Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 50(2), 2023, Pages 342–352, DOI: 10.52846/ami.v50i2.1708
ISSN: 1223-6934

Almost Bi-Quasi-Ideals and Their Fuzzifications in Semigroups

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Abstract. The notion of bi-quasi-ideals of semigroups was introduced by Krishna Rao in 2017, which is a generalization of left ideals, right ideals, bi-ideals, and quasi-ideals of semigroups. In this paper, we introduce the concept of almost bi-quasi ideals as a generalization of bi-quasi ideals of semigroups. We investigate the properties of almost bi-quasi-ideals and fuzzy almost bi-quasi-ideals in semigroups. Moreover, we show some relationships between almost bi-quasi-ideals and fuzzy almost bi-quasi-ideals.

2020 *Mathematics Subject Classification.* 06F05; 08A72.
Key words and phrases. Almost bi-quasi-ideals; minimal bi-quasi ideals; fuzzy almost bi-quasi-ideals.

1. Introduction

Fuzzy sets are a kind of proper mathematical structure to represent a collection of objects whose boundary is vague, which was introduced by Zaden in 1965 [15]. The fuzzy set theory became a phenomenon since its logic can deal with information that is imprecise, vague, partially true, or without sharp boundaries. In 1979, Kuroki [6] firstly applied fuzzy set theory in semigroups. He investigated the properties of fuzzy semigroups. The almost ideal theory in semigroups was studied by Grosek and Satko in 1980 [3]. In 1981, Bogdanvic [1] established the definition of almost bi-ideals in semigroups and studied their properties. Later, Chinram gave the definitions of various types of almost ideals in semigroups such that almost quasi-ideals [14], (m, n)-almost ideals [10]. In 2020, Kaopusek et al. [4] discussed almost interior ideals and weakly almost interior ideals of semigroups by using the concepts of almost ideals and interior ideals of semigroups and investigated their properties. In 2022, Chinram and Nakkhasen [2] studied properties of almost bi-quasi-interior ideals and their fuzzy bi-interior ideals in semigroups. In the same year, Gaketem [7] defined and gave properties of almost bi-interior ideals and fuzzy almost bi-interior ideals in semigroups, and, Tiprachot and Lekkoksung [12] studied almost α-ideals in semigroups. Moreover, the concepts of almost ideals be extended to study in another algebraic structure, for example, in ternary semigroups [11], in LA-semihypergroups [8], in Γ-semigroups [9], in Γ-semihypergroups [13]. Recently, Krishna Rao [5] gave a definition of bi-quasi-ideals in semigroups in 2017.

Received December 5, 2022. Accepted March 21, 2023.
This research project was supported by the Thailand Science Research and Innovation fund and the University of Phayao (Grant No. FF66-UoE017) Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand.

In this paper, we give the definition of almost bi-interior ideals and fuzzy almost bi-quasi-ideals in semigroups. We discuss properties of almost bi-quasi-ideals and fuzzy almost bi-quasi-ideals in semigroups. Moreover, we prove the relation between almost bi-quasi-ideals and fuzzy almost bi-quasi-ideals.

2. Preliminaries

In this section, we give some concepts and results, which will be helpful in later sections.

Definition 2.1. A non-empty subset $\mathcal{L}$ of a semigroup $\mathcal{S}$ is called
(1) a *subsemigroup* of $\mathcal{S}$ if $\mathcal{L}^2 \subseteq \mathcal{L}$,
(2) a *left (right) ideal* of $\mathcal{S}$ if $\mathcal{SL} \subseteq \mathcal{L}$ ($\mathcal{LS} \subseteq \mathcal{L}$). By an *ideal* $\mathcal{L}$ of a semigroup $\mathcal{S}$ we mean a left ideal and a right ideal of $\mathcal{S}$,
(3) a *bi-ideal* of $\mathcal{S}$ if $\mathcal{L}$ is a subsemigroup of $\mathcal{S}$ and $\mathcal{LSL} \subseteq \mathcal{L}$,
(4) an *interior ideal* of $\mathcal{S}$ if $\mathcal{L}$ is a subsemigroup of $\mathcal{S}$ and $\mathcal{SLS} \subseteq \mathcal{L}$,
(5) a *quasi-ideal* of $\mathcal{S}$ if $\mathcal{LS} \cap \mathcal{SL} \subseteq \mathcal{L}$,
(6) a *left (right) almost ideal* of $\mathcal{S}$ if $h\mathcal{L} \cap \mathcal{L} \neq \emptyset$ ($\mathcal{L}h \cap \mathcal{L} \neq \emptyset$) for all $h \in \mathcal{S}$. By an *almost ideal* M of a semigroup $\mathcal{S}$ we mean a left almost ideal and a right almost ideal of $\mathcal{S}$,
(7) a *almost bi-ideal* of $\mathcal{S}$ if $\mathcal{L}h\mathcal{L} \cap \mathcal{L} \neq \emptyset$ for all $h \in \mathcal{S}$,
(8) a *almost interior ideal* of $\mathcal{S}$ if $h\mathcal{L}k \cap \mathcal{L} \neq \emptyset$ for all $h, k \in \mathcal{S}$.
(9) a *almost quasi-ideal* of $\mathcal{S}$ if $(h\mathcal{L} \cap \mathcal{L}h) \cap \mathcal{L} \neq \emptyset$ for all $h \in \mathcal{S}$.

A subsemigroup $\mathcal{L}$ of a semigroup $\mathcal{S}$ is said to be *left (right) bi-quasi ideal* of $\mathcal{S}$ if $\mathcal{SL} \cap \mathcal{LSL} \subseteq \mathcal{L}(\mathcal{LS} \cap \mathcal{SL} \subseteq \mathcal{L})$. A subsemigroup $\mathcal{L}$ of a semigroup $\mathcal{S}$ is said to be *bi-quasi ideal* of $\mathcal{S}$ if it is both a left bi-quasi and right bi-quasi ideal of $\mathcal{S}$. [5].

For any $h_i \in [0,1]$, $i \in \mathcal{F}$, define

$$\underset{i\in\mathcal{F}}{\vee} h_i := \sup_{i\in\mathcal{F}}\{h_i\} \quad \text{and} \quad \underset{i\in\mathcal{F}}{\wedge} h_i := \inf_{i\in\mathcal{F}}\{h_i\}.$$

We see that for any $h, r \in [0,1]$, we have

$$h \vee r = \max\{h, r\} \quad \text{and} \quad h \wedge r = \min\{h, r\}.$$

A *fuzzy set (fuzzy subset)* of a non-empty set E is a function $\rho : E \to [0,1]$.

For any two fuzzy sets ρ and ν of a non-empty set E, define the symbol as follows:
(1) $\rho \subseteq \nu \Leftrightarrow \rho(h) \leq \nu(h)$ for all $h \in E$,
(2) $\rho = \nu \Leftrightarrow \rho \subseteq \nu$ and $\nu \subseteq \rho$,
(3) $(\rho \wedge \nu)(h) = \rho(h) \wedge \nu(h)$ and $(\rho \vee \nu)(h) = \rho(h) \vee \nu(h)$ for all $h \in E$,
(4) the *support* of ρ instead of $\text{supp}(\rho) = \{h \in E \mid \rho(h) \neq 0\}$.
For the symbol $\rho \supseteq \nu$, we mean $\nu \subseteq \rho$.

For any two fuzzy sets ρ and ν of a semigroup $\mathcal{S}$. The product of fuzzy subsets ρ and ν of $\mathcal{S}$ is defined as follow, for all $h \in \mathcal{S}$

$$(\rho \circ \nu)(h) = \begin{cases} \displaystyle\bigvee_{h=kr} \{\rho(k) \wedge \nu(r)\} & \text{if} \quad h = kr, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic function of a subset $\mathcal{L}$ of a nonempty set $\mathcal{S}$ is a fuzzy set of $\mathcal{S}$

$$\lambda_{\mathcal{L}}(h) = \begin{cases} 1 & \text{if} \quad h \in \mathcal{L}, \\ 0 & \text{if} \quad h \notin \mathcal{L}. \end{cases}$$

for all $h \in \mathcal{S}$.

Lemma 2.1. [2] *Let K and L be non-empty subsets of a semigroup E. Then the following holds.*
(1) $\lambda_{\mathcal{K}} \wedge \lambda_{\mathcal{L}} = \lambda_{\mathcal{K}\cap\mathcal{L}}$.
(2) $\lambda_{\mathcal{K}} \circ \lambda_{\mathcal{L}} = \lambda_{\mathcal{K}\mathcal{L}}$.

Definition 2.2. A fuzzy set ρ of a semigroup $\mathcal{S}$ is said to be
(1) a *fuzzy subsemigroup* of $\mathcal{S}$ if $\rho(h) \wedge \rho(r) \leq \rho(hr)$, for all $h, r \in \mathcal{S}$,
(2) a *fuzzy left (right) ideal* of $\mathcal{S}$ if $\rho(r) \leq \rho(hr)$ $(\rho(h) \leq \rho(hr))$, for all $h, r \in \mathcal{S}$. A *fuzzy ideal* of $\mathcal{S}$ if it is both a fuzzy left ideal and a fuzzy right ideal of $\mathcal{S}$,
(3) a *fuzzy bi-ideal* of $\mathcal{S}$ if ρ is a fuzzy subsemigroup of $\mathcal{S}$ and $\rho(h) \wedge \rho(k) \leq \rho(hrk)$ for all $h, r, k \in \mathcal{S}$,
(4) a *fuzzy interior ideal* of $\mathcal{S}$ if ρ is a fuzzy subsemigroup of $\mathcal{S}$ and $\rho(r) \leq \rho(hrk)$ for all $h, r, k \in \mathcal{S}$,
(5) a *fuzzy quasi-ideal* of $\mathcal{S}$ if $(\lambda_{\mathcal{S}} \circ \rho)(h) \wedge (\rho \circ \lambda_{\mathcal{S})}(h) \leq \rho(h)$ for all $h \in \mathcal{S}$ where $\lambda_{\mathcal{S}}$ is a fuzzy subset of $\mathcal{S}$ mapping every element of $\mathcal{S}$ to 1,
(6) a *fuzzy bi-interior ideal* of $\mathcal{S}$ if $\rho \supseteq (\lambda_{\mathcal{L}} \circ \rho \circ \lambda_{\mathcal{L}}) \wedge (\rho \circ \lambda_{\mathcal{L}} \circ \rho)$,
(7) a *fuzzy left (right) bi-quasi-ideal* of $\mathcal{S}$ if $\rho \supseteq \lambda_{\mathcal{L}} \circ \rho \wedge \rho \circ \lambda_{\mathcal{L}} \circ \rho (\rho \supseteq \rho \circ \lambda_{\mathcal{L}} \wedge \rho \circ \lambda_{\mathcal{L}} \circ \rho)$. A fuzzy set ρ of semigroup $\mathcal{S}$ is called a *fuzzy bi-quasi ideal* of S if it is both a fuzzzy left bi-quasi ideal and a fuzzy right bi-quasi ideal of $\mathcal{S}$.

Next, we recall the definition of fuzzy point of a set $\mathcal{S}$. For $h \in \mathcal{S}$ and $t \in (0, 1]$, a fuzzy point w_δ of a set $\mathcal{S}$ is a fuzzy subset of $\mathcal{S}$ defined by

$$w_\delta(h) = \begin{cases} \delta & \text{if } h = r, \\ 0 & \text{if } h \neq r. \end{cases}$$

Definition 2.3. A fuzzy set ρ of a semigroup $\mathcal{S}$ is said to be
(1) a *fuzzy left (right) almost ideal* of $\mathcal{S}$ if $(w_\delta \circ \rho) \wedge \rho \neq 0$ $((\rho \circ w_\delta) \wedge \rho \neq 0)$. A *fuzzy almost ideal* of $\mathcal{S}$ if it is both a fuzzy left almost ideal and a fuzzy right almost ideal of $\mathcal{S}$,
(2) a *fuzzy almost bi-ideal* of $\mathcal{S}$ if $(w_\delta \circ \rho \circ w_\delta) \wedge \rho \neq 0$,
(3) a *fuzzy almost interior ideal* of $\mathcal{S}$ if $(\rho \circ w_\delta \circ \rho) \wedge \rho \neq 0$,
(4) a *fuzzy almost quasi-ideal* of $\mathcal{S}$ if $[(w_\delta \circ \rho) \wedge (\rho \circ w_\delta)] \wedge \rho \neq 0$.

3. Almost bi-quasi ideals in semigroups

In this section, we define the notions of almost bi-quasi ideals in semigroups and some properties of them are investigated.

Definition 3.1. A nonempty set $\mathcal{L}$ of a semigroup $\mathcal{S}$ is called a *almost left (right) bi-quasi-ideal* of $\mathcal{S}$ if $(h\mathcal{L} \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \neq \emptyset((\mathcal{L}h \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \neq \emptyset)$, for all $h, k \in \mathcal{S}$. An *almost bi-quasi-ideal* $\mathcal{L}$ of $\mathcal{S}$ if it is both an almost left bi-quasi-ideal and an almost right bi-quasi-ideal of $\mathcal{S}$.

Theorem 3.1. *Every left bi-quasi-ideal of a semigroup $\mathcal{S}$ is an almost left bi-qausi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{L}$ is a left bi-quasi-ideal of $\mathcal{S}$. Then $(h\mathcal{L} \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L}$. Thus, $(h\mathcal{L} \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \subseteq \mathcal{L} \cap \mathcal{L} \neq \mathcal{L}$. Hence, $\mathcal{L}$ is an almost left bi-quasi-ideal of $\mathcal{S}$. □

Theorem 3.2. *Every right bi-quasi-ideal of a semigroup $\mathcal{S}$ is an almost right bi-qausi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{L}$ is a right bi-quasi-ideal of $\mathcal{S}$. Then $(\mathcal{L}h \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L}$. Thus, $(\mathcal{L}h \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \subseteq \mathcal{L} \cap \mathcal{L} \neq \mathcal{L}$. Hence, $\mathcal{L}$ is an almost right bi-quasi-ideal of $\mathcal{S}$. □

Corollary 3.3. *Every bi-quasi-ideal of a semigroup $\mathcal{S}$ is an almost bi-qausi-ideal of $\mathcal{S}$.*

Lemma 3.4. *Let $\mathcal{L}$ and $\mathcal{Q}$ be nonempty subsets of a semigroup $\mathcal{S}$ with $\mathcal{L} \subseteq \mathcal{Q}$. If $\mathcal{L}$ is an almost left bi-quasi-ideal of $\mathcal{S}$, then $\mathcal{Q}$ is an almost left bi-quasi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{L}$ is an almost left bi-quasi-ideal of $\mathcal{S}$ and $h, k \in \mathcal{S}$. Then $(h\mathcal{L} \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \neq \emptyset$. Thus, $(h\mathcal{L} \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \subseteq (h\mathcal{Q} \cap \mathcal{Q}k\mathcal{Q}) \cap \mathcal{Q} \neq \emptyset$. Thus, $(h\mathcal{Q} \cap \mathcal{Q}k\mathcal{Q}) \cap \mathcal{Q} \neq \emptyset$. Hence $\mathcal{Q}$ is an almost left bi-quasi-ideal of $\mathcal{S}$. □

Lemma 3.5. *Let $\mathcal{L}$ and $\mathcal{Q}$ be nonempty subsets of a semigroup $\mathcal{S}$ with $\mathcal{L} \subseteq \mathcal{Q}$. If $\mathcal{L}$ is an almost right bi-quasi-ideal of $\mathcal{S}$, then $\mathcal{Q}$ is an almost right bi-quasi-ideal of $\mathcal{S}$.*

Proof. It is similar to Lemma 3.4. □

The next theorem follows from Lemma 3.4 and 3.5.

Theorem 3.6. *Let $\mathcal{L}$ and $\mathcal{Q}$ be nonempty subsets a semigroup of $\mathcal{S}$ with $\mathcal{L} \subseteq \mathcal{Q}$. If $\mathcal{L}$ is an almost bi-quasi-ideal of $\mathcal{S}$, then $\mathcal{Q}$ is an almost bi-quasi-ideal of $\mathcal{S}$.*

Lemma 3.7. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are almost left bi-quasi-ideals of $\mathcal{S}$, then $\mathcal{L}_1 \cup \mathcal{L}_2$ is an almost left bi-quasi-ideal of $\mathcal{G}$.*

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be any two almost left bi-quasi-ideals of $\mathcal{S}$. Then $\mathcal{L}_1 \subseteq \mathcal{L}_1 \cup \mathcal{L}_2$. Thus by Lemma 3.4, $\mathcal{L}_1 \cup \mathcal{L}_2$ is an almost left bi-quasi-ideal of $\mathcal{G}$. □

Lemma 3.8. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are almost right bi-quasi-ideals of $\mathcal{S}$, then $\mathcal{L}_1 \cup \mathcal{L}_2$ is an almost right bi-quasi-ideal of $\mathcal{G}$.*

Proof. It is similar to Lemma 3.7. □

The next theorem follows from Lemma 3.7 and 3.8.

Theorem 3.9. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are almost bi-quasi-ideals of $\mathcal{S}$, then $\mathcal{L}_1 \cup \mathcal{L}_2$ is an almost bi-quasi-ideal of $\mathcal{G}$.*

Corollary 3.10. *Let $\mathcal{S}$ be a semigroup. Then the following statements hold.*

(1) The finite union of almost left bi-quasi-ideals of $\mathcal{S}$ is an almost left bi-quasi-ideal of $\mathcal{S}$.
(2) The finite union of almost right bi-quasi-ideals of $\mathcal{S}$ is an almost right bi-quasi-ideal of $\mathcal{S}$.
(3) The finite union of almost bi-quasi-ideals of $\mathcal{S}$ is an almost bi-quasi-ideal of $\mathcal{S}$.

Theorem 3.11. *Every almsot bi-qausi-ideal of a semigroup $\mathcal{S}$ is an almost quasi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{L}$ is an almost bi-quasi-ideal of $\mathcal{S}$ and $h, k \in \mathcal{S}$. Then $(\mathcal{L}h \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \neq \emptyset$. Thus,

$$(\mathcal{L}h \cap \mathcal{L}k\mathcal{L}) \subseteq \mathcal{L}h \cap h\mathcal{L}.$$

So, $\emptyset \neq (\mathcal{L}h \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \subseteq (\mathcal{L}h \cap h\mathcal{L}) \cap \mathcal{L}$. Hence, $\mathcal{L}$ is an almsot quasi-ideal of $\mathcal{S}$. □

Theorem 3.12. *Every almsot bi-quasi-ideal of a semigroup $\mathcal{S}$ is an almost ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{L}$ is an almost bi-quasi-ideal of $\mathcal{S}$ and $h, k \in \mathcal{S}$. Then $(\mathcal{L}h \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \neq \emptyset$. Thus,

$$(\mathcal{L}h \cap \mathcal{L}k\mathcal{L}) \subseteq \mathcal{L}h.$$

So, $\emptyset \neq (\mathcal{L}h \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \subseteq \mathcal{L}h \cap \mathcal{L}$. Hence, $\mathcal{L}$ is an almsot ideal of $\mathcal{S}$. □

4. Fuzzy almost bi-quasi-ideals in semigroups

In this section, we define the notions of fuzzy almost bi-quasi-ideals in semigroups and some properties of them are investigated. Moreover, we discuss the relationship between almost bi-quasi-ideals and fuzzy almost bi-quasi-ideals.

Definition 4.1. A nonzero fuzzy set ρ of a semigroup $\mathcal{S}$ is called a *fuzzy almost left (right) bi-quasi-ideal* of $\mathcal{S}$ if $(w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho) \wedge \rho \neq 0((\rho \circ w_{\delta_1} \wedge \rho \circ r_{\delta_2} \circ \rho) \wedge \rho \neq 0))$, for all fuzzy points w_{δ_1} and r_{δ_2} of $\mathcal{S}$.

Lemma 4.1. *Let ρ and ν be a nonzero fuzzy sets of a semigroup $\mathcal{S}$ with $\rho \subseteq \nu$. If ρ is a fuzzy almost left (right) bi-quasi-ideal of $\mathcal{S}$, then ν is a fuzzy almost left (right) bi-quasi-ideal of $\mathcal{S}$.*

Proof. Suppose that ρ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$ and $w_{\delta_1}, r_{\delta_2}$ are fuzzy points of $\mathcal{S}$. Then $(w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho) \wedge \rho \neq 0$. By assumption, $(w_{\delta_1} \circ \rho \wedge \rho \circ v_{\delta_2} \circ \rho) \wedge \rho \subseteq (w_{\delta_1} \circ \nu \wedge \nu \circ r_{\delta_2} \circ \nu) \wedge \nu \neq 0$. Thus $(w_{\delta_1} \circ \nu \wedge \nu \circ r_{\delta_2} \circ \nu) \wedge \nu \neq 0$. Hence ν is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$. □

The next theorem follows from Lemma 4.1.

Theorem 4.2. *Let ρ and ν be nonzero fuzzy sets of a semigroup $\mathcal{S}$ with $\rho \subseteq \nu$. If ρ is a fuzzy almost bi-quasi-ideal of $\mathcal{S}$, then ν is a fuzzy almost bi-quasi-ideal of $\mathcal{S}$.*

Theorem 4.3. *Let ρ and ν be nonzero fuzzy set of a semigroup $\mathcal{S}$. Then the following statement hold.*

(1) $\rho \vee \nu$ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$.
(2) $\rho \vee \nu$ is a fuzzy almost right bi-quasi-ideal of $\mathcal{S}$.
(3) $\rho \vee \nu$ is a fuzzy almost bi-quasi-ideal of $\mathcal{S}$.

Proof. (1) Let ρ and ν be a nonzero fuzzy almost left bi-quasi-ideals of $\mathcal{S}$. Then $\rho \subseteq \rho \vee \nu$. Thus by Lemma 4.1, $\rho \vee \nu$ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$.
(2) It can proved similarly 1.
(3) It is a result of 1 and 2.

□

Corollary 4.4. *Let $\mathcal{S}$ be a semigroup. Then the following statement hold.*

(1) The finite union of fuzzy almost left bi-quasi-ideals of $\mathcal{S}$ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$.

(2) The finite union of fuzzy almost right bi-quasi-ideals of $\mathcal{S}$ is a fuzzy almost right bi-quasi-ideal of $\mathcal{S}$.
(3) The finite union of fuzzy almost bi-quasi-ideals of $\mathcal{S}$ is a fuzzy almost bi-quasi-ideal of $\mathcal{S}$.

Theorem 4.5. *Let $\mathcal{S}$ be a semigroup. Then the following statement hold.*
(1) Every fuzzy almsot left bi-qausi-ideal of a semigroup $\mathcal{S}$ is a fuzzy almost quasi-ideal of $\mathcal{S}$.
(2) Every fuzzy almsot right bi-qausi-ideal of a semigroup $\mathcal{S}$ is a fuzzy almost quasi-ideal of $\mathcal{S}$.
(3) Every fuzzy almsot bi-qausi-ideal of a semigroup $\mathcal{S}$ is a fuzzy almost quasi-ideal of $\mathcal{S}$.

Proof. (1) Suppose that ρ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$ and $w_{\delta_1}, r_{\delta_2}$ are fuzzy points of $\mathcal{S}$. Then $(w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho) \wedge \rho \neq 0$. Thus,

$$(w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho) \subseteq [(w_\delta \circ \rho) \wedge (\rho \circ w_\delta)].$$

So, $0 \neq (w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho) \wedge \rho \subseteq [(w_\delta \circ \rho) \wedge (\rho \circ w_\delta)] \wedge \rho$. Hence, ρ is a fuzzy almsot quasi-ideal of $\mathcal{S}$.
(2) It can proved similarly 1.
(3) It is a result of 1 and 2.

□

Theorem 4.6. *Let $\mathcal{S}$ be a semigroup. Then the following statement hold.*
(1) Every fuzzy almsot left bi-quasi-ideal of a semigroup $\mathcal{S}$ is a fuzzy almost ideal of $\mathcal{S}$.
(2) Every fuzzy almsot right bi-quasi-ideal of a semigroup $\mathcal{S}$ is a fuzzy almost ideal of $\mathcal{S}$.
(3) Every fuzzy almsot left bi-quasi-ideal of a semigroup $\mathcal{S}$ is a fuzzy almost ideal of $\mathcal{S}$.

Proof. (1) Suppose that ρ is a fuzzy almost bi-quasi-ideal of $\mathcal{S}$ and $w_{\delta_1}, r_{\delta_2}$ are fuzzy points of $\mathcal{S}$. Then $(w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho) \wedge \rho \neq 0$. Thus,

$$(w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho) \subseteq (w_\delta \circ \rho).$$

So, $0 \neq (w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho) \wedge \rho \subseteq (w_\delta \circ \rho)$. Hence, ρ is a fuzzy almsot ideal of $\mathcal{S}$.
(2) It can proved similarly 1.
(3) It is a result of 1 and 2.

□

The following theorem shows the connection between almost left (right) bi-quasi-ideals and fuzzy almost left (right) bi-quasi-ideals in semigroups.

Theorem 4.7. *Let $\mathcal{L}$ be a nonempty subset of a semigroup $\mathcal{S}$. Then $\mathcal{L}$ is an almost left (right) bi-quasi-ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{L}}$ is a fuzzy almost left (right) bi-quasi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{L}$ is an almost left bi-quasi-ideal of $\mathcal{S}$, $h, k \in \mathcal{S}$ and $\delta_1, \delta_2 \in (0, 1]$. Then $(h\mathcal{L} \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \neq \emptyset$. Thus there exists $t \in \mathcal{S}$ such that $t \in (h\mathcal{L} \cap \mathcal{L}k\mathcal{L})$ and $t \in \mathcal{L}$. So $t = hs_1$ and $t = s_2ks_3$ for some $s_1, s_2, s_3 \in \mathcal{L}$. It follows that

$$(w_{\delta_1} \circ \lambda_{\mathcal{L}})(t) = \bigvee_{c=hs_1} \{w_{\delta_1}(h) \wedge \lambda_{\mathcal{L}}(s_1)\} \neq 0,$$

$$(\lambda_{\mathcal{L}} \circ r_{\delta_2} \circ \lambda_{\mathcal{L}})(t) = \bigvee_{c=hs_1} \{\lambda_{\mathcal{L}}(s_2) \wedge r_{\delta_2}(k) \wedge \lambda_{\mathcal{L}}(s_3)\} \neq 0$$

and $\lambda_{\mathcal{L}}(t) = 1$. Hence $(w_{\delta_1} \circ \lambda_{\mathcal{L}} \wedge \lambda_{\mathcal{L}} \circ r_{\delta_2} \circ \lambda_{\mathcal{L}}) \wedge \lambda_{\mathcal{L}} \neq 0$. Therefore $\lambda_{\mathcal{L}}$ is a fuzzy almost left bi-quasi ideal of $\mathcal{S}$.

For the converse, assume that $\lambda_{\mathcal{L}}$ is a fuzzy almost left bi-quasi ideal of $\mathcal{S}$, let $h, k \in \mathcal{S}$ and $\delta_1, \delta_2 \in (0, 1]$. Then $(w_{\delta_1} \circ \lambda_{\mathcal{L}} \wedge \lambda_{\mathcal{L}} \circ r_{\delta_2} \circ \lambda_{\mathcal{L}}) \wedge \lambda_{\mathcal{L}} \neq 0$. Thus there exists $t \in \mathcal{S}$ such that $(w_{\delta_1} \circ \lambda_{\mathcal{L}} \wedge \lambda_{\mathcal{L}} \circ r_{\delta_2} \circ \lambda_{\mathcal{L}})(t) \neq 0$ and $\lambda_{\mathcal{L}}(t) \neq 0$. So $t \in (h\mathcal{L} \cap \mathcal{L}k\mathcal{L})$ and $t \in \mathcal{L}$, this implies that $(h\mathcal{L} \cap \mathcal{L}k\mathcal{L}) \cap \mathcal{L} \neq \emptyset$. Therefore $\mathcal{L}$ is an almost left bi-quasi-ideal of $\mathcal{S}$. □

The proof of the next theorem follows from Theorem 4.7.

Theorem 4.8. *Let $\mathcal{L}$ be a nonempty subset of a semigroup $\mathcal{S}$. Then $\mathcal{L}$ is an almost bi-quasi-ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{L}}$ is a fuzzy almost bi-quasi-ideal of $\mathcal{S}$.*

Theorem 4.9. *Let ρ be a nonzero fuzzy set of a semigroup $\mathcal{S}$. Then ρ is a fuzzy almost left (right) bi-quasi-ideal of $\mathcal{S}$ if and only if* $\mathrm{supp}(\rho)$ *is an almost left (right) bi-quasi-ideal of $\mathcal{S}$.*

Proof. Suppose that ρ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$, and $w_{\delta_1}, r_{\delta_2}$ are fuzzy points of $\mathcal{S}$. Then $(w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho) \wedge \rho \neq 0$. Thus there exists $t \in \mathcal{S}$ such that $(w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho)(t) \neq 0$ and $\rho(t) \neq 0$. So there exist $s_1, s_2, s_3 \in \mathcal{L}$ such that $t = hs_1$ and $t = s_2ks_3$. It follows that

$$(w_{\delta_1} \circ \lambda_{\mathcal{L}})(t) = \bigvee_{c=hs_1} \{w_{\delta_1}(h) \wedge \lambda_{\mathcal{L}}(s_1)\} \neq 0,$$

$$(\lambda_{\mathcal{L}} \circ r_{\delta_2} \circ \lambda_{\mathcal{L}})(t) = \bigvee_{c=hs_1} \{\lambda_{\mathcal{L}}(s_2) \wedge r_{\delta_2}(k) \wedge \lambda_{\mathcal{L}}(s_3)\} \neq 0$$

Thus $w_{\delta_1}(h) \neq 0$ and $r_{\delta_2}(k) \neq 0$. So $h, k \in \mathrm{supp}(\rho)$, this implies that $(w_{\delta_1} \circ \lambda_{\mathrm{supp}(\rho)} \wedge \lambda_{\mathrm{supp}(\rho)} \circ r_{\delta_2} \circ \lambda_{\mathrm{supp}(\rho)}) \wedge \lambda_{\mathrm{supp}(\rho)} \neq 0$. Thus $\lambda_{\mathrm{supp}(\rho)}$ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$. By Theorem 4.7, $\mathrm{supp}(\rho)$ is an almost left bi-quasi-ideal of $\mathcal{S}$.

For the converse, assume that $\mathrm{supp}(\rho)$ is an almost left bi-quasi-ideal of $\mathcal{S}$. By Theorem 4.7, $\lambda_{\mathrm{supp}(\rho)}$ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$. Thus for any two fuzzy points $w_{\delta_1}, r_{\delta_2}$ of $\mathcal{S}$ such that $(w_{\delta_1} \circ \lambda_{\mathrm{supp}(\rho)} \wedge \lambda_{\mathrm{supp}(\rho)} \circ r_{\delta_2} \circ \lambda_{\mathrm{supp}(\rho)}) \wedge \lambda_{\mathrm{supp}(\rho)} \neq 0$. So there exists $t \in \mathcal{S}$ such that $(w_{\delta_1} \circ \lambda_{supp(\rho)} \wedge \lambda_{supp(\rho)} \circ r_{\delta_2} \circ \lambda_{\mathrm{supp}(\rho)})(t) \neq 0$ and $\lambda_{supp(\varphi)}(t) \neq 0$. Thus there exist $s_1, s_2, s_3 \in \mathcal{L}$ such that $t = hs_1$ and $t = s_2ks_3$. It follows that

$$(w_{\delta_1} \circ \lambda_{\mathrm{supp}(\rho)})(t) = \bigvee_{c=hs_1} \{w_{\delta_1}(h) \wedge \lambda_{\mathrm{supp}(\rho)}(s_1)\} \neq 0,$$

$$(\lambda_{\mathrm{supp}(\rho)} \circ r_{\delta_2} \circ \lambda_{\mathrm{supp}(\rho)})(t) = \bigvee_{c=hs_1} \{\lambda_{\mathrm{supp}(\rho)}(s_1) \wedge r_{\delta_2}(k) \wedge \lambda_{\mathrm{supp}(\rho)}(s_3)\} \neq 0$$

So $\lambda_{\mathrm{supp}(\rho)}(s_1) \neq 0$, $\lambda_{\mathrm{supp}(\rho)}(s_2) \neq 0$ and $\lambda_{\mathrm{supp}(\rho)}(s_3) \neq 0$, this implies that $\rho(s_1) \neq 0$, $\rho(s_2) \neq 0$ and $\rho(s_3) \neq 0$. Hence

$$(w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho) \wedge \rho(t) = (\bigvee_{t=hs_1} \{w_{\delta_1}(h) \wedge \rho(s_1)\}) \wedge (\bigvee_{t=s_2ks_3} \{\rho(s_2) \wedge r_{\delta_2}(k) \wedge \rho(s_3)\}) \wedge \rho(t) \neq 0.$$

It follows that $(w_{\delta_1} \circ \rho \wedge \rho \circ r_{\delta_2} \circ \rho) \wedge \rho \neq 0$ for any two fuzzy points $w_{\delta_1}, r_{\delta_2}$ of $\mathcal{S}$. Therefore ρ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$. □

Corollary 4.10. *Let ρ be a nonzero fuzzy set of a semigroup $\mathcal{S}$. Then ρ is a fuzzy almost bi-quasi-ideal of $\mathcal{S}$ if and only if supp(ρ) is an almost bi-quasi-ideal of $\mathcal{S}$.*

Next, we define minimal fuzzy almost bi-quasi-ideals in semigroups and study the relationship between minimal almost bi-quasi-ideals and minimal fuzzy almost bi-quasi-ideals of semigroups.

Definition 4.2. An almost left (right) bi-quasi-ideal $\mathcal{L}$ of a semigroup $\mathcal{S}$ is called *minimal* if for any almost left (right) bi-quasi-ideal $\mathcal{K}$ of $\mathcal{S}$ if whenever $\mathcal{K} \subseteq \mathcal{L}$, then $\mathcal{K} = \mathcal{L}$.

Definition 4.3. A fuzzy almost left (right) bi-quasi-ideal ρ of a semigroup $\mathcal{S}$ is called *minimal* if for any fuzzy almost left (right) bi-quasi-ideal ν of $\mathcal{S}$ if whenever $\nu \subseteq \rho$, then sup(ν) =sup(ρ).

Theorem 4.11. *Let $\mathcal{L}$ be a nonempty subset of a semigroup $\mathcal{S}$. Then*

(1) $\mathcal{L}$ is a minimal almost left bi-quasi-ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{L}}$ is a minimal fuzzy almost left bi-quasi-ideal of $\mathcal{S}$.

(2) $\mathcal{L}$ is a minimal almost right bi-quasi-ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{L}}$ is a minimal fuzzy almost right bi-quasi-ideal of $\mathcal{S}$.

(3) $\mathcal{L}$ is a minimal almost bi-quasi-ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{L}}$ is a minimal fuzzy almost bi-quasi-ideal of $\mathcal{S}$.

Proof. Assume that $\mathcal{L}$ is a minimal almost left bi-quasi-ideal of $\mathcal{S}$. Then $\mathcal{L}$ is an almost left bi-quasi-ideal of $\mathcal{S}$. Thus, by Theorem 4.7, $\lambda_{\mathcal{L}}$ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$. Let ν be a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$ such that $\nu \subseteq \lambda_{\mathcal{L}}$. Then $\sup(\nu) \subseteq \sup(\lambda_{\mathcal{L}}) = \mathcal{L}$. Also, $\sup(\nu)$ is an almost left bi-quasi ideal of $\mathcal{S}$. Since $\mathcal{L}$ is minimal, we have $\sup(\nu) = \mathcal{L} = \sup(\lambda_{\mathcal{L}})$. Therefore, $\lambda_{\mathcal{L}}$ is minimal fuzzy almost left bi-quasi-ideal of $\mathcal{S}$.

Conversely, suppose that $\lambda_{\mathcal{L}}$ is a minimal fuzzy almost left bi-quasi-ideal of $\mathcal{S}$. Then $\lambda_{\mathcal{L}}$ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$. By Theorem 4.7, $\mathcal{L}$ is an almost left bi-quasi-ideal of $\mathcal{S}$. Let $\mathcal{K}$ be an almost left bi-quasi-ideal of $\mathcal{S}$ such that $\mathcal{K} \subseteq \mathcal{L}$. Then $\lambda_{\mathcal{K}}$ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$ such that $\lambda_{\mathcal{K}} \subseteq \lambda_{\mathcal{L}}$. By assumption, Hence, $\mathcal{K} = \text{supp}(\lambda_{\mathcal{K}}) = \text{supp}(\lambda_{\mathcal{L}}) = \mathcal{L}$. Therefore, $\mathcal{L}$ is minimal of $\mathcal{S}$.

The proofs of 2 and 3 are similar to the proof 1. □

Next, we define maximal fuzzy almost bi-quasi-ideals in semigroups and study the relationship between maximal almost bi-quasi-ideals and minimal fuzzy almost bi-quasi-ideals of semigroups.

Definition 4.4. An almost left (right) bi-quasi-ideal $\mathcal{L}$ of a semigroup $\mathcal{S}$ is called *maxmal* if for any almost left (right) bi-quasi-ideal $\mathcal{K}$ of $\mathcal{S}$ if whenever $\mathcal{L} \subseteq \mathcal{K}$, then $\mathcal{K} = \mathcal{L}$.

Definition 4.5. A fuzzy almost left (right) bi-quasi-ideal ρ of a semigroup $\mathcal{S}$ is called *maximal* if for any fuzzy almost left (right) bi-quasi-ideal ν of $\mathcal{S}$ if whenever $\rho \subseteq \nu$, then sup(ν) =sup(ρ).

Theorem 4.12. *Let $\mathcal{L}$ be a nonempty subset of a semigroup $\mathcal{S}$. Then*

(1) $\mathcal{L}$ is a maximal almost left bi-quasi-ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{L}}$ is a maximal fuzzy almost left bi-quasi-ideal of $\mathcal{S}$.

(2) $\mathcal{L}$ is a maximal almost right bi-quasi-ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{L}}$ is a maximal fuzzy almost right bi-quasi-ideal of $\mathcal{S}$.
(3) $\mathcal{L}$ is a maximal almost bi-quasi-ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{L}}$ is a maximal fuzzy almost bi-quasi-ideal of $\mathcal{S}$.

Proof. Assume that $\mathcal{L}$ is a maximal almost left bi-quasi-ideal of $\mathcal{S}$. Then $\mathcal{L}$ is an almost left bi-quasi-ideal of $\mathcal{S}$. Thus, by Theorem 4.7, $\lambda_{\mathcal{L}}$ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$. Let ν be a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$ such that $\lambda_{\mathcal{L}} \subseteq \nu$. Then $\mathcal{L} = \sup(\lambda_{\mathcal{L}}) \subseteq \sup(\nu)$. Also, $\sup(\nu)$ is an almost left bi-quasi ideal of $\mathcal{S}$. Since $\mathcal{L}$ is maximal, we have $\mathcal{L} = \sup(\lambda_{\mathcal{L}}) = \sup(\nu)$. Therefore, $\lambda_{\mathcal{L}}$ is maximal fuzzy almost left bi-quasi-ideal of $\mathcal{S}$.

Conversely, suppose that $\lambda_{\mathcal{L}}$ is a maxiimal fuzzy almost left bi-quasi-ideal of $\mathcal{S}$. Then $\lambda_{\mathcal{L}}$ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$. By Theorem 4.7, $\mathcal{L}$ is an almost left bi-quasi-ideal of $\mathcal{S}$. Let $\mathcal{K}$ be an almost left bi-quasi-ideal of $\mathcal{S}$ such that $\mathcal{L} \subseteq \mathcal{K}$. Then $\lambda_{\mathcal{K}}$ is a fuzzy almost left bi-quasi-ideal of $\mathcal{S}$ such that $\lambda_{\mathcal{L}} \subseteq \lambda_{\mathcal{K}}$. By assumption, Hence, $\mathcal{K} = \sup(\lambda_{\mathcal{K}}) = \sup(\lambda_{\mathcal{L}}) = \mathcal{L}$. Therefore, $\mathcal{L}$ is maximal of $\mathcal{S}$.

The proofs of 2 and 3 are similar to the proof 1. □

Corollary 4.13. *Let T be a semigroup. Then T has no proper almost bi-quasi-ideal if and only if for all fuzzy almost bi-quasi-ideals ρ of $\mathcal{S}$, $\mathrm{supp}(\rho) = \mathcal{S}$.*

Definition 4.6. Let $\mathcal{P}$ be an almost bi-quasi-ideal of semigroup $\mathcal{S}$. Then we called

(1) $\mathcal{P}$ is a *prime* if for any alomst bi-quasi-ideals $\mathcal{M}$ and $\mathcal{L}$ of $\mathcal{S}$ such that $\mathcal{M}\mathcal{L} \subseteq \mathcal{P}$ implies that $\mathcal{M} \subseteq \mathcal{P}$ or $\mathcal{L} \subseteq \mathcal{P}$.
(2) $\mathcal{P}$ is a *semiprime* if for any alomst bi-quasi-ideal $\mathcal{M}$ of $\mathcal{S}$ such that $\mathcal{M}^2 \subseteq \mathcal{P}$ implies that $\mathcal{M} \subseteq \mathcal{P}$.
(3) $\mathcal{P}$ is a *strongly prime* if for any almost bi-quasi-ideals $\mathcal{M}$ and $\mathcal{L}$ of $\mathcal{S}$ such that $\mathcal{M}\mathcal{L} \cap \mathcal{L}\mathcal{M} \subseteq \mathcal{P}$ implies that $\mathcal{M} \subseteq \mathcal{P}$ or $\mathcal{L} \subseteq \mathcal{P}$.

Definition 4.7. An fuzzy almost bi-quasi-ideal ρ on a semigroup $\mathcal{S}$. Then we called

(1) ρ is a *prime* if for any two fuzzy almost bi-quasi-ideals ω and ν of $\mathcal{S}$ such taht $\omega \circ \nu$ implies that $\omega \subseteq \rho$ or $\nu \subseteq \rho$.
(2) ρ is a *semiprime* if for any fuzzy almost bi-quasi-ideal ω of $\mathcal{S}$ such taht $\omega \circ \omega$ implies that $\omega \subseteq \rho$.
(3) ρ is a *strongly prime* if for any two fuzzy almost bi-quasi-ideals ω and ν of $\mathcal{S}$ such taht $(\omega \circ \nu) \cap (\nu \circ \omega) \subseteq \rho$ implies that $\omega \subseteq \rho$ or $\nu \subseteq \rho$.

It is clarly, every strongly prime fuzzy almost bi-quasi-ideal of a semigroup is a prime fuzzy almost ideal, and every prime fuzzy almost bi-quasi-ideal of a semigroup is a semiprime fuzzyalmost bi-quasi-ideal.

Theorem 4.14. *Let $\mathcal{P}$ be a nonempty subset of a semigroup $\mathcal{S}$. Then*

(1) $\mathcal{P}$ is a prime (resp., semiprime) almost bi-quasi-ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{P}}$ is a prime (resp., semiprime) fuzzy almost bi-ideal of $\mathcal{S}$.
(2) $\mathcal{P}$ is a strongly prime almost bi-quasi-ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{P}}$ is a fuzzy strongly prime almost bi-quasi-ideal of $\mathcal{S}$.

Proof. (1) Suppose that $\mathcal{P}$ is a prime almost bi-quasi-ideal of a semigroup $\mathcal{S}$. Then $\mathcal{P}$ is an almost bi-quasi-ideal of $\mathcal{S}$. Thus by Theorem 4.7, $\lambda_{\mathcal{P}}$ is a fuzzy almost bi-quasi-ideal of $\mathcal{S}$. Let ρ and ω be fuzzy almost bi-quasi-ideals of $\mathcal{S}$ such that $\rho \circ \omega \subseteq \lambda_{\mathcal{P}}$. Assume that $\rho \nsubseteq \lambda_{\mathcal{P}}$ and $\omega \nsubseteq \lambda_{\mathcal{P}}$. Then there exist $h, r \in \mathcal{S}$ such that $\rho(h) \neq 0$ and $\omega(r) \neq 0$. While $\lambda_{\mathcal{P}}(h) = 0$ and $\lambda_{\mathcal{P}}(r) = 0$. Thus $h \in$supp(ρ) and $r \in$supp(ω), but $h, r \notin \mathcal{P}$. So supp$(\rho) \nsubseteq \mathcal{P}$ and supp$(\omega) \nsubseteq \mathcal{P}$. Since supp$(\rho)$ and supp(ω) are almost bi-quasi-ideals of $\mathcal{S}$ we have supp(ρ) supp$(\omega) \nsubseteq \mathcal{P}$. Thus there exists $m = de$ for some $d \in$ supp(ρ) and $e \in$ supp(ω) such that $m \in \mathcal{P}$. Hence $\lambda_{\mathcal{P}}(m) = 0$ implies that $(\rho \circ \omega)(m) = 0$. Since $\rho \circ \omega \subseteq \lambda_{\mathcal{S}}$. Since $d \in$ supp(ρ) and $e \in$ supp(ω) we have $\rho(d) \neq 0$ and $\omega(e) \neq 0$. Thus $(\rho \circ \omega)(m) = \bigvee_{(de) \in F_m} \{\rho(d) \wedge \omega(e)\} \neq 0$. It is a contradiction so $\rho \subseteq \lambda_{\mathcal{P}}$ or $\omega \subseteq \lambda_{\mathcal{P}}$. Therefore $\lambda_{\mathcal{P}}$ is a prime fuzzy almost bi-quasi-ideal of $\mathcal{S}$.

Conversely, suppose that $\lambda_{\mathcal{P}}$ is a prime almost bi-quasi-ideal of $\mathcal{S}$. Then $\lambda_{\mathcal{P}}$ is an almost bi-quasi-ideal of $\mathcal{S}$. Thus by Theorem 4.7, $\mathcal{P}$ is an almost bi-quasi-ideal of $\mathcal{S}$. Let $\mathcal{M}$ and $\mathcal{L}$ be almost bi-quasi-ideals of $\mathcal{S}$ such that $\mathcal{ML} \subseteq \mathcal{P}$. Then $\lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{L}}$ are fuzzy almost bi-quasi-ideals of $\mathcal{S}$. By Lemma 2.1 $\lambda_{\mathcal{M}} \circ \lambda_{\mathcal{P}} = \lambda_{\mathcal{ML}} \subseteq \lambda_{\mathcal{P}}$. By assumption, $\lambda_{\mathcal{M}} \subseteq \lambda_{\mathcal{P}}$ or $\lambda_{\mathcal{L}} \subseteq \lambda_{\mathcal{P}}$. Thus $\mathcal{M} \subseteq \mathcal{P}$ or $\mathcal{L} \subseteq \mathcal{P}$. We conclude that $\mathcal{P}$ is a prime almost bi-quasi-ideal of $\mathcal{S}$.

(2) Suppose that $\mathcal{P}$ is a strongly prime almost bi-quasi-ideal of a semigroup $\mathcal{S}$. Then $\mathcal{P}$ is an almost bi-quasi-ideal of $\mathcal{S}$. Thus by Theorem 4.7, $\lambda_{\mathcal{P}}$ is a fuzzy almost bi-quasi-ideal of $\mathcal{S}$. Let ρ and ω be fuzzy almost bi-quasi-ideals of $\mathcal{S}$ such that $(\rho \circ \omega) \cap (\omega \circ \rho) \subseteq \lambda_{\mathcal{P}}$. Assume that $\rho \nsubseteq \lambda_P$ and $\omega \nsubseteq \lambda_{\mathcal{P}}$. Then there exist $h, r \in \mathcal{S}$ such that $\rho(h) \neq 0$ and $\omega(r) \neq 0$. While $\lambda_{\mathcal{P}}(h) = 0$ and $\lambda_{\mathcal{P}}(r) = 0$. Thus $h \in$ supp(ρ) and $r \in$ supp(ω), but $h, r \notin K$. So supp$(\rho) \nsubseteq \mathcal{P}$ and supp$(\omega) \nsubseteq \mathcal{P}$. Hence, there exists $m \in [$supp(ρ) supp$(\omega)] \cap [$supp(ρ) supp$(\omega)]$ such that $m \notin \mathcal{P}$. Thus $\lambda_{\mathcal{P}}(m) = 0$. Since $m \in$ supp(ρ) supp(ω) and $m \in$ supp(ω) supp(ρ) we have $m = d_1e_1$ and $m = e_2d_2$ for some $d_1, d_2 \in$ supp(ρ) and for some $e_1, e_2 \in$ supp(ω). Thus $(\rho \circ \omega)(m) = \bigvee_{(d_1e_1) \in F_m} \{\rho^p(d_1) \wedge \omega^p(e_1)\} \neq 0$ and

$(\omega \circ \rho)(m) = \bigvee_{(e_2d_2) \in F_m} \{\omega(e_2) \wedge \rho(d_2)\} \neq 0$. So $(\rho \circ \omega)(m) \cap (\rho \circ \omega)(m) \neq 0$.

It is a contradiction so $(\rho \circ \omega)(m) \wedge (\omega \circ \rho)(m) = 0$. Hence, $\rho \subseteq \lambda_P$ and $\omega \subseteq \lambda_P$. Therefore $\lambda_{\mathcal{P}}$ is a fuzzy strongly prime almost bi-quasi-ideal of $\mathcal{S}$.

Conversely, suppose that $\lambda_{\mathcal{P}}$ is a fuzzy strongly prime almost bi-quasi-ideal of $\mathcal{S}$. Then $\lambda_{\mathcal{P}}$ is a fuzzy almost bi-quasi-ideal of $\mathcal{S}$. Thus by Theorem 4.7, $\mathcal{P}$ is an almost bi-quasi-ideal of $\mathcal{S}$. Let $\mathcal{M}$ and $\mathcal{L}$ be alomst ideals of $\mathcal{S}$ such that $\mathcal{ML} \cap \mathcal{LM} \subseteq \mathcal{P}$. Then $\lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{L}}$ are fuzzy almost bi-quasi-ideals of $\mathcal{S}$. By Lemma 2.1 $\lambda_{\mathcal{ML}} = \lambda_{\mathcal{M}} \circ \lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{LM}} = \lambda_{\mathcal{L}} \circ \lambda_{\mathcal{M}}$. Thus $(\lambda_{\mathcal{M}} \circ \lambda_{\mathcal{L}}) \cap (\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{M}}) = \lambda_{\mathcal{ML}} \cap \lambda_{\mathcal{LM}} = \lambda_{\mathcal{ML} \cap \mathcal{LM}} \subseteq \lambda_{\mathcal{P}}$. By assumption, $\lambda_{\mathcal{M}} \leq \lambda_{\mathcal{P}}$ or $\lambda_{\mathcal{L}} \geq \lambda_{\mathcal{P}}$. Thus $\mathcal{M} \subseteq \mathcal{P}$ or $\mathcal{L} \subseteq \mathcal{P}$. We conclude that $\mathcal{P}$ is a strongly prime almost bi-quasi-ideal of $\mathcal{S}$.

□

5. Conclusion

Every bi-quasi-ideal is an almost bi-qausi-ideal of a semigroup. The union of two almost bi-quasi-ideal is also an almost bi-quasi-ideal in semigroups and results in class fuzzifications is the same. We investigate relationships between almost bi-quasi-ideals and class fuzzifications. In the future work, we can study bi-quasi-ideals and their fuzzifications in other algebraic structures.

Acknowledgment

This research project was supported by the Thailand Science Research and Innovation fund and the University of Phayao (Grant No. FF66-UoE017) Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand.

References

[1] S. Bogdanovic, Semigroups in which some bi-ideal is a group, *Review of Research Faculty of Science-University of Novi Sad* **11** (1981), 261–266.
[2] R. Chinram, W. Nakkhasen, Almsot bi-quasi-interior ideals and fuzzy alomst bi-quasi-interior ideals of semigroups, *Journal of Mathematics and Computer Sciences* **26** (2022), 128–136.
[3] O. Grosek, L. Satko, A new notion in the theory of semigroup, *Semigroup Forum* **23** (1980), 233–240.
[4] N. Kaopusek, T. Kaewnoi, R. Chinram, On almost interior ideals and weakly almost interior ideals of semigroups, *Journal of Discrete Mathematical Sciences and Cryptography* **23** (2020), no. 3, 773–778.
[5] M. Krishna Rao, Left bi-quasi ideals of semigroups, *Southeast Asian Bulletion of Mathematics* **44** (2020), 369–376.
[6] N. Kuroki, Fuzzy bi-ideals in semigroup, *Comment. Math. Univ. St. Paul* **5** (1979), 128–132.
[7] T. Gaketem, Almost bi interior ideal in semigroups and their fuzzifications, *European Journal of Pure and Applied Mathematics* **15** (2022), no. 1, 281–289.
[8] S. Nawaz, M. Gulistan, N. Kausar, Salahuddin, M. Munir, On the left and right almost hyperideals of LA-semihypergroups, *International Journal of Fuzzy Logic and Intelligent Systems*, **21** (2021), 86–92.
[9] A. Simuen, W. Yonthanthum, R. Chinram, Almost interior gamma-ideals and fuzzy almost interior gamma-ideals in gamma semigroups, *Mathematics and Staistics* **9** (2021), 302–308.
[10] S. Suebsung, K. Wattanatripop, R. Chinram, On almost (m, n)-ideals and fuzzy almost (m, n)-ideals in semigroups, *Journal of Taibah University for Science* **13** (2019), 897–902.
[11] S. Suebsung, K. Wattanatripop, R. Chinram, A-ideals and fuzzy A- ideals of ternary semigroups, *Songklanakarin Journal of Science and Technology* **41** (2019), 299–304.
[12] N. Tiprachot, N. Lekkoksung, A note on almost α-ideals in semigroups, *International Journal of Mathematics and Computer Science* **17** (2022), 325–330.
[13] Y. Rao, S. Kosari, Z. Shao, M. Akhoundi, S. Omidi, A study on A-I-Γ-hyperideals and (m, n)-Γ-hyperfilters in ordered Γ-semihypergroups, *Discrete Dynamics in Nature and Society*, **2021** (2021), 6683910.
[14] K. Wattanatripop, R. Chinram, T. Changphas, Quasi-A-ideals and fuzzy A-ideals in semigroups, *Journal of Discrecte Mathematical Science and Cryptography* **21** (2018), 1131–1138.
[15] L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338–353.

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