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Permanent of Toeplitz Matrices with Narayana Entries

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Abstract. In this paper, we investigate the properties of Toeplitz matrices with entries derived from the Narayana sequence. We demonstrate that when constructing Toeplitz matrices using Narayana numbers in a specific manner, their permanents exhibit a unique relationship, characterized as an exponential function. This novel finding offers new insights into the interplay between Toeplitz matrices and the Narayana sequence, expanding our understanding of the mathematical properties and potential applications of both.

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1. Introduction

A finite Toeplitz matrix $T = (b_{i-j})_{i,j=1}^{n}$ is an $n \times n$ matrix with the following structure:

$$\begin{pmatrix} b_0 & b_{-1} & b_{-2} & \cdots & b_{-n+1} \\ b_1 & b_0 & b_{-1} & \cdots & b_{-n+2} \\ b_2 & b_1 & b_0 & \cdots & b_{-n+3} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_0 \end{pmatrix}$$

The entries depend on the difference $i - j$ and hence they are constant down all the diagonals.

The Narayana sequence (or Narayana's cows sequence), $\{b_n\}_{n\geq 0}$, is defined by the following third order recurrence relation:

$$b_n = b_{n-1} + b_{n-3}, b_0 = 0, b_1 = b_2 = 1$$

for $n \geq 3$.

The permanent of a square matrix is a number that is defined in a way similar to the determinant. For an arbitrary matrix $A \in M_{N\times N}(\mathbb{C})$, its permanent is defined by

$$\text{per}(A) = \sum_{\pi \in S_N} \prod_{i=1}^{N} A_{i,\pi(i)}$$

where S_N denotes the permutation group of the set $\{1, \ldots, N\}$. Despite the similarity in definition, the permanent has fewer properties than the determinant. No nice geometric or algebraic interpretation is known for permanent, and it is neither multiplicative nor invariant under linear combinations of rows or columns. It is mainly

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used in combinatorics and in dealing with boson Green's functions in quantum field theory. However, it has two graph-theoretic interpretations: as the sum of weights of a cycle that covers a directed graph, and as the sum of weights of perfect matchings in a bipartite graph. In matrix algebra, computations of determinants and permanents have a great importance in many branches of mathematics. Also, determinants and permanents have many applications in physics, chemistry, electrical engineering, and so on. There are a lot of relationships between determinantal and permanental representations of matrices and these sequences of integer (see, [3],[2],[4],[5],[6]). Toeplitz-Hessenberg determinants with entries that are Fibonacci-Narayana (or Narayana's cows) numbers have been studied (see, [1]). In this paper, we study the permanent of Toeplitz matrices with Narayana entries. We show that the permanent of a Toeplitz matrix of size $n \times n$ can be defined as an exponential function. The paper is organized as follows: In section 1, we give definitions and notations that we use throughout the paper, in section 2, we present a recursive formula for permanent of Toeplitz matrices, and in sections 3, 4 we present some results on the permanent of Toeplitz matrices.

Example 1.1. Let

$$A := \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}.$$

The per(A) for a general 3×3 matrix using all the permutations in S_3 is

$$\operatorname{per}(A) = aei + bfg + cdh + afh + bdi + ceg.$$

We consider the following $n \times n$ Toeplitz matrix with entries from the Narayana sequence:

$$T_n\left(b_0; b_1, b_2, \ldots, b_n\right) = \begin{bmatrix} b_1 & b_0 & 0 & \ldots & 0 & 0 \\ b_2 & b_1 & b_0 & \ldots & 0 & 0 \\ b_3 & b_2 & b_1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots & \ldots \\ b_{n-1} & b_{n-2} & b_{n-3} & \ldots & b_1 & b_0 \\ b_n & b_{n-1} & b_{n-2} & \ldots & b_2 & b_1 \end{bmatrix},$$

where $b_0 = 1$ and $b_k \neq 0$ for at least one $k > 0$. So $b_{ij} = 0$ for $j > i + 1$.

In this section, we investigate the properties of Toeplitz matrices constructed using Narayana numbers. Specifically, we are interested in determining a recursive formula for the permanent of these matrices. Toeplitz matrices have a constant value on each diagonal and are defined by their first row and column. In our case, we build a Toeplitz matrix $T_n\left(b_0; b_1, b_2, \ldots, b_n\right)$ with entries from the Narayana sequences. The main result of this section is the following theorem, which presents a recursive formula for the permanent of these Toeplitz matrices.

Now you can state the theorem:

2. Recursive formula for permanent of Toeplitz matrices with Narayana Entries

Theorem 2.1. *Let $T_n\,(b_0; b_1, b_2, \dots, b_n)$ be a Toeplitz matrix built from Narayana numbers. Given that* $\mathrm{per(T_0)} = \mathrm{per(T_1)} = 1$, *then the recursive formula for permanent of T_n is given by*

$$\mathrm{per(T_n)} = \sum_{j=1}^{n} b_j \,\mathrm{per(T_{n\text{-}j})}$$

Proof. Suppose

$$T_n = (b_0; b_1, b_2, \dots, b_n) = \begin{bmatrix} b_1 & b_0 & 0 & \dots & 0 & 0 \\ b_2 & b_1 & b_0 & \dots & 0 & 0 \\ b_3 & b_2 & b_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ b_{n-1} & b_{n-2} & b_{n-3} & \dots & b_1 & b_0 \\ b_n & b_{n-1} & b_{n-2} & \dots & b_2 & b_1 \end{bmatrix},$$

where $b_0 = 1$.

To get the permanent of T_n, we will expand T_n along its first or last column as follows.

$$\mathrm{per(T_n)} = 1 \cdot \mathrm{per} \begin{bmatrix} b_1 & 1 & 0 & \dots & 0 & 0 \\ b_2 & b_1 & 1 & \dots & 0 & 0 \\ b_3 & b_2 & b_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ b_{n-2} & b_{n-3} & b_{n-4} & \dots & b_1 & 1 \\ b_n & b_{n-1} & b_{n-2} & \dots & b_3 & b_2 \end{bmatrix}$$

$$+ b_1 \cdot \mathrm{per} \begin{bmatrix} b_1 & 1 & 0 & \dots & 0 & 0 \\ b_2 & b_1 & 1 & \dots & 0 & 0 \\ b_3 & b_2 & b_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ b_{n-2} & b_{n-3} & b_{n-4} & \dots & b_1 & 1 \\ b_{n-1} & b_{n-2} & b_{n-3} & \dots & b_2 & b_1 \end{bmatrix}$$

$$= 1 \cdot \mathrm{per} \begin{bmatrix} b_1 & 1 & 0 & \dots & 0 & 0 \\ b_2 & b_1 & 1 & \dots & 0 & 0 \\ b_3 & b_2 & b_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ b_{n-2} & b_{n-3} & b_{n-4} & \dots & b_1 & 1 \\ b_n & b_{n-1} & b_{n-2} & \dots & b_3 & b_2 \end{bmatrix} + b_1 \cdot \mathrm{per(T_{n\text{-}1})}$$

$$= 1 \cdot \mathrm{per} \begin{bmatrix} b_1 & 1 & 0 & \dots & 0 & 0 \\ b_2 & b_1 & 1 & \dots & 0 & 0 \\ b_3 & b_2 & b_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ b_{n-2} & b_{n-3} & b_{n-4} & \dots & b_1 & 1 \\ b_n & b_{n-1} & b_{n-2} & \dots & b_4 & b_3 \end{bmatrix} + b_2 \cdot \mathrm{per(T_{n\text{-}2})} + b_1 \cdot \mathrm{per(T_{n\text{-}1})}$$

$$= 1 \cdot \text{per} \begin{bmatrix} b_1 & 1 & 0 & \dots & 0 & 0 \\ b_2 & b_1 & 1 & \dots & 0 & 0 \\ b_3 & b_2 & b_1 & \dots & 0 & 0 \\ & & & \ddots & & \\ \dots & \dots & \dots & & \dots & \dots \\ b_{n-2} & b_{n-3} & b_{n-4} & \dots & b_1 & 1 \\ b_n & b_{n-1} & b_{n-2} & \dots & b_5 & b_4 \end{bmatrix}$$
$$+ b_3 \cdot \text{per}(\mathrm{T_{n\text{-}3}}) + b_2 \cdot \text{per}(\mathrm{T_{n\text{-}2}}) + b_1 \cdot \text{per}(\mathrm{T_{n\text{-}1}})$$

If we continue this process, we will get

$$\begin{aligned} \text{per}(\mathrm{T_n}) &= b_1 \cdot \text{per}(\mathrm{T_{n\text{-}1}}) + b_2 \cdot \text{per}(\mathrm{T_{n\text{-}2}}) + \cdots + b_n \cdot \text{per}(\mathrm{T_0}) \\ &= \sum_{j=1}^{n} b_j \, \text{per}(\mathrm{T_{n\text{-}j}}) \end{aligned}$$

□

3. Odd Toeplitz matrices with Narayana entries

We define odd Toeplitz matrix with Narayana entries as follows:

$$T_n\left(1; b_1, b_3, \dots, b_{2n-1}\right) = \begin{bmatrix} b_1 & 1 & 0 & \dots & 0 & 0 \\ b_3 & b_1 & 1 & \dots & 0 & 0 \\ b_5 & b_3 & b_1 & \dots & 0 & 0 \\ & & & \ddots & & \\ \dots & \dots & \dots & & \dots & \dots \\ b_{2n-3} & b_{2n-5} & b_{2n-7} & \dots & b_1 & 1 \\ b_{2n-1} & b_{2n-3} & b_{2n-5} & \dots & b_3 & b_1 \end{bmatrix},$$

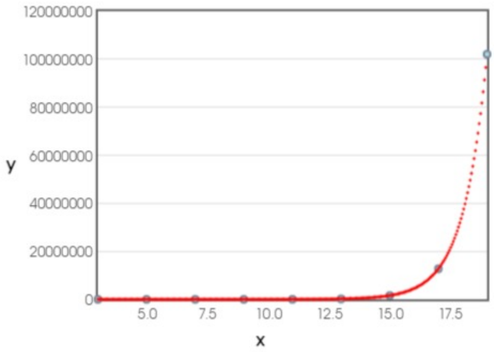
Example 3.1. A 5×5 odd Toeplitz matrix with Narayana entries.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 1 & 0 \\ 6 & 3 & 1 & 1 & 1 \\ 13 & 6 & 3 & 1 & 1 \end{bmatrix}$$

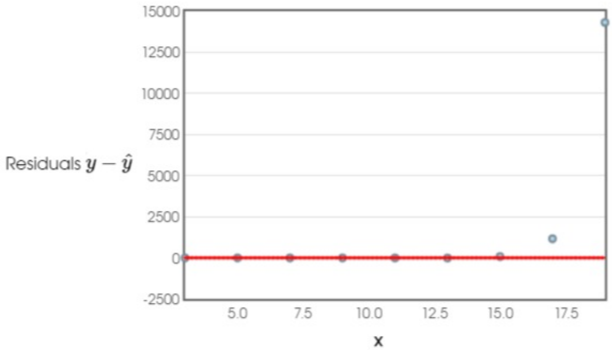
In table below, we include permanent of Toeplitz matrices with different odd sizes.

Table 1. Permanent of Odd-Sized Toeplitz Matrices

Size (n)	**Permanent (Per(T_n))**
3	6
5	48
7	385
9	3,086
11	24,736
13	198,270
15	1,589,300
17	12,739,000
19	102,110,000



Regression Equation: $\operatorname{per}\left(\widehat{T}_n\right) = (0.2641473372) \cdot (2.8311041625)^n$
Correlation: $r = 0.9999999964$
R-squared: $r^2 = 0.9999999928$



In this section, we further explore the properties of Toeplitz matrices with entries from the Narayana sequences, focusing on the behavior of their permanents. We consider Toeplitz matrices $T_n\left(1; b_1, b_3, \ldots, b_{2n-1}\right)$ with Narayana entries $b_1, b_3, \ldots, b_{2n-1}$, and their modified counterparts $\widehat{T}_n$. Our aim is to examine the differences in the permanents of these matrices when n is an odd integer greater than or equal to 3. The main result of this section is the following theorem, which establishes an upper bound on the absolute difference between the permanents of these matrices.

Now you can state the theorem:

Theorem 3.1. *Let $T_n\left(1; b_1, b_3, \ldots, b_{2n-1}\right)$ be a Toeplitz matrix with Narayana entries $b_1, b_3, \ldots, b_{2n-1}$. Whenever $n \geq 3$ is odd, we then have the following.*

$$|\operatorname{per}\left(T_n\right) - \operatorname{per}(\widehat{T}_n)| < \varepsilon$$

for any $\varepsilon > 0$.

Proof. We use induction on n. For the base case, let $n = 3$. Then $\operatorname{per}(\widehat{T}_3) = (0.2641473372) \cdot (2.8311041625)^3 = 5.9939591931$. Now it is clear that

$$|\operatorname{per}\left(T_3\right) - \operatorname{per}(\widehat{T}_3)| < \varepsilon$$

for any $\varepsilon > 0$.

Next suppose that

$$|\operatorname{per}(T_n) - \operatorname{per}(\widehat{T}_n)| < \frac{\varepsilon}{b^2}$$

for any $\varepsilon > 0$ is true for $n = k$. We will prove that the result also holds for $n = k+2$.

Notice that

$$\begin{aligned}\operatorname{per}(\widehat{T}_{k+2}) &= (0.2641473372) \cdot (2.8311041625)^{k+2} \\ &= (0.2641473372) \cdot (2.8311041625)^{k} \cdot (2.8311041625)^{2} \\ &= \operatorname{per}(\widehat{T}_k) \cdot (2.8311041625)^2\end{aligned}$$

we let $b = (2.8311041625)^2$, then $\operatorname{per}(\widehat{T}_{k+2}) = b^2 \operatorname{per}(\widehat{T}_k)$.

Similarly, we can show that

$$\operatorname{per}(T_{k+2}) = b^2 \operatorname{per}(T_k).$$

For $n = k+2$, we have

$$\begin{aligned}|\operatorname{per}(T_{k+2}) - \operatorname{per}(\widehat{T}_{k+2})| &= |b^2 \operatorname{per}(T_k) - b^2 \operatorname{per}(\widehat{T}_k)| \\ &= b^2 |\operatorname{per}(T_k) - \operatorname{per}(\widehat{T}_k)| \\ &< b^2 (\frac{\varepsilon}{b^2}) \\ &= \varepsilon.\end{aligned}$$

□

4. Even Toeplitz matrices with Narayana entries

We define even Toeplitz matrix with Narayana entries as follows:

$$T_n(1; b_2, b_4, \ldots, b_{2n-2}) = \begin{bmatrix} b_2 & 1 & 0 & \ldots & 0 & 0 \\ b_4 & b_2 & 1 & \ldots & 0 & 0 \\ b_6 & b_4 & b_2 & \ldots & 0 & 0 \\ & & & \ddots & & \\ \ldots & \ldots & \ldots & & \ldots & \ldots \\ b_{2n-4} & b_{2n-6} & b_{2n-8} & \ldots & b_2 & 1 \\ b_{2n} & b_{2n-4} & b_{2n-6} & \ldots & b_4 & b_2 \end{bmatrix},$$

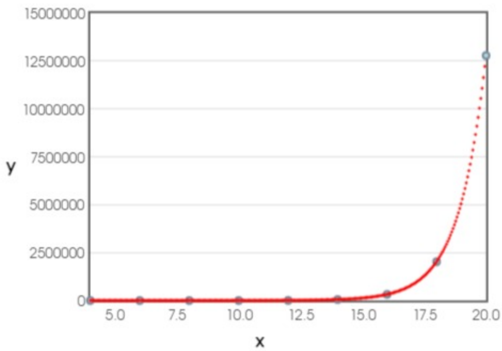
Example 4.1. A 4×4 even Toeplitz matrix with Narayana entries.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 1 \\ 9 & 4 & 2 & 1 \end{bmatrix}$$

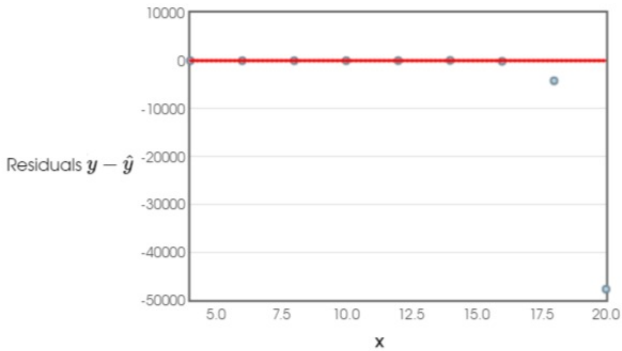
In table below, we include permanent of Toeplitz matrices with different even sizes.

TABLE 2. Permanent of Even-Sized Toeplitz Matrices

Size (n)	**Permanent (Per(T_n))**
4	5
6	32
8	203
10	1,281
12	8,080
14	50,967
16	321,490
18	2,027,900
20	12,792,000



Regression Equation: per $\left(\widehat{T}_n\right) = (0.1266043096) \cdot (2.5136533511)^n$
Correlation: $r = 0.9999994894$
R-squared: $r^2 = 0.9999989787$



In this section, we continue our investigation into the properties of Toeplitz matrices with entries from the Narayana sequences by examining the behavior of their permanents under different conditions. We now consider Toeplitz matrices $T_n\left(1; b_2, b_4, \ldots, b_{2n-2}\right)$ with Narayana entries $b_2, b_4, \ldots, b_{2n-2}$, and their modified counterparts $\widehat{T}_n$. Our focus is on the differences in the permanents of these matrices when n is an even integer greater than or equal to 4. The main result of this section is the following theorem, which establishes an upper bound on the absolute difference between the permanents of these matrices.

Now you can state the theorem:

Theorem 4.1. *Let $T_n\,(1; b_2, b_4, \ldots, b_{2n-2})$ be a Toeplitz matrix with Narayana entries $b_2, b_4, \ldots, b_{2n-2}$. Whenever $n \geq 4$ is even, we then have the following:*

$$|\operatorname{per}(T_n) - \operatorname{per}(\widehat{T}_n)| < \varepsilon$$

for any $\varepsilon > 0$.

Proof. We use induction on n. For the base case, let $n = 4$. Then $\operatorname{per}(\widehat{T}_4) = (0.1266043096) \cdot (2.5136533511)^4 = 5.2409607409$. Now it is clear that

$$|\operatorname{per}(T_4) - \operatorname{per}(\widehat{T}_4)| < \varepsilon$$

for any $\varepsilon > 0$.

Next suppose that

$$|\operatorname{per}(T_n) - \operatorname{per}(\widehat{T}_n)| < \frac{\varepsilon}{b^2}$$

for any $\varepsilon > 0$ is true for $n = k$. We will prove that the result also holds for $n = k + 2$.

Notice that

$$\begin{aligned}
\operatorname{per}(\widehat{T}_{k+2}) &= (0.1266043096) \cdot (2.5136533511))^{k+2} \\
&= (0.1266043096) \cdot (2.5136533511)^k \cdot (2.5136533511)^2 \\
&= \operatorname{per}(\widehat{T}_k) \cdot (2.8311041625)^2
\end{aligned}$$

we let $b = (2.5136533511)^2$, then $\operatorname{per}(\widehat{T}_{k+2}) = b^2 \operatorname{per}(\widehat{T}_k)$. And similarly, we can show that

$$\operatorname{per}(T_{k+2}) = b^2 \operatorname{per}(T_k).$$

For $n = k + 2$, we have

$$\begin{aligned}
|\operatorname{per}(T_{k+2}) - \operatorname{per}(\widehat{T}_{k+2})| &= |b^2 \operatorname{per}(T_k) - b^2 \operatorname{per}(\widehat{T}_k)| \\
&= b^2 |\operatorname{per}(T_k) - \operatorname{per}(\widehat{T}_k)| \\
&< b^2 (\frac{\varepsilon}{b^2}) \\
&= \varepsilon.
\end{aligned}$$

□

5. Conclusion

In this study, we have introduced and explored a new class of Toeplitz matrices with entries derived from the Narayana sequences. We have rigorously established that the permanent of Toeplitz matrices belonging to this class can be characterized as an exponential function. To further substantiate our findings, we have provided several numerical examples for both even and odd-sized Toeplitz matrices.

Our work sheds new light on the intricate relationship between Toeplitz matrices and the Narayana sequences, thereby expanding the current understanding of their mathematical properties and potential applications. As a future direction, we plan to extend this research to encompass other well-known sequences, further broadening the scope and applicability of our results. Details of these extensions will be presented in an upcoming paper.

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