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# Mathematical Analysis and Numerical Simulation of a Strongly Nonlinear Singular Model

Maryem Hourri, Laila Taourirte, and Nour Eddine Alaa

Abstract. In this paper, we are interested in the one-dimensional singular optimization problem with constraints:

$$\text{Min}\left\{\mathcal{J}(v) = \frac{1}{p}\int_{-1}^{1}|v_x|^p + \frac{1}{\gamma-1}\int_{-1}^{1}v^{1-\gamma}, v(\pm 1) = 0 \text{ and } v(0) = d\right\},$$

where $1 < p < \infty$, $1 < \gamma < \frac{2p-1}{p-1}$ and $d > 0$.

In the first part of the paper, we show the existence of a critical value $d^* > 0$ such that if $d \leq d^*$, $\mathcal{J}$ admits a minimum in a carefully chosen closed convex set of $W_0^{1,p}(-1,1)$. The second part of the paper is dedicated to numerical simulations. We elaborate a numerical algorithm that transforms our constrained optimization problem into the solution of a system of ordinary differential equations. Illustrative examples are given to verify the efficiency and accuracy of the proposed numerical method to test the relevance of the proposed approach. We point out that the numerical results obtained are in good agreement with the physical phenomenon of pleated graphene in the particular case $p = 4$ and $\gamma = 9/5$ [12].

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## 1. Introduction

The aim of this work is to study the following constrained optimization problem

$$\begin{cases} \text{minimize } \mathcal{J}(v) = \frac{1}{p}\int_{-1}^{1}|v_x|^p + \dfrac{1}{\gamma-1}\int_{-1}^{1}v^{1-\gamma}, \\ \\ \qquad \text{subject to } v(\pm 1) = 0 \text{ and } v(0) = d > 0, \end{cases} \tag{1}$$

where $1 < p < \infty$ and $1 < \gamma < \frac{2p-1}{p-1}$.

This type of problem naturally arises in many physical phenomena among which we cite, for appropriate values of $p$ and $\gamma$, the folding of graphene. Since then, the latter has attracted a lot of attention because of its exceptional mechanical and electrical characteristics, which are exploited in many projects,[18] including nanomechanical systems, nanoelectronics, [20, 13] and nano-composites, [14, 19, 8].
The mathematical analysis of this type of singular equations has attracted the attention of several authors in recent years. We cite in particular the following works [1], [2], [3], [4], [10].

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In the present work, we pursue the natural question of whether there exists a minimizer that satisfies the smoothness of the catenary form. In particular, we will focus on a numerical algorithm for minimizing the elastic energy Eq. (1), with a view to identifying a smooth approximate numerical solution that can be used to improve the numerical results of the reference [12].

For this purpose, we propose a numerical algorithm to minimize the equation (1), in order to identify a smooth (local) minimizer. This optimization problem may lead to singular minimizers that do not satisfy the Euler-Lagrange equation [**?**] or in other cases to the appearance of Lavrentiev phenomena [6, 5]. This could make the numerical problem very difficult to handle. Concerning the above point, we seek a simple analysis by imposing a different optimization space. Remind that the case $p = 4$ and $\gamma = 9/5$ comes from the crumpled graphene problem [11]. The idea is to determine a function $\phi \in W_0^{1,p}(-1,1)$ to be specified later, which satisfies the requirement of smoothing the catenary profile, and such that the minimum sought noted $v^*$ satisfies $v^* \geq \phi$. We will then minimize the energy functional Eq. (1) over the following convex admissible set

$$K = \left\{ v \in W_0^{1,p}(-1,1), v \geq \phi \text{ and } v(0) = d \right\}. \tag{2}$$

Therefore, the problem can be formulated as follows: find the range of parameters $d$ such that the energy function Eq. (1) has a minimum $v^*$ for a given $\phi > 0$ in $(-1,1)$ that satisfies $\phi(0) = 1$.

As we will later see, one of the candidates for the function $\phi$ is $v_1 = d\, \phi_1^{\frac{p}{\gamma+p-1}}$, in which $\phi_1$ is the first eigenfunction associated to the $p$-laplacian given in Definition 1.1 below.

A particular form of the problem consists in finding a (minimizing) function $v^\star$ that solves the following problem (see below)

$$\begin{cases} -(|v^*{}_x|^{p-2}\, v^*{}_x)_x = v^{*-\gamma} + \Lambda^*\ \delta_0 & \mathcal{D}'(-1,1), \\ v^* \in K, \end{cases} \tag{3}$$

where $\Lambda^*$ is a nonnegative constant and $\delta_0$ is a Dirac mass at the origin. Let us note that there are many works that investigate this kind of problems involving the singular $p-$Laplacian operator [11, 9] but,to our knowledge, there is no publication of such a double-constrained problem with a nonlinear singular source term.
In what follows, we aim to show the existence of a suitable weak solution to Eq. (3). The first step is to precise in which sense we want to solve our problem. Let us first recall the definition of the first eigenfunction of the $p$-Laplacian for $1 < p < \infty$.

**Definition 1.1.** For $a < b$, the first eigenfunction associated to the smallest eigenvalue $\lambda_1 > 0$ is the unique solution $\phi_1 \in W_0^{1,p}(a,b)$ such that $||\phi_1||_{L^p} = 1$, satisfying

$$\begin{cases} -\left(|\phi_{1x}|^{p-2}\, \phi_{1x}\right)_x = \lambda_1\, |\phi_1|^{p-2}\, \phi_1 \quad \text{in } (a,b), \\ \phi_1 > 0 \text{ in } (a,b), \\ \phi_1(a) = \phi_1(b) = 0. \end{cases} \tag{4}$$

Moreover, $\lambda_1 = \left(\frac{\pi_p}{b-a}\right)^p$, where $\pi_p := \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin(\frac{\pi}{p})}$.

In what follows, we adopt $\phi = \phi_1^{\frac{p}{\gamma+p-1}}$ and we define the following non-empty closed convex subset of $W_0^{1,p}(-1,1)$

$$K = \left\{ v \in W_0^{1,p}(-1,1) \; / \; v \geq d\phi_1^{\frac{p}{\gamma+p-1}} \text{ and } v(0) = d \right\}. \tag{5}$$

**Remark 1.1.** It should be noted that if $v \in K$, then $0 \leq \mathcal{J}(v) < \infty$. In fact, $v \geq d\phi_1^{\frac{p}{\gamma+p-1}}$, and then by a Lemma of Lazer and McKenna in Ref. [10], we conclude that $\phi_1^{\frac{p(1-\gamma)}{\gamma+p-1}} \in L^1(-1,1)$ if and only if $\gamma < \frac{2p-1}{p-1}$, i.e $v^{1-\gamma} \in L^1(-1,1)$ under the constraint imposed on $\gamma$. Furthermore, $v^{-\gamma} \in L^\infty_{loc}(-1,1)$ and $v_x(\pm 1) = \infty$.

**Definition 1.2.** A weak solution to Eq. (3) is a function $v$ such that

$$\begin{cases} v \in K, \\ \displaystyle\int_{-1}^{1} |v_x|^{p-2} v_x \varphi_x = \int_{-1}^{1} v^{-\gamma}\varphi + \Lambda^* d, \; \forall \varphi \in C_c^1(-1,1). \end{cases} \tag{6}$$

This article is structured as shown below: Section 2 is dedicated to proving the existence of a minimizer of the energy functional that satisfies the smoothness requirement of the catenary form, in a suitable convex optimization set and formulating the Euler-Lagrange equation. In section 3, we propose a numerical algorithm using the Lagrange multiplier approach of [21], which converts the constrained minimization problem into the solution of a system of partial differential equations. We discuss the implementation of the algorithm using a finite difference method for the approximation in space and a Runge-Kutta 4 method for the approximation in time. Finally, in the section 4 we show the results obtained and we conclude with a discussion and some remarks.

## 2. Existence result: sufficient condition for existence

The objective of this section is to study the existence of an optimal solution to the minimization problem (1). In the following theorem, we state our main existence result.

**Theorem 2.1.** *Let $1 < p < \infty$ and $1 < \gamma < \frac{2p-1}{p-1}$. There exists $d^* > 0$ such that for all $0 < d \leq d^*$,*

$$\inf_{v \in K} \mathcal{J}(v) = \min_{v \in K} \mathcal{J}(v) = \mathcal{J}(v^*). \tag{7}$$

*Moreover, $v^*$ satisfies*

$$\begin{cases} -(|{v^*}_x|^{p-2}\, {v^*}_x)_x = {v^*}^{-\gamma} + \Lambda^* \; \delta_0 \qquad \mathcal{D}'(-1,1), \\ \qquad\qquad v^* \in K, \end{cases} \tag{8}$$

*where $\Lambda^* = \frac{1}{d}\left[ \int_{-1}^{1} |{v^*}_x|^p - {v^*}^{1-\gamma} \right]$ and $\delta_0$ is a Dirac mass at the origin.*

*Proof.* Let $\beta < 1$ a constant to be determined and let $v_1 = d{\phi_1}^{\beta}$.
Recalling that $\phi_1$ is the first eigenfunction of the $p$-Laplacian, direct computations

shows that

$$\begin{aligned}
-\left(|v_{1x}|^{p-2}\, v_{1x}\right)_x &= -d^{p-1}\beta^{p-1}\left(|\phi_{1x}|^{p-2}\, \phi_{1x}\ {\phi_1}^{(p-1)(\beta-1)}\right)_x \\
&= d^{p-1}\beta^{p-1}[-\left(|\phi_{1x}|^{p-2}\, \phi_{1x}\right)_x {\phi_1}^{(p-1)(\beta-1)} \\
&\quad + (p-1)(1-\beta){\phi_1}^{(p-1)(\beta-1)-1}\,|\phi_{1x}|^p] \\
&= d^{p-1}\beta^{p-1}\left[\lambda_1\phi_1^{\beta(p-1)} + (p-1)(1-\beta){\phi_1}^{(p-1)(\beta-1)-1}\,|\phi_{1x}|^p\right] \\
&= {v_1}^{-\gamma} d^{\gamma+p-1}\beta^{p-1}[\lambda_1\phi_1^{\beta(\gamma+p-1)} \\
&\quad + (p-1)(1-\beta){\phi_1}^{\beta(\gamma+p-1)-p}\,|\phi_{1x}|^p].
\end{aligned}$$

Thus, for $\beta = \frac{p}{\gamma+p-1}$, we get

$$-\left(|v_{1x}|^{p-2}\, v_{1x}\right)_x = g(x,d)\ {v_1}^{-\gamma}, \tag{9}$$

where

$$g(x,d) = d^{\gamma+p-1}\left[\beta^{p-1}(1-\beta)(p-1)\,|\phi_{1x}|^p + \lambda_1\beta^{p-1}{\phi_1}^p)\right]. \tag{10}$$

Note that the strong maximum and boundary point principles from Vasquez [16] guarantee $\phi_1 > 0$ in $(-1,1)$ and $|\phi_{1x}| \neq 0$ on the boundary. Hence

$$\Gamma_1 := max\left[\beta^{p-1}(1-\beta)(p-1)\,|\phi_{1x}|^p + \lambda_1\beta^{p-1}{\phi_1}^p)\right] > 0, \tag{11}$$

which means that $g(x,d) \leq 1$ if and only if $d \leq \left(\dfrac{1}{\Gamma_1}\right)^{\frac{1}{\gamma+p-1}} := d^*$.

Finally, we deduce that for $d \leq d^*$, the function $v_1$ satisfies

$$\begin{cases} -(|v_{1x}|^{p-2}\, v_{1x})_x \leq {v_1}^{-\gamma} & \mathcal{D}'(-1,1), \\ \qquad\qquad v_1 \in K. \end{cases} \tag{12}$$

$\Lambda^* > 0$ and $\delta_0$ is a nonnegative measure, hence, $v_1$ is clearly a sub-solution of (3) that verifies

$$\begin{cases} -(|v_{1x}|^{p-2}\, v_{1x})_x \leq {v_1}^{-\gamma} + \Lambda^*\delta_0, & \mathcal{D}'(-1,1), \\ \qquad\qquad v_1 \in K. \end{cases} \tag{13}$$

We insert $v_1$ as a test function in the weak formulation of (13) and we obtain

$$\int_{-1}^{1} |v_{1x}|^p \leq \int_{-1}^{1} {v_1}^{1-\gamma} + \Lambda^* d. \tag{14}$$

Consequently

$$\begin{aligned}
\mathcal{J}(v_1) &= \frac{1}{p}\int_{-1}^{1} |v_{1x}|^p + \frac{1}{\gamma-1}\int_{-1}^{1} {v_1}^{1-\gamma} && (15)\\
&\leq \left(\frac{1}{p} + \frac{1}{\gamma-1}\right)\int_{-1}^{1} {v_1}^{1-\gamma} + \Lambda^* d && (16)\\
&\leq \left(\frac{1}{p} + \frac{1}{\gamma-1}\right)\int_{-1}^{1} d^{1-\gamma}\phi_1^{\frac{p(1-\gamma)}{\gamma+p-1}} + \Lambda^* d. && (17)
\end{aligned}$$

$v_1 \in K$, thus $K$ is a non-empty closed convex of $L^p(-1,1)$. Therefore one can take a minimizing sequence $(v_n)_n$ in $K$ i.e a sequence such that $\mathcal{J}(v_n) \to \inf\limits_{v\in K} \mathcal{J}(v) := L$.

Hence, for all $\varepsilon > 0$, there exists $\eta_0 > 0$, such that for all $\eta > \eta_0$, we have

$$\frac{L}{2} \leq \mathcal{J}(v_n) \leq \frac{3L}{2}, \tag{18}$$

thus

$$\int_{-1}^{1} |v_{nx}|^p \, dx \leq p \left( \frac{3L}{2} - \frac{1}{\gamma - 1} \int_{-1}^{1} v_n^{1-\gamma} \right) dx. \tag{19}$$

Again, since $1 < \gamma < \frac{2p-1}{p-1}$, then ${v_n}^{1-\gamma} \leq d^{1-\gamma} {\phi_1}^{\frac{p(1-\gamma)}{\gamma+p-1}}$ which belongs to $L^1(-1,1)$. This ensures the uniform boundedness of $v_n$ in $W_0^{1,p}(-1,1)$.
We pick a subsequence, still denoted $v_n$ that converges to $v^*$ weakly in $W_0^{1,p}(-1,1)$, strongly in $L^p(-1,1)$ and $v_n(x) \longrightarrow v^*(x)$ a.e in $(-1,1)$.
Since $W_0^{1,p}$ is injected in the space of Holder continuous functions $C^\alpha$, it follows that $v_n$ converges uniformly to $v^*$ in $K$, and $v_n^{1-\gamma}$ converges to ${v^*}^{1-\gamma}$ in $L^1(-1,1)$.
Finally, we deduce that $L \leq \mathcal{J}(v^*) \leq \liminf_{n \to +\infty} J(v_n) = L$, i.e. $\mathcal{J}(v^*) = \inf_{v \in K} \mathcal{J}(v)$.
Next, we prove the second result of the theorem.
Let $\phi \in \mathcal{C}_c^\infty(-1,1)$. We can easily check that there exists $\delta > 0$ sufficiently small, and $\alpha \geq d$ such that $\frac{v^* + t\phi}{1 + t\phi(0)} \geq \alpha \, \phi_1^\beta$, $\forall t \in (-\delta, \delta)$.

Thus

$$\forall t \in (-\delta, \delta) \quad \frac{v^* + t\phi}{1 + t\phi(0)} \in K. \tag{20}$$

We have

$$\begin{aligned} \frac{\partial}{\partial t} \left( \mathcal{J} \left( \frac{v^* + t\phi}{1 + t\phi(0)} \right) \right)_{t=0} &= \int_{-1}^{1} |v_x^*|^{p-2} \, v_x^* \, \frac{\partial}{\partial t} \left( \frac{v_x^* + t\phi_x}{1 + t\phi(0)} \right) - {v^*}^{-\gamma} \, \frac{\partial}{\partial t} \left( \frac{v^* + t\phi}{1 + t\phi(0)} \right) \\ &= \int_{-1}^{1} |{v^*}_x|^{p-2} \, {v^*}_x \, \phi_x - \int_{-1}^{1} {v^*}^{-\gamma} \, \phi \\ &\quad - \frac{1}{d} \left( \int_{-1}^{1} |{v^*}_x|^p - {v^*}^{1-\gamma} \right) \phi(0) \end{aligned}$$

Since $v^*$ is a minimum, we obtain for all $\phi \in \mathcal{C}_c^\infty(-1,1)$

$$\int_{-1}^{1} \left( |{v^*}_x|^{p-2} \, {v^*}_x \, \phi_x - {v^*}^{-\gamma} \, \phi \right) = \frac{1}{d} \left( \int_{-1}^{1} |{v^*}_x|^p - {v^*}^{1-\gamma} \right) <\delta_0, \phi>, \tag{21}$$

which means that equation (8) is satisfied in the sense of distributions. $\square$

## 3. Numerical algorithm and results

**3.1. Determination of the first eigenfunction of the p-Laplacian.** In order to numerically determine the first eigenfunction $\phi_1$ associated to the smallest eigenvalue that verifies Eq. (4), we use the Lagrange multiplier approach, which is based on solving the system of equations which constitute the necessary conditions of optimality for the equality constrained problem given by

$$\begin{aligned} &\text{Minimize} \quad \frac{1}{p}\int_{-1}^{1} |\phi_x|^p \\ &\text{subject to} \quad \int_{-1}^{1} |\phi|^p - 1 = 0. \end{aligned} \tag{22}$$

The Lagrange function associated to Eq. (22) is defined by

$$\mathcal{L}(\phi,\lambda) = \frac{1}{p}\int_{-1}^{1} |\phi_x|^p + \lambda\left(\int_{-1}^{1} |\phi|^p - 1\right), \tag{23}$$

where $\lambda \in \mathbb{R}$ is referred to as the Lagrange multiplier.
The first-order necessary condition of optimality can be obtained as a stationary point $(\phi^*, \lambda^*)$ of $\mathcal{L}(\phi,\lambda)$ over $\phi$ and $\lambda$ described by

$$\begin{cases} \left.\dfrac{\partial \phi}{\partial t}\right|_{(\phi^*,\lambda^*)} = 0, \\ \left.\dfrac{d\lambda}{dt}\right|_{(\phi^*,\lambda^*)} = 0. \end{cases} \tag{24}$$

That is a stationary point of the system

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(t,x) = -\dfrac{\partial \mathcal{L}}{\partial \phi}(\phi,\lambda), \\ \dfrac{d\lambda}{dt}(t) = \dfrac{\partial \mathcal{L}}{\partial \lambda}(\phi,\lambda). \end{cases} \tag{25}$$

In our case, we have

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(t,x) = \left(|\phi_x|^{p-2}\,\phi_x\right)_x - \lambda p |\phi|^{p-2}\phi, \\ \phi(t,-1) = \phi(t,1) = 0, \\ \dfrac{d\lambda}{dt}(t) = \displaystyle\int_{-1}^{1} |\phi|^p - 1, \end{cases} \tag{26}$$

with $\phi(0,x) = (1-x^2)$ and $\lambda(0) = \lambda_0 > 0$.
Concerning the numerical simulation of the system (26), we use a finite difference method for the approximation in space and a method of Runge-Kutta 4 for the approximation in time. Numerical results are given in Fig. 1, Fig. 2, Fig. 3 and Fig. 4.

**3.2. Determination of the solution $v^*$.** The aim of this part is to numerically determine a minimizer $v^*$ of the energy functional Eq. (1) given by

$$\begin{cases} \mathcal{J}(v) = \frac{1}{p}\displaystyle\int_{-1}^{1} |v_x|^p + \frac{1}{\gamma-1}\int_{-1}^{1} v^{1-\gamma}, \\ v \in K. \end{cases}$$

This optimization problem involves both equality and inequality constraints. We first transform it into an equivalent problem that includes only equalities so that the

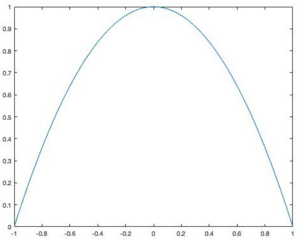

FIGURE 1. First eigenfunction of the $p$-Laplacian for $p = 2$.

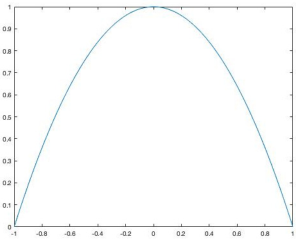

FIGURE 2. First eigenfunction of the $p$-Laplacian for $p = 3$.

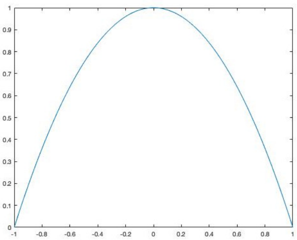

FIGURE 3. First eigenfunction of the $p$-Laplacian for $p = 4$.

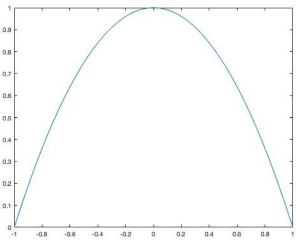

FIGURE 4. First eigenfunction of the $p$-Laplacian for $p = 5$.

theory of [21] can be applied in a practical way. First, we observe that $K$ can be rewritten as

$$K = \left\{ v \in W_0^{1,p}(-1,1) \;/\; v \geq d\phi_1^{\frac{p}{\gamma+p-1}} \text{ and } v(0) \leq d \right\}.$$

The problem can then be expressed as

$$\begin{aligned} &\text{Minimize} \qquad \mathcal{J}(v) = \tfrac{1}{p}\int_{-1}^{1} |v_x|^p + \frac{1}{\gamma-1}\int_{-1}^{1} v^{1-\gamma}, \\ &\text{subject to} \qquad d\phi_1^{\frac{p}{\gamma+p-1}} - v \leq 0 \text{ and } v(0) - d \leq 0. \end{aligned} \tag{27}$$

Besides the nonlinear nature of the energy, the constraints are not standard, which adds some complexity. To avoid this, we introduce additional variables $y_1$ and $y_2$, and consider the following nonlinear programming problem in which the two inequality constraints are transformed into two equalities.

$$\begin{aligned} &\text{Minimize} \qquad \mathcal{J}(v) = \tfrac{1}{p}\int_{-1}^{1} |v_x|^p + \frac{1}{\gamma-1}\int_{-1}^{1} v^{1-\gamma}, \\ &\text{subject to} \qquad d\phi_1^{\frac{p}{\gamma+p-1}} - v + y_1^2 = 0 \text{ and } v(0) - d + y_2^2 = 0. \end{aligned} \tag{28}$$

The terms $y_i^2$, $i = 1, 2$ can be replaced by any differentiable positive functions of $y_i$ with suitable dynamic range [21]. But for simplicity, $y_i^2$ are adopted in what follows. Based on the new constraints ( with equalities instead of inequalities), we define the associated Lagrangian function $\mathcal{L}(v, y_1, \mu, y_2, \sigma)$ as follows

$$\begin{aligned} \mathcal{L}(v, y_1, \mu, y_2, \sigma) = \frac{1}{p}\int_{-1}^{1} |v_x|^p \quad &+ \quad \frac{1}{\gamma-1}\int_{-1}^{1} v^{1-\gamma} + \int_{-1}^{1} \mu\left(d\phi_1^{\frac{p}{\gamma+p-1}} - v + y_1^2\right) \\ &+ \quad \int_{-1}^{1} \sigma\left(v(0) - d + y_2^2\right). \end{aligned}$$

We point out that the Lagrangian function $\mathcal{L}(v, y_1, \mu, y_2, \sigma)$ is a function of $W_0^{1,p}(-1,1)\times W_0^{1,p}(-1,1) \times L^{p'}(-1,1) \times \mathbb{R} \times \mathbb{R}$ with values in $\mathbb{R}$.

The stationary point $(v^*, {y_1}^*, \mu^*, {y_2}^*, \sigma^*)$ of $\mathcal{L}(v, y_1, \mu, y_2, \sigma)$ is described by the stationary point of the following dynamic equations that can be briefly written as

$$\begin{cases} \dfrac{\partial v}{\partial t} = -\dfrac{\partial \mathcal{L}}{\partial v}(v, y_1, \mu, y_2, \sigma), \\ \dfrac{\partial y_1}{\partial t} = -\dfrac{\partial \mathcal{L}}{\partial y_1}(v, y_1, \mu, y_2, \sigma), \\ \dfrac{\partial \mu}{\partial t} = \dfrac{\partial \mathcal{L}}{\partial \mu}(v, y_1, \mu, y_2, \sigma), \\ \dfrac{\partial y_2}{\partial t} = -\dfrac{\partial \mathcal{L}}{\partial y_2}(v, y_1, \mu, y_2, \sigma), \\ \dfrac{\partial \sigma}{\partial t} = \dfrac{\partial \mathcal{L}}{\partial \sigma}(v, y_1, \mu, y_2, \sigma). \end{cases} \tag{29}$$

That is

$$\begin{cases} \dfrac{\partial v}{\partial t} = (|v_x|^{p-2}\, v_x)_x + v^{-\gamma} + \mu - \sigma\delta_0, \\ \dfrac{\partial y_1}{\partial t} = -2\mu y_1, \\ \dfrac{\partial \mu}{\partial t} = d\phi_1^{\frac{p}{\gamma+p-1}} - v + y_1^2, \\ \dfrac{\partial y_2}{\partial t} = -2\sigma y_2, \\ \dfrac{\partial \sigma}{\partial t} = v(0) - d + y_2^2. \end{cases} \tag{30}$$

**3.3. Numerical results.** As previously, for the numerical simulation of the system (30), we employ a finite difference method for the spatial approximation and a Runge-Kutta 4 method for the temporal approximation. We introduce $\tau$ to be the time step size $t_n = n \times \tau$, $\quad n = 0, 1, 2, ....,$ and we denote by $h$ a step of space. We define the nodes of a regular meshing of $[-1, 1]$ by $x_i = (i - N - 1) \times h$, $\quad i = 0, 1, 2, ...., 2N + 1$. Let's denote by $(v_i^n)$ the approximation of $v(t_n, x_i)$. Then the discrete approximation of the p-laplacien operator is as follows:

- if $p \geq 2$

$$
\begin{aligned}
\left(|v_x|^{p-2} v_x\right)_x(t_n, x_i) &\simeq \frac{1}{h^p}\left(\left|v_{i+1}^n - v_i^n\right|^{p-2}(v_{i+1}^n - v_i^n)\right) \\
&- \frac{1}{h^p}\left(\left|v_i^n - v_{i-1}^n\right|^{p-2}(v_i^n - v_{i-1}^n)\right)
\end{aligned}
$$

- if $1 < p < 2$

$$
\begin{aligned}
\left(|v_x|^{p-2} v_x\right)_x(t_n, x_i) &\simeq \frac{1}{h^p}\left(\frac{1}{\sqrt{\left|v_{i+1}^n - v_i^n\right|^{\frac{p-2}{2}} + \varepsilon}}(v_{i+1}^n - v_i^n)\right) \\
&- \frac{1}{h^p}\left(\frac{1}{\sqrt{\left|v_i^n - v_{i-1}^n\right|^{\frac{p-2}{2}} + \varepsilon}}(v_i^n - v_{i-1}^n)\right)
\end{aligned}
$$

The other terms of the system are approximated in the following way:

- The singular term $v^{-\gamma}$ is approached by the $C^1$ regularization $f_\varepsilon$ given for each $\varepsilon \in (0,1]$ by

$$
f_\varepsilon(s) = \begin{cases} s^{-\gamma} & \text{if } s > \varepsilon, \\ (1+\gamma)\varepsilon^{-\gamma} - \gamma\varepsilon^{-\gamma-1}s & \text{if } s \leq \varepsilon. \end{cases}
$$

  We note that $f_\varepsilon$ is decreasing and positive on $\mathbb{R}$.

- The Dirac mass $\delta_0$ is approached as usual by

$$
\delta_\varepsilon(x) = \frac{1}{\pi} \cdot \frac{\varepsilon}{\varepsilon^2 + x^2}. \tag{31}
$$

  It is known that $\lim\limits_{\varepsilon \to 0} \delta_\varepsilon = \delta_0$ in the sense of distributions [7].

In all the numerical simulations, we choose $\varepsilon = 3\mathrm{e}-4$ and $h = 0.005$.
In Fig. 5 to Fig. 12, we present some numerical results obtained for different values of $1 < p < \infty$ and $1 < \gamma < \frac{2p-1}{p-1}$. For a useful comparison, three different values of $d$ are considered, namely $d = d^*$, $d = \dfrac{d^*}{2}$ and $d = \dfrac{d^*}{3}$. Recall that $d^*$ is the parameter obtained analytically as follows: $d^* = \left(\dfrac{1}{\Gamma_1}\right)^{\frac{1}{\gamma+p-1}}$, where

$$
\Gamma_1 := max\left[\beta^{p-1}(1-\beta)(p-1)\left|\phi_{1x}\right|^p + \lambda_1\beta^{p-1}{\phi_1}^p\right).
$$

In Fig. 13 to Fig. 15, for each value of $p$, we choose a value of $\gamma$ associated to $p$, and we plot the three-dimensional representation of the solution $v^*$ for $d = d^*$.

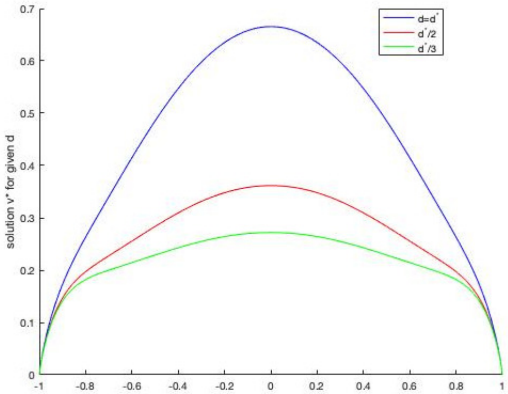


FIGURE 5. Numerical solution for $v^*$, for $p = 2$, $\gamma = 1.2$, $d^* = 0.6633$ and given $d$: $d = d^*$ (blue curve), $d = d^*/2$ (red curve) and $d = d^*/3$ (green curve).

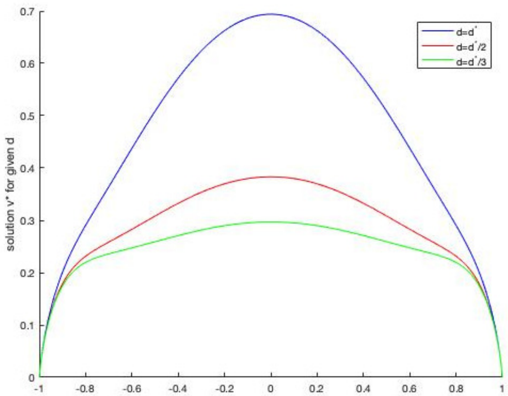


FIGURE 6. Numerical solution for $v^*$ for $p = 2$, $\gamma = 1.45$, $d^* = 0.6917$ and given $d$: $d = d^*$ (blue curve), $d = d^*/2$ (red curve) and $d = d^*/3$ (green curve)

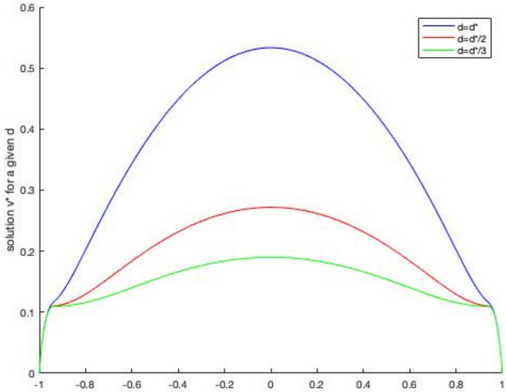


FIGURE 7. Numerical solution for $v^*$ for $p = 3$, $\gamma = 2$, $d^* = 0.5318$ and given $d$: $d = d^*$ (blue curve), $d = d^*/2$ (red curve) and $d = d^*/3$ (green curve)

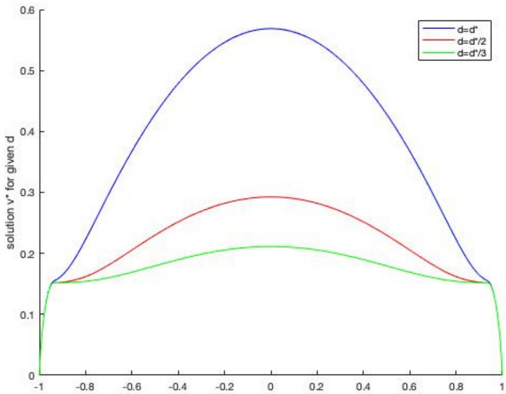


FIGURE 8. Numerical solution for $v^*$ for $p = 3$, $\gamma = 2.45$, $d^* = 0.5669$ and given $d$: $d = d^*$ (blue curve), $d = d^*/2$ (red curve) and $d = d^*/3$ (green curve)

## 4. Discussion and conclusion

The major objective of this paper is to study analytically and numerically the one-dimensional constrained singular optimization problem Eq. (1). The goal of our study

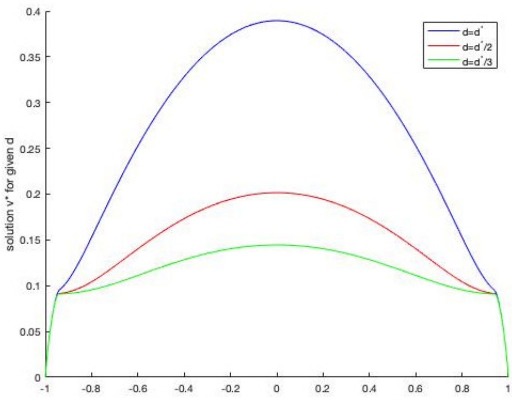


FIGURE 9. Numerical solution for $v^*$ for $p = 4$, $\gamma = 9/5$, $d^* = 0.3171$ and given $d$: $d = d^*$ (blue curve), $d = d^*/2$ (red curve) and $d = d^*/3$ (green curve)

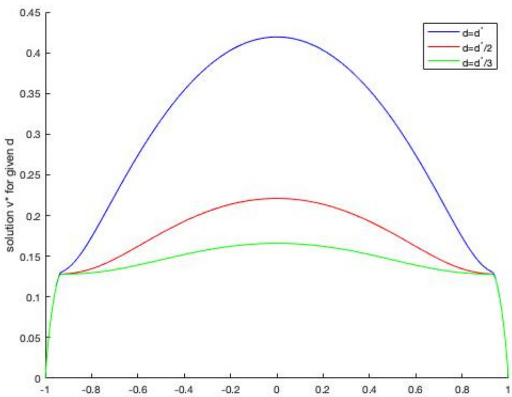


FIGURE 10. Numerical solution for $v^*$ for $p = 4$, $\gamma = 2.2$, $d^* = 0.4164$ and given $d$: $d = d^*$ (blue curve), $d = d^*/2$ (red curve) and $d = d^*/3$ (green curve)

is twofold: on the one hand, we show the existence of a minimum of $\mathcal{J}$ under the condition $d \leq d^*$, and on the other hand, we provide an algorithm that transforms the constrained optimization problem into the solution of the system of equations

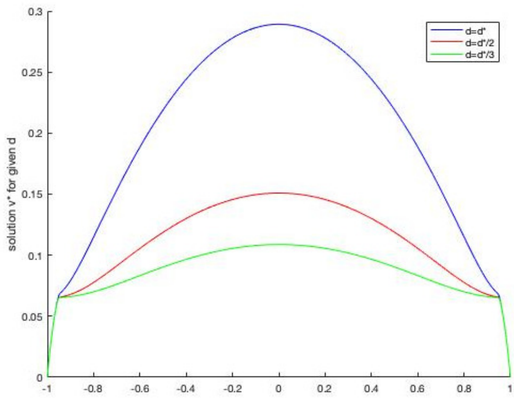


FIGURE 11. Numerical solution for $v^*$ for $d$, $p = 5$, $\gamma = 1.5$, $d^* = 0.2863$ and given $d$: $d = d^*$ (blue curve), $d = d^*/2$ (red curve) and $d = d^*/3$ (green curve)

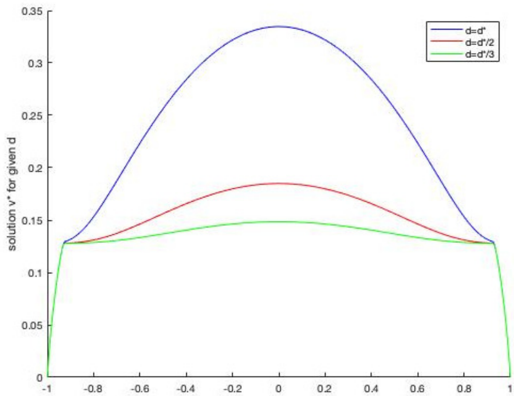


FIGURE 12. Numerical solution for $v^*$ for $p = 5$, $\gamma = 2.2$, $d^* = 0.3297$ and given $d$: $d = d^*$ (blue curve), $d = d^*/2$ (red curve) and $d = d^*/3$ (green curve)

that constitute the necessary conditions for optimality, using the Lagrange multiplier approach.

In figures 5 to 12, we have summarized our numerical results for different values

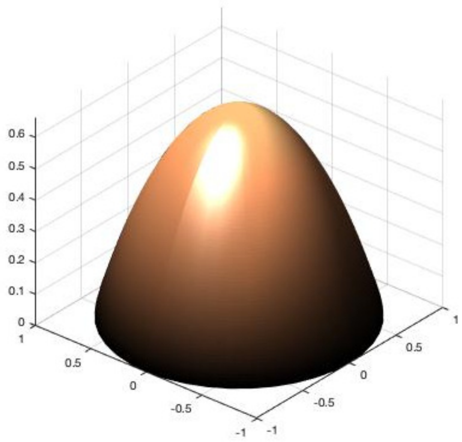


FIGURE 13. The 3d-representation of $v^*$ for $p = 2$, $\gamma = 1.2$, $d^* = 0.6633$ and $d = d^*$

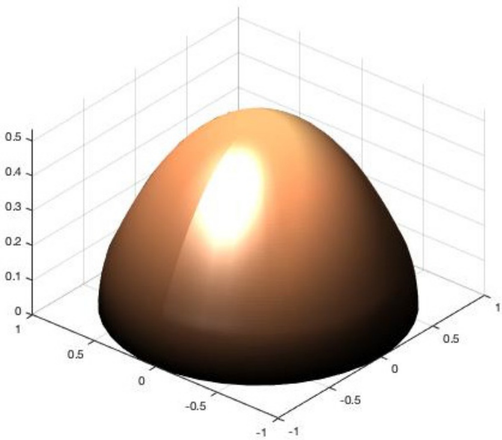


FIGURE 14. The 3d-representation of $v^*$ for $p = 3$, $\gamma = 2$, $d^* = 0.5318$ and $d = d^*$

of the parameters $p$, $gamma$ and $d$. For any fixed $p$ and $gamma$, the numerical

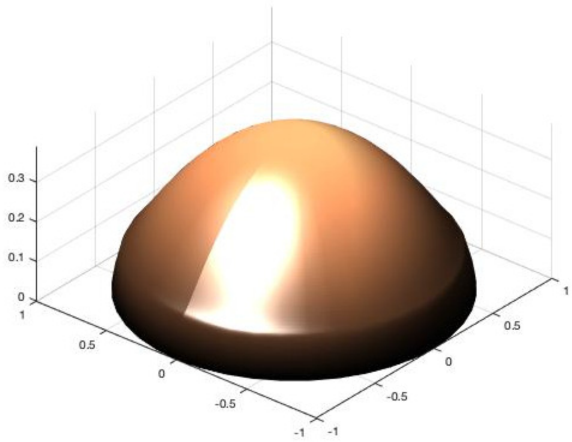


FIGURE 15. The 3d-representation of $v^*$ for $p = 4$, $\gamma = 9/5$, $d^* = 0.3171$ and $d = d^*$

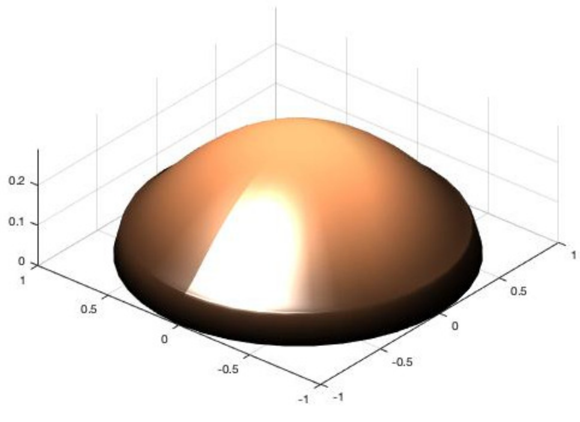


FIGURE 16. The 3d-representation of $v^*$ for $p = 5$, $\gamma = 1.5$, $d^* = 0.2863$ and $d = d^*$

solutions obtained for $d \leq d^\star$ are not singular at the origin. More importantly, it appears from the numerical simulations that as $d$ increases, the numerical solution seems to approach the desired solution $v^\star$, which describes the catenary profile. This confirms the relevance of the $K$ optimization space. In particular, our analytical and numerical results provide an accurate analytical estimate, for the parameter $d$ and provide reasonably convincing evidence that for $d = d^\star$ the numerical solution can be used as a good approximation to a catenary-like profile that satisfies our expectation. We point out that the equations Eq. (30) involve the $p$-Laplacian with a singular term and the Dirac mass at the origin, which is a very interesting prospect. In fact, a detailed analysis will be carried out by the authors in the near future for the mathematical analysis as well as the asymptotic behavior of the system Eq. (30).

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(Maryem Hourri) Laboratory LAMAI, Cadi Ayyad University, Marrakesh 40000, Morocco
*E-mail address*: maryem.hourri@ced.uca.ma

(Laila Taourirte) IBNOU ZOUHR University, ESEF Agadir, Morocco
*E-mail address*: l.taourirte@uiz.ac.ma

(Nour Eddine Alaa) Laboratory LAMAI, Cadi Ayyad University, Marrakesh 40000, Morocco
*E-mail address*: n.alaa@uca.ac.ma