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# Some inequalities for sums involving the distance in metric spaces

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Abstract. Let $(X, d)$ be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$. In this paper we show among others that

$$2^{p-1} \left[ \sum_{k=1}^{n} p_k d^2 (x_k, x) - \left( \sum_{k=1}^{n} p_k d (x_k, x) \right)^2 \right]^p$$
$$\leq \sum_{1 \leq i < j \leq n} p_i p_j d^{2p} (x_i, x_j) \leq 2^{2p-1} \sum_{i=1}^{n} p_i d^{2p} (x_i, x)$$

for $p \geq 1$ and $x \in X$. Some examples for normed and inner product spaces are also given.

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## 1. Introduction

Let $X$ be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a *distance* on $X$ if the following properties are satisfied:

(d) $d(x, y) = 0$ if and only if $x = y$;
(dd) $d(x, y) = d(y, x)$ for any $x, y \in X$ (the *symmetry* of the distance);
(ddd) $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$ (the *triangle inequality*).

The pair $(X, d)$ is called in the literature a *metric space.*

Important examples of metric spaces are normed linear spaces. We recall that, a linear space $E$ over the real or complex number field $\mathbb{K}$ endowed with a function $\|\cdot\| : E \to [0, \infty)$, is called a *normed space* if $\|\cdot\|$, the *norm*, satisfies the properties:

(n) $\|x\| = 0$ if and only if $x = 0$;
(nn) $\|\alpha x\| = |\alpha| \|x\|$ for any scalar $\alpha \in \mathbb{K}$ and any vector $x \in E$;
(nnn) $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in E$ (the triangle inequality).

Further, we recall that, the linear space $H$ over the real or complex number field $\mathbb{K}$ endowed with an application $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}$ is called an *inner product space*, if the function $\langle \cdot, \cdot \rangle$, called the *inner product*, satisfies the following properties:

(i) $\langle x, x \rangle \geq 0$ for any $x \in H$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;
(ii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for any scalars $\alpha, \beta$ and any vectors $x, y, z$;
(iii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for any $x, y \in H$.

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It is well know that the function $\|x\| := \sqrt{\langle x, x\rangle}$ defines a norm on $H$ and thus an important example of normed spaces are the inner product spaces.

A fundamental inequality in metric spaces, which obviously follows by the triangle inequality and mathematical induction, is the *generalised triangle inequality*, or the *polygonal inequality* which states that: for any points $x_1, x_2, ..., x_{n-1}, x_n$ $(n \geq 3)$ in a metric space $(X, d)$, we have the inequality

$$d(x_1, x_n) \leq d(x_1, x_2) + ... + d(x_{n-1}, x_n). \tag{1}$$

The following result in the general setting of metric spaces holds [3].

**Theorem 1.1.** *Let* $(X, d)$ *be a metric space and* $x_i \in X$, $p_i \geq 0$, $i \in \{1, ..., n\}$ *with* $\sum_{i=1}^{n} p_i = 1$. *Then we have the inequality*

$$\sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \inf_{x \in X} \left[ \sum_{i=1}^{n} p_i d(x_i, x) \right]. \tag{2}$$

*The inequality is sharp in the sense that the multiplicative constant* $c = 1$ *in front of "inf " cannot be replaced by a smaller quantity.*

We have:

**Corollary 1.2.** *Let* $(X, d)$ *be a metric space and* $x_i \in X$, $i \in \{1, ..., n\}$. *If there exists a closed ball of radius* $r > 0$ *centered in a point* $x$ *containing all the points* $x_i$, *i.e.,* $x_i \in \overline{B}(x, r) := \{y \in X : d(x, y) \leq r\}$, *then for any* $p_i \geq 0$, $i \in \{1, ..., n\}$ *with* $\sum_{i=1}^{n} p_i = 1$ *we have the inequality*

$$\sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq r. \tag{3}$$

The inequality (2) and its consequences were extended to the case of $b$-metric spaces in [4] and for natural powers of the distance in [1].

In the recent note [2] we provided some upper and lower bounds for the sum $\sum_{1 \leq i < j \leq n} p_i p_j d^s(x_i, x_j)$,

$$\sum_{1 \leq i < j \leq n} p_i p_j d^s(x_i, x_j) \leq \begin{cases} 2^{s-1} \inf_{x \in X} \left[\sum_{i=1}^{n} p_i d^s(x_i, x)\right], \text{ if } s \geq 1, \\ \\ \inf_{x \in X} \left[\sum_{i=1}^{n} p_i d^s(x_i, x)\right], \text{ if } 0 < s < 1 \end{cases} \tag{4}$$

and

$$\left(\frac{2}{1 - \sum_{i=1}^{n} p_i^2}\right)^{s-1} \left(\sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j)\right)^s \leq \sum_{1 \leq i < j \leq n} p_i p_j d^s(x_i, x_j), \tag{5}$$

for $s > 1$, where $(X, d)$ is a metric space, $x_i \in X$, $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$.

In this paper we show among others that

$$2^{p-1} \left[ \sum_{k=1}^{n} p_k d^2(x_k, x) - \left( \sum_{k=1}^{n} p_k d(x_k, x) \right)^2 \right]^p$$
$$\leq \sum_{1 \leq i < j \leq n} p_i p_j d^{2p}(x_i, x_j) \leq 2^{2p-1} \sum_{i=1}^{n} p_i d^{2p}(x_i, x)$$

for $p \geq 1$ and $x$, $x_i \in X$, $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$. Some examples for normed and inner product spaces are also given.

## 2. Main results

We have the following results:

**Theorem 2.1.** *Let $(X, d)$ be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequalities*

$$0 \leq \sum_{k=1}^{n} p_k d^2 (x_k, x) - \left( \sum_{k=1}^{n} p_k d (x_k, x) \right)^2 \leq \sum_{1 \leq i < j \leq n} p_i p_j d^2 (x_i, x_j) \tag{6}$$
$$\leq \sum_{k=1}^{n} p_k d^2 (x_k, x) + \left( \sum_{k=1}^{n} p_k d (x_k, x) \right)^2$$

*for all $x \in X$.*

*Proof.* By the triangle inequality we have

$$|d (x_i, x) - d (x_j, x)| \leq d (x_i, x_j) \leq d (x_i, x) + d (x_j, x) \tag{7}$$

for all $x \in X$ and $i$, $j \in \{1, ..., n\}$.

By taking the square in (7), we get

$$(d (x_i, x) - d (x_j, x))^2 \leq d^2 (x_i, x_j) \leq (d (x_i, x) + d (x_j, x))^2 ,$$

namely

$$d^2 (x_i, x) - 2d (x_i, x) d (x_j, x) + d^2 (x_j, x) \tag{8}$$
$$\leq d^2 (x_i, x_j)$$
$$\leq d^2 (x_i, x) + 2d (x_i, x) d (x_j, x) + d^2 (x_j, x) ,$$

for all $x \in X$ and $i$, $j \in \{1, ..., n\}$.

If we multiply (8) by $p_i p_j \geq 0$ and sum over $i$, $j$ from 1 to $n$ we get

$$\sum_{i,j=1}^{n} p_i p_j \left[ d^2 (x_i, x) - 2d (x_i, x) d (x_j, x) + d^2 (x_j, x) \right] \tag{9}$$
$$\leq \sum_{i,j=1}^{n} p_i p_j d^2 (x_i, x_j)$$
$$\leq \sum_{i,j=1}^{n} p_i p_j \left[ d^2 (x_i, x) + 2d (x_i, x) d (x_j, x) + d^2 (x_j, x) \right] ,$$

for all $x \in X$.

Since

$$\sum_{i,j=1}^{n} p_i p_j d^2 (x_i, x) = \sum_{k=1}^{n} p_k d^2 (x_k, x) = \sum_{i,j=1}^{n} p_i p_j d^2 (x_j, x) ,$$

$$\sum_{i,j=1}^{n} p_i p_j d (x_i, x) d (x_j, x) = \left( \sum_{k=1}^{n} p_k d (x_k, x) \right)^2$$

and

$$\sum_{i,j=1}^{n} p_i p_j d^2(x_i, x_j) = 2 \sum_{1 \le i < j \le n} p_i p_j d^2(x_i, x_j),$$

hence by (9) we get

$$\begin{aligned} 0 &\le 2\sum_{k=1}^{n} p_k d^2(x_k, x) - 2\left(\sum_{k=1}^{n} p_k d(x_k, x)\right)^2 \\ &\le 2 \sum_{1 \le i < j \le n} p_i p_j d^2(x_i, x_j) \\ &\le 2\sum_{k=1}^{n} p_k d^2(x_k, x) + 2\left(\sum_{k=1}^{n} p_k d(x_k, x)\right)^2 \end{aligned}$$

for all $x \in X$, and the inequality (6) is proved. □

**Remark 2.1.** We observe that the second and third inequalities in (6) are equivalent to

$$\left| \sum_{1 \le i < j \le n} p_i p_j d^2(x_i, x_j) - \sum_{k=1}^{n} p_k d^2(x_k, x) \right| \le \left(\sum_{k=1}^{n} p_k d(x_k, x)\right)^2 \tag{10}$$

for all $x \in X$.

The case $p_i = \frac{1}{n}$, $i \in \{1, ..., n\}$ in (6) and (10) produces the inequalities

$$\begin{aligned} 0 &\le n\sum_{k=1}^{n} d^2(x_k, x) - \left(\sum_{k=1}^{n} d(x_k, x)\right)^2 \le \sum_{1 \le i < j \le n} d^2(x_i, x_j) \\ &\le n\sum_{k=1}^{n} d^2(x_k, x) + \left(\sum_{k=1}^{n} d(x_k, x)\right)^2 \end{aligned} \tag{11}$$

and

$$\left| \sum_{1 \le i < j \le n} d^2(x_i, x_j) - n\sum_{k=1}^{n} d^2(x_k, x) \right| \le \left(\sum_{k=1}^{n} d(x_k, x)\right)^2 \tag{12}$$

for $x_i, x \in X$, $i \in \{1, ..., n\}$.

**Corollary 2.2.** *Let $(X, d)$ be a metric space and $x_i \in X$, $p_i \ge 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$. Assume that $S \subset X$ such that*

$$\sup_{x \in S} \left[\sum_{k=1}^{n} p_k d^2(x_k, x)\right] < \infty,$$

*then we have the inequalities*

$$
\begin{aligned}
0 &\le \sup_{x\in S}\left[\sum_{k=1}^{n} p_k d^2(x_k,x)\right] - \left(\sup_{x\in S}\sum_{k=1}^{n} p_k d(x_k,x)\right)^2 \qquad (13)\\
&\le \sum_{1\le i<j\le n} p_i p_j d^2(x_i,x_j)\\
&\le \sup_{x\in S}\left[\sum_{k=1}^{n} p_k d^2(x_k,x)\right] + \left(\sup_{x\in S}\sum_{k=1}^{n} p_k d(x_k,x)\right)^2\\
&\le 2\sup_{x\in S}\left[\sum_{k=1}^{n} p_k d^2(x_k,x)\right].
\end{aligned}
$$

*Proof.* If we take the supremum over $x \in S$ in (6), then we get

$$
\begin{aligned}
&\sup_{x\in S}\left[\sum_{k=1}^{n} p_k d^2(x_k,x) - \left(\sum_{k=1}^{n} p_k d(x_k,x)\right)^2\right]\\
&\le \sum_{1\le i<j\le n} p_i p_j d^2(x_i,x_j)\\
&\le \sup_{x\in S}\left[\sum_{k=1}^{n} p_k d^2(x_k,x) + \left(\sum_{k=1}^{n} p_k d(x_k,x)\right)^2\right]. \qquad (14)
\end{aligned}
$$

By the properties of supremum we have

$$
\begin{aligned}
&\sup_{x\in S}\left[\sum_{k=1}^{n} p_k d^2(x_k,x) + \left(\sum_{k=1}^{n} p_k d(x_k,x)\right)^2\right]\\
&\le \sup_{x\in S}\left[\sum_{k=1}^{n} p_k d^2(x_k,x)\right] + \sup_{x\in S}\left(\sum_{k=1}^{n} p_k d(x_k,x)\right)^2\\
&= \sup_{x\in S}\left[\sum_{k=1}^{n} p_k d^2(x_k,x)\right] + \left(\sup_{x\in S}\sum_{k=1}^{n} p_k d(x_k,x)\right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{x\in S}\left[\sum_{k=1}^{n} p_k d^2(x_k,x) - \left(\sum_{k=1}^{n} p_k d(x_k,x)\right)^2\right]\\
&\ge \sup_{x\in S}\left[\sum_{k=1}^{n} p_k d^2(x_k,x)\right] - \sup_{x\in S}\left(\sum_{k=1}^{n} p_k d(x_k,x)\right)^2 \ge 0
\end{aligned}
$$

since, by Cauchy-Buniakowski-Schwarz inequality we have

$$
\sum_{k=1}^{n} p_k d^2(x_k,x) \ge \left(\sum_{k=1}^{n} p_k d(x_k,x)\right)^2
$$

for all $x \in X$. □

**Remark 2.2.** Assume that $S \subset X$ such that

$$\sup_{x \in S} \left[ \sum_{k=1}^{n} d^2(x_k, x) \right] < \infty,$$

then we have the inequalities

$$0 \leq n \sup_{x \in S} \left[ \sum_{k=1}^{n} d^2(x_k, x) \right] - \left( \sup_{x \in S} \sum_{k=1}^{n} d(x_k, x) \right)^2 \tag{15}$$
$$\leq \sum_{1 \leq i < j \leq n} d^2(x_i, x_j)$$
$$\leq n \sup_{x \in S} \left[ \sum_{k=1}^{n} d^2(x_k, x) \right] + \left( \sup_{x \in S} \sum_{k=1}^{n} d(x_k, x) \right)^2$$
$$\leq 2n \sup_{x \in S} \left[ \sum_{k=1}^{n} d^2(x_k, x) \right].$$

We have the following generalization

**Theorem 2.3.** *Let $(X, d)$ be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequalities*

$$2^{p-1} \left[ \sum_{k=1}^{n} p_k d^2(x_k, x) - \left( \sum_{k=1}^{n} p_k d(x_k, x) \right)^2 \right]^p \tag{16}$$
$$\leq \sum_{1 \leq i < j \leq n} p_i p_j d^{2p}(x_i, x_j) \leq 2^{2p-1} \sum_{i=1}^{n} p_i d^{2p}(x_i, x)$$

*for $p \geq 1$ and $x \in X$. The second inequality also holds for $p \geq \frac{1}{2}$.*

*Proof.* From (7) we get

$$|d(x_i, x) - d(x_j, x)|^{2p} \leq d^{2p}(x_i, x_j) \leq [d(x_i, x) + d(x_j, x)]^{2p},$$

namely

$$\left[ d^2(x_i, x) - 2d(x_i, x) d(x_j, x) + d^2(x_j, x) \right]^p \leq d^{2p}(x_i, x_j) \tag{17}$$
$$\leq [d(x_i, x) + d(x_j, x)]^{2p},$$

for all $x \in X$ and $i, j \in \{1, ..., n\}$.

If we multiply (17) by $p_i p_j \geq 0$ and sum over $i, j$ from 1 to $n$ we get

$$\sum_{i,j=1}^{n} p_i p_j \left[ d^2(x_i, x) - 2d(x_i, x) d(x_j, x) + d^2(x_j, x) \right]^p \tag{18}$$
$$\leq \sum_{i,j=1}^{n} p_i p_j d^{2p}(x_i, x_j) = 2 \sum_{1 \leq i < j \leq n} p_i p_j d^{2p}(x_i, x_j)$$
$$\leq \sum_{i,j=1}^{n} p_i p_j [d(x_i, x) + d(x_j, x)]^{2p},$$

for all $x \in X$.

By the convexity of power function, we have

$$\left[\frac{d(x_i,x)+d(x_j,x)}{2}\right]^{2p} \le \frac{1}{2}\left[d^{2p}(x_i,x)+d^{2p}(x_j,x)\right],$$

namely

$$[d(x_i,x)+d(x_j,x)]^{2p} \le 2^{2p-1}\left[d^{2p}(x_i,x)+d^{2p}(x_j,x)\right], \tag{19}$$

for all $x \in X$ and $i,\, j \in \{1,...,n\}$.

Therefore

$$\begin{aligned}\sum_{i,j=1}^{n} p_i p_j [d(x_i,x)+d(x_j,x)]^{2p} &\le 2^{2p-1}\sum_{i,j=1}^{n} p_i p_j \left[d^{2p}(x_i,x)+d^{2p}(x_j,x)\right] \\ &= 2^{2p}\sum_{i=1}^{n} p_i d^{2p}(x_i,x).\end{aligned} \tag{20}$$

By Jensen's discrete inequality for the power function we also have

$$\begin{aligned}&\left\{\frac{\sum_{i,j=1}^{n} p_i p_j \left[d^2(x_i,x)-2d(x_i,x)\,d(x_j,x)+d^2(x_j,x)\right]}{\sum_{i,j=1}^{n} p_i p_j}\right\}^{p} \\ &\le \frac{\sum_{i,j=1}^{n} p_i p_j \left[d^2(x_i,x)-2d(x_i,x)\,d(x_j,x)+d^2(x_j,x)\right]^{p}}{\sum_{i,j=1}^{n} p_i p_j},\end{aligned}$$

namely

$$\begin{aligned}&2^{p}\left[\sum_{k=1}^{n} p_k d^2(x_k,x)-\left(\sum_{k=1}^{n} p_k d(x_k,x)\right)^{2}\right]^{p} \\ &\le \sum_{i,j=1}^{n} p_i p_j \left[d^2(x_i,x)-2d(x_i,x)\,d(x_j,x)+d^2(x_j,x)\right]^{p}\end{aligned} \tag{21}$$

for all $x \in X$.

By making use of (18), (20) and (21), we get (16).

Since the inequality (19) also holds for $p \ge \frac{1}{2}$, hence the last part of the theorem is also proved. □

**Corollary 2.4.** *With the assumptions of Theorem 2.3 and if*

$$\sup_{x\in S}\left[\sum_{k=1}^{n} p_k d^2(x_k,x)\right] < \infty,$$

*for $S \subset X$, then we have*

$$\begin{aligned}&2^{p-1}\left[\sup_{x\in S}\left(\sum_{k=1}^{n} p_k d^2(x_k,x)\right)-\left(\sup_{x\in S}\sum_{k=1}^{n} p_k d(x_k,x)\right)^{2}\right]^{p} \\ &\le \sum_{1\le i<j\le n} p_i p_j d^{2p}(x_i,x_j) \le 2^{2p-1}\sup_{x\in S}\left[\sum_{i=1}^{n} p_i d^{2p}(x_i,x)\right].\end{aligned} \tag{22}$$

*Proof.* By taking the supremum in (16) we derive

$$2^{p-1}\sup_{x\in S}\left[\sum_{k=1}^{n}p_{k}d^{2}\left(x_{k},x\right)-\left(\sum_{k=1}^{n}p_{k}d\left(x_{k},x\right)\right)^{2}\right]^{p} \tag{23}$$
$$\leq\sum_{1\leq i<j\leq n}p_{i}p_{j}d^{2p}\left(x_{i},x_{j}\right)\leq 2^{2p-1}\sup_{x\in S}\left[\sum_{i=1}^{n}p_{i}d^{2p}\left(x_{i},x\right)\right].$$

Since

$$\sup_{x\in S}\left[\sum_{k=1}^{n}p_{k}d^{2}\left(x_{k},x\right)-\left(\sum_{k=1}^{n}p_{k}d\left(x_{k},x\right)\right)^{2}\right]^{p}$$
$$=\left[\sup_{x\in S}\left\{\sum_{k=1}^{n}p_{k}d^{2}\left(x_{k},x\right)-\left(\sum_{k=1}^{n}p_{k}d\left(x_{k},x\right)\right)^{2}\right\}\right]^{p}$$
$$\geq\left[\sup_{x\in S}\left(\sum_{k=1}^{n}p_{k}d^{2}\left(x_{k},x\right)\right)-\left(\sup_{x\in S}\sum_{k=1}^{n}p_{k}d\left(x_{k},x\right)\right)^{2}\right]^{p},$$

hence by (23) we obtain (22). □

**Remark 2.3.** The case of uniform weights $p_i=\frac{1}{n}$, $i\in\{1,...,n\}$ in (16) and (22) is as follows

$$\left(\frac{2}{n^{2}}\right)^{p-1}\left[n\sum_{k=1}^{n}d^{2}\left(x_{k},x\right)-\left(\sum_{k=1}^{n}d\left(x_{k},x\right)\right)^{2}\right]^{p} \tag{24}$$
$$\leq\sum_{1\leq i<j\leq n}d^{2p}\left(x_{i},x_{j}\right)\leq 2^{2p-1}n\sum_{i=1}^{n}p_{i}d^{2p}\left(x_{i},x\right)$$

and

$$\left(\frac{2}{n^{2}}\right)^{p-1}\left[n\sup_{x\in S}\left(\sum_{k=1}^{n}d^{2}\left(x_{k},x\right)\right)-\left(\sup_{x\in S}\sum_{k=1}^{n}d\left(x_{k},x\right)\right)^{2}\right]^{p} \tag{25}$$
$$\leq\sum_{1\leq i<j\leq n}d^{2p}\left(x_{i},x_{j}\right)\leq 2^{2p-1}n\sup_{x\in S}\left(\sum_{i=1}^{n}d^{2p}\left(x_{i},x\right)\right).$$

## 3. Applications

If $(E, \|\cdot\|)$ is a normed linear space and $x_i \in E$, $i \in \{1, ..., n\}$, $p_i \geq 0$ $(i \in \{1, ..., n\})$ with $\sum_{i=1}^{n} p_i = 1$, then by (6) we have the inequalities

$$\begin{aligned} 0 &\leq \sum_{k=1}^{n} p_k \|x_k - x\|^2 - \left(\sum_{k=1}^{n} p_k \|x_k - x\|\right)^2 \\ &\leq \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^2 \\ &\leq \sum_{k=1}^{n} p_k \|x_k - x\|^2 + \left(\sum_{k=1}^{n} p_k \|x_k - x\|\right)^2 \end{aligned} \tag{26}$$

and

$$\begin{aligned} 0 &\leq \sup_{x \in S} \sum_{k=1}^{n} p_k \|x_k - x\|^2 - \left(\sup_{x \in S} \sum_{k=1}^{n} p_k \|x_k - x\|\right)^2 \\ &\leq \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^2 \\ &\leq \sup_{x \in S} \sum_{k=1}^{n} p_k \|x_k - x\|^2 + \left(\sup_{x \in S} \sum_{k=1}^{n} p_k \|x_k - x\|\right)^2 \end{aligned} \tag{27}$$

for $x \in X$.

If we consider the weighted centre of gravity $\overline{x_p} := \sum_{i=1}^{n} p_i x_i$, then by taking $x = \overline{x_p}$ in (26) we get

$$\begin{aligned} 0 &\leq \sum_{k=1}^{n} p_k \|x_k - \overline{x_p}\|^2 - \left(\sum_{k=1}^{n} p_k \|x_k - \overline{x_p}\|\right)^2 \\ &\leq \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^2 \\ &\leq \sum_{k=1}^{n} p_k \|x_k - \overline{x_p}\|^2 + \left(\sum_{k=1}^{n} p_k \|x_k - \overline{x_p}\|\right)^2 . \end{aligned} \tag{28}$$

From (28) we get

$$\left| \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^2 - \sum_{k=1}^{n} p_k \|x_k - \overline{x_p}\|^2 \right| \leq \left(\sum_{k=1}^{n} p_k \|x_k - \overline{x_p}\|\right)^2 \tag{29}$$

for $x_i \in E$, $i \in \{1, ..., n\}$, $p_i \geq 0$ $(i \in \{1, ..., n\})$ with $\sum_{i=1}^{n} p_i = 1$.

If we take $p_i = \frac{1}{n}$, $i \in \{1, ..., n\}$, then by (29) we get

$$\left| \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 - n \sum_{k=1}^{n} \|x_k - \overline{x}\|^2 \right| \leq \left(\sum_{k=1}^{n} \|x_k - \overline{x}\|\right)^2 , \tag{30}$$

where $\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$.

We observe that if $(E,\langle\cdot,\cdot\rangle)$ is an inner product space, then

$$\begin{aligned}\sum_{1\leq i<j\leq n} p_i p_j \|x_i-x_j\|^2 &= \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \|x_i-x_j\|^2 \\ &= \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \left[\|x_i\|^2 - 2\mathrm{Re}\langle x_i,x_j\rangle + \|x_j\|^2\right] \\ &= \frac{1}{2}\left[\sum_{i=1}^{n} p_i\|x_i\|^2 - 2\mathrm{Re}\left\langle \sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j\right\rangle + \sum_{j=1}^{n} p_j \|x_j\|^2\right] \\ &= \sum_{i=1}^{n} p_i\|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n} p_k\|x_k-\overline{x_p}\|^2 &= \sum_{k=1}^{n} p_k\left[\|x_k\|^2 - 2\mathrm{Re}\langle x_k,\overline{x_p}\rangle + \|\overline{x_p}\|^2\right] \\ &= \sum_{k=1}^{n} p_k\|x_k\|^2 - 2\mathrm{Re}\left\langle \sum_{k=1}^{n} p_k x_k, \overline{x_p}\right\rangle + \|\overline{x_p}\|^2 \\ &= \sum_{k=1}^{n} p_k\|x_k\|^2 - \left\|\sum_{k=1}^{n} p_k x_k\right\|^2,\end{aligned}$$

which shows that, in the case of inner product spaces

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i-x_j\|^2 = \sum_{k=1}^{n} p_k\|x_k-\overline{x_p}\|^2, \tag{31}$$

for $x_i\in E$, $i\in\{1,...,n\}$, $p_i\geq 0$ $(i\in\{1,...,n\})$ with $\sum_{i=1}^{n} p_i = 1$.

However, in the general case of normed linear spaces the identity (31) does not hold for all sequences of vectors and probability densities as above. Therefore the inequality (29) can be seen as an error bound in approximating $\sum_{1\leq i<j\leq n} p_i p_j \|x_i-x_j\|^2$ by $\sum_{k=1}^{n} p_k\|x_k-\overline{x_p}\|^2$ in the general case of normed linear spaces.

From the inequality (16) we also obtain

$$\begin{aligned}&2^{p-1}\left[\sum_{k=1}^{n} p_k\|x_k-x\|^2 - \left(\sum_{k=1}^{n} p_k\|x_k-x\|\right)^2\right]^p \\ &\leq \sum_{1\leq i<j\leq n} p_i p_j \|x_i-x_j\|^{2p} \leq 2^{2p-1}\sum_{i=1}^{n} p_i\|x_i-x\|^{2p}\end{aligned} \tag{32}$$

for $x$, $x_i\in E$, $i\in\{1,...,n\}$, $p_i\geq 0$ $(i\in\{1,...,n\})$ with $\sum_{i=1}^{n} p_i = 1$. The second inequality in (32) also holds for $p\geq\frac{1}{2}$.

In particular, we derive

$$2^{p-1}\left[\sum_{k=1}^{n} p_k \left\|x_k-\overline{x_p}\right\|^2-\left(\sum_{k=1}^{n} p_k \left\|x_k-\overline{x_p}\right\|\right)^2\right]^p \tag{33}$$

$$\leq \sum_{1\leq i<j\leq n} p_i p_j \left\|x_i-x_j\right\|^{2p} \leq 2^{2p-1}\sum_{i=1}^{n} p_i \left\|x_i-\overline{x_p}\right\|^{2p}$$

for $x_i\in E$, $i\in\{1,...,n\}$, $p_i\geq 0$ $(i\in\{1,...,n\})$ with $\sum_{i=1}^{n} p_i=1$.

Finally, in the case of inner product spaces, we may point out an upper bound as follows.

**Proposition 3.1.** *Let $(E,\langle\cdot,\cdot\rangle)$ be an inner product space, $x_i\in E,(i\in\{1,...,n\})$ and assume that there exists the vectors $a$, $A\in E$ so that either*

$$Re\langle A-x_i,x_i-a\rangle\geq 0,\ \textit{for } i\in\{1,...,n\},$$

*or, equivalently,*

$$\left\|x_i-\frac{a+A}{2}\right\|\leq\frac{1}{2}\left\|A-a\right\|,\ \textit{for } i\in\{1,...,n\}.$$

*Then for any $p_i\geq 0$ $(i\in\{1,...,n\})$ with $\sum_{i=1}^{n} p_i=1$ one has the inequality*

$$\sum_{1\leq i<j\leq n} p_i p_j \left\|x_i-x_j\right\|^{2p}\leq\frac{1}{2}\left\|A-a\right\|^{2p} \tag{34}$$

*for $p\geq\frac{1}{2}$.*

Indeed, we have by the last inequality in (32) that

$$\sum_{1\leq i<j\leq n} p_i p_j \left\|x_i-x_j\right\|^{2p}\leq 2^{2p-1}\sum_{i=1}^{n} p_i\left\|x_i-\frac{a+A}{2}\right\|^{2p}\leq\frac{1}{2}\left\|A-a\right\|^{2p},$$

which proves the statement.

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