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Neutrosophic $\mathfrak{Y}$-Cesàro Summability of a Sequence of Order α of Neutrosophic Random Variables in Probability

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Abstract. In this paper, we define the notions of neutrosophic $\mathfrak{Y}$-Cesàro summability of a sequence of order α, neutrosophic $\mathfrak{Y}$-lacunary statistical convergence of order α, neutrosophic strongly $\mathfrak{Y}$-lacunary statistical convergence of order α and neutrosophic strongly $\mathfrak{Y}$-Cesàro summability of order α in neutrosophic probability. Besides, we prove some relations among them.

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1. Introduction

The notion of ideal was introduced by Kuratowski in 1933 on [16], an ideal $\mathfrak{Y}$ on a space $\mathfrak{X}$ is a collection of elements of $\mathfrak{X}$ which satisfies: (1) $\emptyset \in \mathfrak{Y}$; (2) If $A, B \in \mathfrak{Y}$ then $A \cup B \in \mathfrak{Y}$; and (3) if $B \subset \mathfrak{Y}$ and $A \subset B$, then $A \in \mathfrak{Y}$. This notion has been grown in several concepts of general topology. On the other hand, the idea of statistical convergence was known to A. Zygmund as early as 1935 and in particular after 1951 when Steinhaus [5] and Fast [6] reintroduced statistical convergence for sequences of real numbers, several generalizations and applications of this notion have been investigated. Recently, many studies associated to statistical convergence on ideal spaces have formulated. One of then was presented by Kisi and Guler [17] in which they studied Cesàro summability of random variables in probability.

Smarandache [4] presented a new branch of philosophy, with dealing with the begin, nature and scope of neutralities, as well as their interactions with different situations. The principal thesis of neutrosophy is that every idea has not only a certain degree of truth, as is generally assumed in many-valued logic contexts, but also a falsity degree and an indeterminacy degree that have to be considered independently from each other. Smarandache looks like to understand such indeterminacy both in a subjective and an objective sense, i.e. as uncertainty as well as imprecision, vagueness, error and so on. Neutrosophy is a new mathematical theory which is a generalizing from classical logic and fuzzy logic, such as neutrosophic set theory, neutrosophic probability, neutrosophic statistics and neutrosophic logic.

Recently, Bisher and Hatip in 2020 [7] used the notion of random variable and indeterminacy of a neutrosophic set and they gave the first view of neutrosophic random variables in which they presented some basics notions. later on, Granados in 2021 [8] proved new notions on neutrosophic random variables and then Granados and

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Sanabria [9] studied independence neutrosophic random variables. In 2020, Granados et al. [10, 11] studied some neutrosophic probabilities distributions (discrete [10] and continuous [11]) based on neutrosophic random variables.

In this paper, we define the notion of neutrosophic $\mathfrak{Y}$-Cesàro Summability of a Sequence of Order α of Neutrosophic Random Variables in Probability. Granados [12] defined the notion of neutrosophic statistical convergence as follows:

Let $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ be a sequence of neutrosophic random variables where each X_{N_n} is defined on the same event space $\mathcal{S}$ with respect to a given class of subsets of $\mathcal{S}$ as the class Λ of events and a given probability function $\mathcal{P}:\Lambda\to\mathbb{R}$. The sequence $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ is said to be neutrosophic statistically convergent in probability to a neutrosophic random variable $\mathcal{X}$, where $\mathcal{X}:\mathcal{S}\to\mathbb{R}$, if for any $\varepsilon,\delta>0$,

$$\lim_{n\to\infty}\frac{1}{n}|k\leq n:\mathcal{P}(|\mathcal{X}_{N_k}-\mathcal{X}_N|\geq\varepsilon)\geq\delta\}|=0, \tag{1}$$

or equivalently,

$$\lim_{n\to\infty}\frac{1}{n}|k\leq n:1-\mathcal{P}(|\mathcal{X}_{N_k}-\mathcal{X}_N|<\varepsilon)\geq\delta\}|=0. \tag{2}$$

2. Preliminaries

In this section, we present some well-known notions which will be useful for the development of this paper.

Definition 2.1. (see [15]) Let $\mathcal{X}$ be a non-empty fixed set. A neutrosophic set $\mathcal{A}$ is an object having the form $\{x,(\mu\mathcal{A}(x),\delta\mathcal{A}(x),\gamma\mathcal{A}(x)):x\in\mathcal{X}\}$, where $\mu\mathcal{A}(x),\delta\mathcal{A}(x)$ and $\gamma\mathcal{A}(x)$ represent the degree of membership, the degree of indeterminacy , and the degree of non-membership respectively of each element $x\in\mathcal{X}$ to the set $\mathcal{A}$.

Definition 2.2. (see [14]) Let $\mathcal{K}$ be a field, the neutrosophic filed generated by $\mathcal{K}$ and I is denoted by $\langle\mathcal{K}\cup I\rangle$ under the operations of $\mathcal{K}$, where I is the neutrosophic element with the property $I^2=I$.

Definition 2.3. (see [13]) Classical neutrosophic number has the form $a+bI$ where a,b are real or complex numbers and I is the indeterminacy such that $0.I=0$ and $I^2=I$ which results that $I^n=I$ for all positive integers n.

Definition 2.4. (see [13]) The neutrosophic probability of event $\mathcal{A}$ occurrence is $NP(\mathcal{A})=(ch(\mathcal{A}),ch(neut\mathcal{A}),ch(anti\mathcal{A}))=(T,I,F)$ where T,I,F are standard or non-standard subsets of the non-standard unitary interval $]^-0,1^+[$.

Now, we present some notions of neutrosophic random variables [7].

Definition 2.5. Consider the real valued crisp random variable $\mathfrak{X}$ which is defined as follows:

$$\mathfrak{X}:\Omega\to\mathbb{R}$$

where Ω is the events space. Now, they defined a neutrosophic random variable $\mathfrak{X}_N$ as follows:

$$\mathfrak{X}_N:\Omega\to\mathbb{R}(I)$$

and

$$\mathfrak{X}_N=\mathfrak{X}+I$$

where I is indeterminacy.

Theorem 2.1. *Consider the neutrosophic random variable $\mathfrak{X}_N = \mathfrak{X} + I$ where cumulative distribution function of $\mathfrak{X}_N$ is $F_{\mathfrak{X}_N}(x) = P(\mathfrak{X}_N \leq x)$. Then, the following statements hold:*
(1) $F_{\mathfrak{X}_N}(x) = F_{\mathfrak{X}}(x - I)$,
(2) $f_{\mathfrak{X}_N}(x) = f_{\mathfrak{X}}(x - I)$.
Where $F_{\mathfrak{X}_N}$ and $f_{\mathfrak{X}_N}$ are cumulative distribution function and probability density function of $\mathfrak{X}_N$, respectively.

3. Main results

Definition 3.1. $\{\mathfrak{X}_{N_k}\}$ is called neutrosophic $\mathfrak{Y}$-statistically convergent of order α in probability to a neutrosophic random variable $\mathfrak{X}_N$ if for any $\varepsilon, \delta, \gamma > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha} |\{k \leq m : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \geq \gamma\right\} \in \mathfrak{Y},$$

and it will be denoted by $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PSN}(\mathfrak{Y})^\alpha} \mathfrak{X}_N$.

Definition 3.2. $\{\mathfrak{X}_{N_k}\}$ is called neutrosophic $\mathfrak{Y}$-lacunary statistically convergent of order α in probability to a neutrosophic random variable $\mathfrak{X}_N$ if for any $\varepsilon, \delta, \gamma > 0$,

$$\left\{t \in \mathbb{N} : \frac{1}{h_t^\alpha} |\{k \leq I_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \geq \gamma\right\} \in \mathfrak{Y},$$

and it will be denoted by $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PSN}_\theta(\mathfrak{Y})^\alpha} \mathfrak{X}_N$.

Definition 3.3. $\{\mathfrak{X}_{N_k}\}$ is called neutrosophic strongly $\mathfrak{Y}$-lacunary convergent or $\mathcal{PV}_\theta(\mathfrak{Y})$-convergent of order α in probability to a neutrosophic random variable $\mathfrak{X}_N$ if for any $\varepsilon, \delta > 0$,

$$\left\{t \in \mathbb{N} : \frac{1}{h_t^\alpha} \sum_{k \in I_t} \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}\right\} \in \mathfrak{Y},$$

and it will be denoted by $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PVN}_\theta(\mathfrak{Y})^\alpha} \mathfrak{X}_N$.

Definition 3.4. $\{\mathfrak{X}_{N_k}\}$ is called neutrosophic strongly $\mathfrak{Y}$-Cesàro summable of order α in probability to a neutrosophic random variable $\mathfrak{X}_N$ if for any $\varepsilon, \delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha} \sum_{k=1}^{n} \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}\right\} \in \mathfrak{Y},$$

and it will be denoted by $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PC}_\infty\mathcal{N}[\mathfrak{Y}]^\alpha} \mathfrak{X}_N$.

Theorem 3.1. *If $0 < \alpha \leq \beta \leq 1$ then $\mathcal{PSN}(\mathfrak{Y})^\alpha \subseteq \mathcal{PSN}(\mathfrak{Y})^\beta$.*

Proof. From the assumption, we can say that

$$\frac{1}{n^\beta} |\{k \leq n : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \leq \frac{1}{n^\alpha} |\{k \leq n : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}|.$$

Therefore,

$$\{n \in \mathbb{N} : \frac{1}{n^\beta}|\{k \le n : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon) \ge \delta\}| \ge \gamma\}$$
$$\subseteq \{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \le n : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon) \ge \delta\}| \ge \gamma\}.$$

For $\gamma > 0$. Hence, we have $\mathcal{PSN}(\mathfrak{Y})^\alpha \subseteq \mathcal{PSN}(\mathfrak{Y})^\beta$. □

Theorem 3.2. *If* $\liminf_t q_t > 1$. *If* $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PC}_\infty\mathcal{N}[\mathfrak{Y}]^\alpha} \mathfrak{X}_N$, *then* $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PVN}_\theta(\mathfrak{Y})^\alpha} \mathfrak{X}_N$.

Proof. If $\liminf_t q_t > 1$, there exists $\gamma > 0$ such that $q_t \ge 1 + \gamma$ for all $t \ge 1$. Since $h_t = k_t - k_{t-1}$, we have $\frac{k_t^\alpha}{h_t^\alpha} \le \left(\frac{1+\gamma}{\gamma}\right)^\alpha$ and $\frac{k_{t-1}^\alpha}{h_t^\alpha} \le \left(\frac{1}{\gamma}\right)^\alpha$. Let $\varepsilon > 0$ and we define the set $\mathcal{S}$ by

$$\mathcal{S} = \left\{k_t \in \mathbb{N} : \frac{1}{k_t^\alpha}\sum_{k=1}^{k_t}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon) < \delta\right\}.$$

Hence, $\mathcal{S} \in \mathfrak{F}(\mathfrak{Y})$. Thus,

$$\frac{1}{h_t^\alpha}\sum_{k\in I_t}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon) = \frac{1}{h_t^\alpha}\sum_{k=1}^{k_t}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon) - \frac{1}{h_t^\alpha}\sum_{k=1}^{k_{t-1}}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon)$$
$$= \frac{k_t^\alpha}{h_t^\alpha k_t^\alpha}\sum_{k=1}^{k_t}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon) - \frac{k_{t-1}^\alpha}{h_t^\alpha k_{t-1}^\alpha}\sum_{k=1}^{k_{t-1}}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon)$$
$$\le \left(\frac{1+\gamma}{\gamma}\right)^\alpha \delta - \left(\frac{1}{\gamma\delta}\right)^\alpha \delta^l.$$

For each $k_t \in \mathcal{S}$. Take, $\rho = \left(\frac{1+\gamma}{\gamma}\right)^\alpha \delta - \left(\frac{1}{\gamma\delta}\right)^\alpha \delta^l$. Hence,

$$\left\{t \in \mathbb{N} : \frac{1}{h_t^\alpha}\sum_{k\in I_t}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon) < \rho\right\} \in \mathfrak{F}(\mathfrak{Y}).$$

□

Theorem 3.3. *If* $\{\mathfrak{X}_{N_k}\}$ *is neutrosophic strongly* $\mathfrak{Y}$*-Cesàro summable of order* α*, then it is neutrosophic* $\mathfrak{Y}$*-statistical convergent of order* α *in probability to a neutrosophic random variable* $\mathfrak{X}_N$.

Proof. Let $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PC}_\infty\mathcal{N}[\mathfrak{Y}]^\alpha} \mathfrak{X}_N$ and $\varepsilon > 0$ given. Then,

$$\frac{1}{n^\alpha}\sum_{k=1}^{n}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon) \ge \frac{\delta}{n^\alpha}|k \le n : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon) \ge \delta\}|,$$

and so

$$\frac{1}{n^\alpha\delta}\sum_{k=1}^{n}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon) \ge \frac{1}{n^\alpha}|k \le n : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \ge \varepsilon) \ge \delta\}|.$$

Thus, for a given $\rho > 0$,

$$\{n \in \mathbb{N} : \frac{\delta}{n^\alpha}|k \leq n : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \geq \rho\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\sum_{k=1}^{n}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\rho\right\} \in \mathfrak{Y}.$$

Hence, $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PSN}(\mathfrak{Y})^\alpha} \mathfrak{X}_N$. □

Theorem 3.4. *Let a bounded $\{\mathfrak{X}_{N_k}\}$ be neutrosophic $\mathfrak{Y}$-statistical convergent of order α to $\mathfrak{X}_N$. Therefore, $\{\mathfrak{X}_{N_k}\}$ is neutrosophic strongly $\mathfrak{Y}$-Cesàro summable of order α to $\mathfrak{X}_N$.*

Proof. Let's consider $\{\mathfrak{X}_{N_k}\}$ is bounded and $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PSN}(\mathfrak{Y})^\alpha} \mathfrak{X}_N$. Since $\{\mathfrak{X}_{N_k}\}$ is bounded, we obtain $\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| > \varepsilon) \leq \mathcal{M}$ for all $k \in \mathbb{N}$. For $\varepsilon > 0$, we have

$$\frac{1}{n^\alpha}\sum_{k=1}^{n}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \leq \frac{1}{n^\alpha}\mathcal{M}|\{k \leq n : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| + \delta.$$

Therefore, for any $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\sum_{k=1}^{n}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \gamma\right\}$$
$$\subseteq \{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \geq \frac{\delta}{\mathcal{M}}\} \in \mathfrak{Y}.$$

Hence, $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PC}_\infty\mathcal{N}[\mathfrak{Y}]^\alpha} \mathfrak{X}_N$. □

Theorem 3.5. *For $\theta = \{k_t\}$,*

(1) *If $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PVN}_\theta(\mathfrak{Y})^\alpha} \mathfrak{X}_N$, then $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PSN}_\theta(\mathfrak{Y})^\alpha} \mathfrak{X}_N$.*
(2) *$\mathcal{PVN}_\theta(\mathfrak{Y})^\alpha$ is proper subset of $\mathcal{PSN}_\theta(\mathfrak{Y})^\alpha$.*

Proof. We begin proving (1): Let $\varepsilon, \delta > 0$ and $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PVN}_\theta(\mathfrak{Y})^\alpha} \mathfrak{X}_N$. Then, it can be written

$$\frac{1}{h_t^\alpha}\sum_{k \in I_t}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \frac{\delta}{h_t^\alpha}|\{k \in I_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}|.$$

Hence,

$$\frac{1}{h_t^\alpha\delta}\sum_{k \in I_t}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \frac{1}{h_t^\alpha}|\{k \in I_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}|,$$

which implies that for any $\gamma > 0$,

$$\{t \in \mathbb{N} : \frac{1}{h_t^\alpha}|\{k \in I_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \geq \gamma\}$$
$$\subseteq \left\{t \in \mathbb{N} : \frac{1}{h_t^\alpha}\sum_{k \in I_t}\mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\gamma\right\} \in \mathfrak{Y}.$$

Therefore, $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PSN}_\theta(\mathfrak{Y})^\alpha} \mathfrak{X}_N$.

Then, we prove (2): Let $\{\mathfrak{X}_{N_k}\}$ be defined by

$$\mathfrak{X}_{N_k} = \begin{cases} \{-1+I, 1-I\} & \textit{with probability } \frac{1}{2}-I,\ \textit{if } n \textit{ is the first } [\sqrt{h_r^\alpha}] \textit{ integers in the interval } I_t \\ \{I-1, I\} & \textit{with probability } \mathfrak{P}(\mathfrak{X}_{N_n}=0) = 1-\frac{1}{n}-I \textit{ and } \mathfrak{P}(\mathfrak{X}_{N_n}=1) = \frac{1}{n}-I,\ \textit{if } n \textit{ is other than the first} \\ \sqrt{h_r^\alpha} & \textit{integers in the interval } I_t. \end{cases}$$

Let $0 < \varepsilon < 1$ and $\delta > 1$. Then, we have

$$\mathfrak{P}(|\mathfrak{X}_{N_k} - 0| \geq \varepsilon) = \begin{cases} 1-I & \textit{if } n \textit{ is the first } \sqrt{h_t^\alpha} \textit{ integers in the interval } I_t \\ \frac{1}{n}-I & \textit{if } n \textit{ is other than the first } \sqrt{h_t^\alpha} \textit{ integers in the interval } I_t. \end{cases}$$

Now,

$$\frac{1}{h_t^\alpha}|\{k \in I_t : \mathfrak{P}(|\mathfrak{X}_k - 0| \geq \varepsilon) \geq \delta\}| \leq \frac{[\sqrt{h_t^\alpha}]}{h_t^\alpha}$$

and for any $\gamma > 0$ we obtain

$$\left\{t \in \mathbb{N} : \frac{1}{h_t^\alpha}|\{k \in I_t : \mathfrak{P}(|\mathfrak{X}_k - 0| \geq \varepsilon) \geq \delta\}|\gamma\right\}$$
$$\subseteq \left\{t \in \mathbb{N} : \frac{[\sqrt{h_t^\alpha}]}{h_t^\alpha} \geq \gamma\right\}.$$

Since the set

$$\left\{t \in \mathbb{N} : \frac{[\sqrt{h_t^\alpha}]}{h_t^\alpha} \geq \gamma\right\}$$

is finite and so belong to $\mathbb{F}$, hence we have

$$\left\{t \in \mathbb{N} : \frac{1}{h_t^\alpha}|\{k \in I_t : \mathfrak{P}(|\mathfrak{X}_k - 0| \geq \varepsilon) \geq \delta\}|\gamma\right\} \in \mathfrak{Y}$$

which means that $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PSN}_\theta(\mathfrak{Y})^\alpha} 0$. Besides,

$$\frac{1}{h_t^\alpha}\sum_{k \in I_t} \mathfrak{P}(|\mathfrak{X}_{N_k} - 0| \geq \varepsilon) = \frac{[\sqrt{h_t^\alpha}]([\sqrt{h_t^\alpha}]+1)}{2h_t^\alpha}$$

then

$$\left\{t \in \mathbb{N} : \frac{1}{h_t^\alpha}\sum_{k \in I_t} \mathfrak{P}(|\mathfrak{X}_{N_k} - 0| \geq \varepsilon) \geq \frac{1}{4} + I\right\}$$
$$= \left\{t \in \mathbb{N} : \frac{[\sqrt{h_t^\alpha}]([\sqrt{h_t^\alpha}]+1)}{2h_t^\alpha} \geq \frac{1}{2}\right\}$$
$$= \{m+I, m+1+I, m+2+I\} \in \mathfrak{F}(\mathfrak{Y}).$$

For some $m \in \mathbb{N}$. Therefore, $\mathfrak{X}_{N_k}$ does not converge to 0 in $\mathcal{PSN}_\theta(\mathfrak{Y})^\alpha$.

□

Theorem 3.6. *Neutrosophic $\mathfrak{Y}$-statistical convergence in probability of order α implies neutrosophic $\mathfrak{Y}$-lacunary statistical convergence in probability of order α if* $\liminf_t q_t > 1$.

Proof. By the assumption $\liminf_t q_t > 1$, there exists a $\sigma > 0$ such that $q_t > 1+\sigma$ for sufficiently large t, this means

$$\frac{h_t}{k_t} \geq \frac{\sigma}{\sigma+1},\ then\ \frac{1}{h_t} \leq \frac{1}{k_t^\alpha}\left(\frac{1+\sigma}{\sigma}\right)^\alpha.$$

If $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PSN}(\mathfrak{Y})^\alpha} \mathfrak{X}_N$, then for $\varepsilon > 0$ and for $t > 0$, we obtain

$$\begin{aligned}&\frac{1}{h_t^\alpha}|\{k \in I_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}|\\ &\leq \frac{1}{k_t^\alpha}\left(\frac{1+\sigma}{\sigma}\right)^\alpha |\{k \leq k_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}|.\end{aligned}$$

Then, for any $\gamma > 0$, we have

$$\begin{aligned}&\left\{t \in \mathbb{N} : \frac{1}{h_t^\alpha}|\{k \in I_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta| \geq \gamma\right\}\\ &\subseteq \left\{t \in \mathbb{N} : \frac{1}{k_t^\alpha}|\{k \leq k_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\} \geq \gamma| \geq \frac{\gamma\sigma^\alpha}{(1+\sigma)^\alpha}\right\} \in \mathfrak{Y}.\end{aligned}$$

□

Theorem 3.7. *Neutrosophic $\mathfrak{Y}$-lacunary statistical convergence in probability of order α implies neutrosophic $\mathfrak{Y}$-statistical convergence in probability of order α, $0 < \alpha < 1$, if $\sup_t \sum_{i=0}^{t-1} \frac{h_{i+1}^\alpha}{(k_{t-1})^\alpha} = \mathcal{B} < \infty$.*

Proof. Consider $\mathfrak{X}_{N_k} \xrightarrow{\mathcal{PSN}_\theta(\mathfrak{Y})^\alpha} \mathfrak{X}_N$, and for $\varepsilon, \delta, \gamma_1, \gamma_2 > 0$ define the sets

$$\mathcal{C} = \left\{t \in \mathbb{N} : \frac{1}{h_t^\alpha}|k \in I_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| < \gamma_1\right\}$$

and

$$\mathcal{T} = \left\{n \in \mathbb{N} : \frac{1}{n_t}|\{k \leq n : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| < \gamma_2\right\}.$$

From assumption, we obtain $\mathcal{C} \in \mathfrak{F}(\mathfrak{Y})$. Moreover, we can see that

$$\mathcal{K}_j = \frac{1}{h_j^\alpha}|\{k \in I_j : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| < \gamma_1,$$

for all $j \in \mathcal{C}$. Now, let $n \in \mathbb{N}$ be such that $k_{t-1} < n \leq k_t$ for some $t \in \mathcal{C}$. Therefore, we get

$$\begin{aligned}
\frac{1}{n^\alpha}|\{k \leq n : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \leq & \frac{1}{k_{t-1}^\alpha}|\{k \leq k_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \\
&= \frac{1}{k_{t-1}^\alpha}|\{k \in I_1 : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \\
&\quad + \frac{1}{k_{t-1}^\alpha}|\{k \in I_2 : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \\
&\quad + \ldots + \frac{1}{k_{t-1}^\alpha}|\{k \in I_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \\
&= \frac{k_1^\alpha}{k_{t-1}^\alpha h_1^\alpha}|\{k \in I_1 : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \\
&\quad + \frac{(k_2 - k_1)^\alpha}{k_{t-1}^\alpha h_2^\alpha}|\{k \in I_2 : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \\
&\quad + \ldots + \frac{(k_t - k_{t-1})^\alpha}{k_{t-1}^\alpha h_t^\alpha}|\{k \in I_t : \mathfrak{P}(|\mathfrak{X}_{N_k} - \mathfrak{X}_N| \geq \varepsilon) \geq \delta\}| \\
&= \frac{k_1^\alpha}{k_{t-1}^\alpha h_1^\alpha}\mathcal{K}_1 + \frac{(k_2 - k_1)^\alpha}{k_{t-1}^\alpha h_2^\alpha}\mathcal{K}_2 + \ldots + \frac{(k_t - k_{t-1})^\alpha}{k_{t-1}^\alpha h_t^\alpha}\mathcal{K}_t \\
&\leq \{\sup_{j \in \mathcal{C}} \mathcal{K}_j\} \sup_t \sum_{i=0}^{t-1} \frac{h_{i+1}^\alpha}{(k_{t-1})^\alpha} \\
&< \gamma_1 \mathcal{B}.
\end{aligned}$$

Taking $\gamma_2 = \dfrac{\gamma_1}{\mathcal{B}}$ and by $\cup\{n : k_{t-1} < n \leq k_t, t \in \mathcal{C}\} \subset \mathcal{T}$ where $\mathcal{C} \in \mathfrak{F}(\mathfrak{Y})$. Then, the set $\mathcal{T}$ belongs to $\mathfrak{F}(\mathfrak{Y})$. □

4. Conclusion

In this paper, the notions of neutrosophic $\mathfrak{Y}$-Cesàro summability of a sequence of order α, neutrosophic $\mathfrak{Y}$-lacunary statistical convergence of order α, neutrosophic strongly $\mathfrak{Y}$-lacunary statistical convergence of order α and neutrosophic strongly $\mathfrak{Y}$-Cesàro summability of order α in neutrosophic probability were defined and some relations between them were proved. For future works, we suggest the reader to define new sort of neutrosophic statistical convergence, even though these notions can be extended for double sequences, triple sequences and higher sequences.

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