Banach space, Probability space, Measurable space, and so on. This concept is quite valuable in Real Analysis, Functional Analysis, Harmonic Analysis, and other related areas. Here, in this connection, we choose to refer the interested readers to the recent works [5], [6], [9], [10], [11], [13], [17], [18], [19] and [21].

Let $[0,1] \subset \mathbb{R}$ and suppose $\mathcal{C}[0,1]$ be the space of all continuous real-valued functions defined on $[0,1]$. Also, it is a complete normed linear space (Banach space) with the norm $\|.\|_\infty$. Then for $g \in \mathcal{C}[0,1]$, the norm of g is given by,

$$\|g\|_\infty = \sup\{|g(\zeta)| \ : \ 0 \leqq \zeta \leqq 1\}.$$

We say that a sequence of linear operators $\mathfrak{A}_j : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ is positive if

$$\mathfrak{A}_j(g;\zeta) \geqq 0 \quad \text{as} \quad g \geqq 0.$$

Now, in view of our proposed mean, we use the notion of statistical Riemann integrability ($\mathrm{DNR}_{\mathrm{stat}}$) and statistical Riemann summability ($\mathrm{stat}_{\mathrm{DNR}}$) for sequence of functions to establish and prove the following new Korovkin-type approximation theorems.

Theorem 3.1. *Let*

$$\mathfrak{A}_j : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$$

be a sequence of positive linear operators. Then, for all $g \in \mathcal{C}[0,1]$,

$$\mathrm{DNR}_{\mathrm{stat}} \lim_{j\to\infty} \|\mathfrak{A}_j(g;\zeta) - g(\zeta)\|_\infty = 0 \tag{4}$$

if and only if

$$\mathrm{DNR}_{\mathrm{stat}} \lim_{j\to\infty} \|\mathfrak{A}_j(1;\zeta) - 1\|_\infty = 0, \tag{5}$$

$$\mathrm{DNR}_{\mathrm{stat}} \lim_{j\to\infty} \|\mathfrak{A}_j(\zeta;\zeta) - \zeta\|_\infty = 0 \tag{6}$$

and

$$\mathrm{DNR}_{\mathrm{stat}} \lim_{j\to\infty} \|\mathfrak{A}_j(\zeta^2;\zeta) - \zeta^2\|_\infty = 0. \tag{7}$$

Proof. Since each of the following functions:

$$g_0(\zeta) = 1, \quad g_1(\zeta) = 2\zeta \quad \text{and} \quad g_2(\zeta) = 3\zeta^2$$

belongs to $\mathcal{C}[0,1]$ and is continuous, the implication given by (4) implies (5) to (7) is trivial.

In order to complete the proof of Theorem 3.1, we first assume that the conditions (5) to (7) hold true. If $g \in \mathcal{C}[0,1]$, then there exists a constant $\mathcal{J} > 0$ such that

$$|g(\zeta)| \leqq \mathcal{J} \quad (\forall\, \zeta \in [0,1]).$$

We thus find that

$$|g(r) - g(\zeta)| \leqq 2\mathcal{J} \quad (r, \zeta \in [0,1]). \tag{8}$$

Clearly, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|g(r) - g(\zeta)| < \epsilon \tag{9}$$

whenever

$$|r - \zeta| < \delta, \quad \text{for all} \quad r, \zeta \in [0,1].$$

Let us choose

$$\mu_1 = \mu_1(r,\zeta) = (2r - 2\zeta)^2.$$

If

$$|r - \zeta| \geqq \delta,$$

then we obtain

$$|g(r) - g(\zeta)| < \frac{2\mathcal{J}}{\theta^2}\mu_1(r,\zeta). \tag{10}$$

From equation (9) and (10), we get

$$|g(r) - g(\zeta)| < \epsilon + \frac{2\mathcal{J}}{\theta^2}\mu_1(r,\zeta),$$

which implies that

$$-\epsilon - \frac{2\mathcal{J}}{\theta^2}\mu_1(r,\zeta) \leqq g(r) - g(\zeta) \leqq \epsilon + \frac{2\mathcal{J}}{\theta^2}\mu_1(r,\zeta). \tag{11}$$

Now, since $\mathfrak{A}_m(1;\zeta)$ is monotone and linear, by applying the operator $\mathfrak{A}_m(1;\zeta)$ to this inequality, we have

$$\mathfrak{A}_m(1;\zeta)\left(-\epsilon-\frac{2\mathcal{J}}{\theta^2}\mu_1(r,\zeta)\right) \leqq \mathfrak{A}_m(1;\zeta)(g(r)-g(\zeta)) \leqq \mathfrak{A}_m(1;\zeta)\left(\epsilon+\frac{2\mathcal{J}}{\theta^2}\mu_1(r,\zeta)\right).$$

We note that ζ is fixed and so $g(\zeta)$ is a constant number. Therefore, we have

$$-\epsilon\mathfrak{A}_m(1;\zeta)-\frac{2\mathcal{J}}{\theta^2}\mathfrak{A}_m(\mu_1;\zeta) \leqq \mathfrak{A}_m(g;\zeta)-g(\zeta)\mathfrak{A}_m(1;\zeta) \leqq \epsilon\mathfrak{A}_m(1;\zeta)+\frac{2\mathcal{J}}{\theta^2}\mathfrak{A}_m(\mu_1;\zeta). \tag{12}$$

Also, we know that

$$\mathfrak{A}_m(g;\zeta)-g(\zeta)=[\mathfrak{A}_m(g;\zeta)-g(\zeta)\mathfrak{A}_m(1;\zeta)]+g(\zeta)[\mathfrak{A}_m(1;\zeta)-1]. \tag{13}$$

Using (12) and (13), we have

$$\mathfrak{A}_m(g;\zeta)-g(\zeta)<\epsilon\mathfrak{A}_m(1;\zeta)+\frac{2\mathcal{J}}{\theta^2}\mathfrak{A}_m(\mu_1;\zeta)+g(\zeta)[\mathfrak{A}_m(1;\zeta)-1]. \tag{14}$$

We now estimate $\mathfrak{A}_m(\mu_1;\zeta)$ as follows:

$$\begin{aligned}\mathfrak{A}_m(\mu_1;\zeta)&=\mathfrak{A}_m((2r-2\zeta)^2;\zeta)=\mathfrak{A}_m(2r^2-8\zeta r+4\zeta^2;\zeta)\\&=\mathfrak{A}_m(4r^2;\zeta)-8t\mathfrak{A}_m(r;\zeta)+4\zeta^2\mathfrak{A}_m(1;\zeta)\\&=4[\mathfrak{A}_m(r^2;\zeta)-\zeta^2]-8t[\mathfrak{A}_m(r;\zeta)-\zeta]\\&\quad+4\zeta^2[\mathfrak{A}_m(1;\zeta)-1].\end{aligned}$$

Using (14), we obtain

$$\begin{aligned}\mathfrak{A}_m(g;\zeta)-g(\zeta)&<\epsilon\mathfrak{A}_m(1;\zeta)+\frac{2\mathcal{J}}{\theta^2}\{4[\mathfrak{A}_m(r^2;\zeta)-\zeta^2]\\&\qquad-8\zeta[\mathfrak{A}_m(r;\zeta)-\zeta]+4\zeta^2[\mathfrak{A}_m(1;\zeta)-1]\}\\&\qquad+g(\zeta)[\mathfrak{A}_m(1;\zeta)-1].\\&=\epsilon[\mathfrak{A}_m(1;\zeta)-1]+\epsilon+\frac{2\mathcal{L}}{\theta^2}\{4[\mathfrak{A}_m(r^2;\zeta)-\zeta^2]\\&\qquad-8\zeta[\mathfrak{A}_m(r;\zeta)-\zeta]+4\zeta^2[\mathfrak{A}_m(1;\zeta)-1]\}\\&\qquad+g(\zeta)[\mathfrak{A}_m(1;\zeta)-1].\end{aligned}$$

Since $\epsilon>0$ is arbitrary, we can write

$$\begin{aligned}|\mathfrak{A}_m(g;\zeta)-g(\zeta)|&\leqq\epsilon+\left(\epsilon+\frac{8\mathcal{J}}{\theta^2}+\mathcal{J}\right)|\mathfrak{A}_m(1;\zeta)-1|\\&\quad+\frac{16\mathcal{J}}{\theta^2}|\mathfrak{A}_m(r;\zeta)-\zeta|+\frac{8\mathcal{J}}{\theta^2}|\mathfrak{A}_m(r^2;\zeta)-\zeta^2|\\&\leqq\mathcal{A}(|\mathfrak{A}_m(1;\zeta)-1|+|\mathfrak{A}_m(r;\zeta)-\zeta|\\&\qquad+|\mathfrak{A}_m(r^2;\zeta)-\zeta^2|),\end{aligned} \tag{15}$$

where

$$\mathcal{A}=\max\left(\epsilon+\frac{8\mathcal{J}}{\theta^2}+\mathcal{J},\frac{16\mathcal{J}}{\theta^2},\frac{8\mathcal{J}}{\theta^2}\right).$$

Now, for a given $\omega > 0$, there exists $\epsilon > 0$ $(\epsilon < \omega)$ such that

$$\mathfrak{L}_m(\zeta;\omega) = \{m : m \leqq P_k \quad \text{and} \quad p_{\varphi_k - \varrho} \left|\mathfrak{A}_m(g;\zeta) - g(\zeta)\right| \geqq \omega\}.$$

Furthermore, for $\nu = 0, 1, 2$, we have

$$\mathfrak{L}_{\nu,m}(\zeta;\omega) = \left\{m : m \leqq P_k \quad \text{and} \quad p_{\varphi_k - \varrho} \left|\mathfrak{A}_m(g;\zeta) - g_\nu(\zeta)\right| \geqq \frac{\omega - \epsilon}{3\mathcal{A}}\right\},$$

so that

$$\mathfrak{L}_m(\zeta;\omega) \leqq \sum_{\nu=0}^{2} \mathfrak{L}_{\nu,m}(\zeta;\omega).$$

Clearly, we obtain

$$\frac{\|\mathfrak{L}_m(\zeta;\omega)\|_{\mathcal{C}[0,1]}}{P_k} \leqq \sum_{\nu=0}^{2} \frac{\|\mathfrak{L}_{\nu,m}(\zeta;\omega)\|_{\mathcal{C}[0,1]}}{P_k}. \tag{16}$$

Now, using the above assumption about the implications in (5) to (7) and by Definition 2.1, the right-hand side of (16) tends to zero as $n \to \infty$. Consequently, we get

$$\lim_{k\to\infty} \frac{\|\mathfrak{L}_m(\zeta;\omega)\|_{\mathcal{C}[0,1]}}{P_k} = 0 \ (\delta, \omega > 0).$$

Therefore, the implication (4) holds true. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let*

$$\mathfrak{A}_j : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$$

be a sequence of positive linear operators. Then, for all $g \in \mathcal{C}[0,1]$,

$$\text{stat}_{\text{DNR}} \lim_{j\to\infty} \|\mathfrak{A}_j(g;\zeta) - g(\zeta)\|_\infty = 0 \tag{17}$$

if and only if

$$\text{stat}_{\text{DNR}} \lim_{j\to\infty} \|\mathfrak{A}_j(1;\zeta) - 1\|_\infty = 0, \tag{18}$$

$$\text{stat}_{\text{DNR}} \lim_{j\to\infty} \|\mathfrak{A}_j(\zeta;\zeta) - \zeta\|_\infty = 0 \tag{19}$$

and

$$\text{stat}_{\text{DNR}} \lim_{j\to\infty} \|\mathfrak{A}_j(\zeta^2;\zeta) - \zeta^2\|_\infty = 0. \tag{20}$$

Proof. The proof of Theorem 3.2 is similar to the proof of Theorem 3.1. We, therefore, choose to skip the details involved. □

In view of Theorem 3.2, here we consider an example that a sequence of positive linear operators that does not work via the statistical version of the deferred Nörlund Riemann integrable sequence of functions (Theorem 3.1). However, it fairly works on Theorem 3.2. In this sense we say that Theorem 3.2 is a non-trivial extension of the statistical Nörlund Riemann integrable sequence of functions (Theorem 3.1).

We now recall the operator

$$\zeta(1 + \zeta D) \qquad \left(D = \frac{d}{d\zeta}\right), \tag{21}$$

which was used by Al-Salam [2] and, more recently, by Viskov and Srivastava [26] (see [14] and [23]).

Example 3.1. Consider the *Bernstein polynomial* $\mathfrak{B}_n(g;\beta)$ on $\mathcal{C}[0,1]$ given by (see also [24])

$$\mathfrak{B}_k(g;\beta)=\sum_{\varrho=0}^{k} g\left(\frac{\varrho}{k}\right)\binom{k}{\varrho}\beta^{\varrho}(1-b)^{k-\varrho} \quad (\beta\in[0,1];k=0,1,\cdots). \tag{22}$$

We now introduce the positive linear operators on $\mathcal{C}[0,1]$ under the composition of Bernstein polynomial and the operators given by (21) as follows:

$$\mathfrak{A}_{\varrho}(g;\beta)=[1+g_{\varrho}]\beta(1+\beta D)\mathfrak{B}_{\varrho}(g;\beta) \quad (\forall\ g\in C[0,1]), \tag{23}$$

where (g_{ϱ}) is the same as mentioned in Example 2.1.

We now estimate the values of each of the testing functions 1, β and β^2 by using our proposed operators (23) as follows:

$$\mathfrak{A}_{\varrho}(1;\beta)=[1+g_{\varrho}]\beta(1+\beta D)1=[1+g_{\varrho}]\beta,$$

$$\mathfrak{A}_{\varrho}(t;\beta)=[1+g_{\varrho}]\beta(1+\beta D)\beta=[1+g_{\varrho}]\beta(1+\beta)$$

and

$$\begin{aligned}\mathfrak{A}_{\varrho}(t^2;\beta)&=[1+g_{\varrho}]\beta(1+\beta D)\left\{\beta^2+\frac{\beta(1-\beta)}{\varrho}\right\}\\&=[1+g_{\varrho}]\left\{\beta^2\left(2-\frac{3\beta}{\varrho}\right)\right\}.\end{aligned}$$

Consequently, we have

$$\text{stat}_{\text{DNR}}\lim_{\varrho\to\infty}\|\mathfrak{A}_{\varrho}(1;\beta)-1\|_{\infty}=0, \tag{24}$$

$$\text{stat}_{\text{DNR}}\lim_{\varrho\to\infty}\|\mathfrak{A}_{\varrho}(\beta;\beta)-\beta\|_{\infty}=0 \tag{25}$$

and

$$\text{stat}_{\text{DNR}}\lim_{\varrho\to\infty}\|\mathfrak{A}_{\varrho}(\beta^2;\beta)-\beta^2\|_{\infty}=0, \tag{26}$$

that is, the sequence $\mathfrak{A}_{\varrho}(g;\beta)$ satisfies the conditions (18) to (20). Therefore, by Theorem 3.2, we have

$$\text{stat}_{\text{DNR}}\lim_{\varrho\to\infty}\|\mathfrak{A}_{\varrho}(g;\beta)-g\|_{\infty}=0.$$

The given sequence (g_k) of the functions mentioned in Example 2.1 is statistically deferred Nörlund Riemann summable, but not deferred Nörlund statistically Riemann integrable. Therefore, our proposed operators defined by (23) satisfy Theorem 3.2; however, they do not satisfy the statistical versions of deferred Nörlund Riemann integrable sequence of functions (Theorem 3.1).

4. Conclusion

In this concluding section of our investigation, we further observe the potentiality of our Theorem 3.2 over Theorem 3.1 as well as over classical version of the Korovkin-type approximation theorems.

Let us consider sequence $(g_\varrho)_{\varrho\in\mathbb{N}}$ of functions in Example 2.1 and also that (g_ϱ) is statistically deferred Nörlund Riemann summable, so that

$$\mathrm{stat}_{\mathrm{DNR}} \lim_{\varrho\to\infty} \delta(g_\varrho; \dot{\mathcal{P}}) = \frac{1}{2} \text{ on } [0,1].$$

Then we have

$$\mathrm{stat}_{\mathrm{DNR}} \lim_{k\to\infty} \|\mathfrak{A}_k(g_\nu;\zeta) - g_\nu(\zeta)\|_\infty = 0 \quad (\nu = 0,1,2). \tag{27}$$

Thus, by Theorem 3.2, we immediately get

$$\mathrm{stat}_{\mathrm{DNR}} \lim_{j\to\infty} \|\mathfrak{A}_k(g;\zeta) - g(\zeta)\|_\infty = 0, \tag{28}$$

where

$$g_0(\zeta) = 1, \quad g_1(\zeta) = \zeta \quad \text{and} \quad g_2(\zeta) = \zeta^2.$$

As the given sequence (g_k) of functions is statistically deferred Nörlund Riemann summable, but neither deferred Nörlund statistically Riemann integrable nor classically Riemann integrable. Therefore, our Korovkin-type approximation Theorem 3.2 properly works under the operators defined in the equation (23), but the classical as well as statistical versions of deferred Nörlund Riemann integrable sequence of functions do not work for the same operators. In view of this observation, we certainly say that our Theorem 3.2 is a non-trivial extension of Theorem 3.1 as well as the classical Korovkin-type approximation theorem [12].

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