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Δ^f-Lacunary Statistical Boundedness of Order β for Sequences of Fuzzy Numbers

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Abstract. In the present paper, we introduce the concept of Δ^f-lacunary statistical boundedness of order β with respect to a modulus function f for sequences of fuzzy numbers and give some relations between Δ^f-lacunary statistical boundedness of order β and Δ^f-statistical boundedness with respect to a modulus function f with the help of many examples and figures. Furthermore, we study some properties like solidity, symmetricity, etc.

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1. Introduction, definitions and preliminaries

In the traditional approach to analysis of convergence, almost all of the terms of a sequence are required to belong to an arbitrarily small neighborhood of the limit. Fast [15] and Steinhaus [27] first proposed the idea of statistical convergence and later Schoenberg [26] gave a formal definition of that concept, independently. The essential tenet of statistical convergence is to loosen the restrictions of this condition and to insist that the convergence requirement be valid only for the vast majority of the elements. Although the notion was developed about seventy years ago, it has only lately become an active topic of study. Several mathematicians studied properties of this kind of convergence and applied this notion in different areas ([29],[10],[21],[9],[2],[18]). Another important kind of statistical convergence is the concept of lacunary statistical convergence introduced by Fridy and Orhan [16]. For the lacunary statistical convergence in classical real sequences and associated topics, the reader may refer to ([11],[13],[14],[25]).

Recent years Gadjiev and Orhan [17] have extended the concept of statistical convergence into the ordered statistical convergence. Following this, Çolak [7] and Çolak and Bektaş [8] conducted research on the concept of statistical convergence. These investigations show that the principles of statistical convergence give a significant addition to the enhancement of classical analysis. In the foundational work that was authored by Zadeh [28], the idea of fuzziness was first discovered and presented to the scientific world. In 1986, Matloka [20] provided the notion of fuzzy number sequence, and then in 1995, Nuray and Savaş [23] established the statistical convergence of these sequences.

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The statistical boundedness in sequences of fuzzy numbers, was first described by Aytar and Pehlivan [5]. Altinok and Mursaleen [3] then used a difference operator to generalize the statistical boundedness.

To clarify, a lacunary sequence is an increasing integer sequence $\theta = (k_r)$ of positive (including zero) $\theta = (k_r)$ with $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. We note that intervals $I_r = (k_{r-1}, k_r]$ are determined by sequence θ and q_r denote the fraction $\frac{k_r}{k_{r-1}}$ with $q_1 = k_1$ in terms of suitability.

Nakano [22] introduced the concept of modulus function. According to this definition, a mapping $f : [0, \infty) \to [0, \infty)$ is said to be a modulus if following conditions hold: $i)$ $f(x) = 0$ iff $x = 0$, $ii)$ $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$, $iii)$ f is increasing, $iv)$ f is right-continuous at 0. It is well-known that bounded or unbounded modulus functions exist. For example $f(x) = x^p, (0 < p \leq 1)$ is unbounded and $f(x) = \frac{x}{1+x}$ is bounded. The modulus function was applied to fuzzy number sequences by Sarma [24]. With an unbounded modulus function, Aizpuru *et al.* [1] introduced the notion of the $f-$density of a set A, where A is a subset of natural numbers set. To generalize this study of Aizpuru *et al.* [1], Bhardwaj [6] defined the $f-$statistical convergence of order α according to f for real sequences.

Kızmaz [19] defined the difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, which consist of any real-valued sequences $x = (x_k)$ such that $\Delta x = \Delta^1 x = (x_k - x_{k+1})$ in the sequence spaces ℓ_∞, c and c_0. Et and Çolak [12] expanded the concept of difference sequences by making the difference m times such that $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ for $(m = 1, 2, 3, ...)$.

A fuzzy set consists of elements with degrees of membership. The idea of membership function is the most significant aspect of characterizing and defining a fuzzy set, and it is essential to the field of fuzzy sets. If a fuzzy set u on the set of real number $\mathbb{R}$ possesses the criteria listed below, then we refer to that set as a fuzzy number: $i)$ u is normal, $ii)$ u is fuzzy convex, $iii)$ u is upper semi-continuous, $iv)$ $\operatorname{supp} u = cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact. $L(\mathbb{R})$ will denote the set of all fuzzy numbers on $\mathbb{R}$.

In this sense, a fuzzy number is a specific case of a normal, convex fuzzy set of the real numbers line and is an extension of real number. For a fuzzy number u, $\alpha-$level set $[u]^\alpha$ is described by

$$[u]^\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0, 1] \\ \operatorname{supp} u, & \text{if } \alpha = 0 \end{cases}.$$

When $[u]^\alpha$ is a closed interval for each $\alpha \in [0, 1]$ and $[u]^1 \neq \emptyset$, it is obvious that u is a fuzzy number. Let u and v be two fuzzy numbers, $k \in \mathbb{R}$ and $\alpha-$level sets $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ and $[v]^\alpha = [\underline{v}^\alpha, \overline{v}^\alpha]$ be given. In general, the metric $d(u, v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha)$ will be used in our studies, where d_H is the Hausdorff metric defined by

$$d_H([u]^\alpha, [v]^\alpha) = \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}.$$

The order relation on fuzzy numbers is defined by

$$u \leq v \text{ if and only if } \underline{u}^\alpha \leq \overline{u}^\alpha \text{ and } \underline{v}^\alpha \leq \overline{v}^\alpha.$$

for any $\alpha \in [0, 1]$. If neither $u \leq v$ nor $v \leq u$, then we say that u and v are incomparable and we make use of the notation $u \not\sim v$ to show this situation.

Let $E(F)$ be a fuzzy sequence class and consider the sequences $X_k, Y_k \in E(F)$. Then, $E(F)$ is

$i)$ Normal (or solid), if $d(Y_k, \bar{0}) \leq d(X_k, \bar{0})$ for all $k \in \mathbb{N}$ implies $(Y_k) \in E(F)$, whenever $(X_k) \in E(F)$,

$ii)$ Monotone, if $E(F)$ contains the canonical pre-images of all its step spaces,

$iii)$ Symmetric, if $\left(X_{\pi(n)}\right) \in E(F)$, whenever $(X_k) \in E(F)$, where π is a permutation of $\mathbb{N}$.

It is well known that; If $E(F)$ is solid, then $E(F)$ is monotone.

Within the scope of this investigation, we broaden the application of the concept of "statistical boundedness" and "$f-$statistical convergence" present various inclusion relations by making use of the generalized difference operator. Moreover, in order to contribute to the field of the fuzzy numbers theory, we present certain relation theorems as a means of filling in the gaps that currently exist.

2. Main results

Specifically, we focus on $\Delta^{f}-$lacunary statistical boundedness of order β and look at various inclusion theorems between $\Delta^{f}-$statistical boundedness and $\Delta^{f}-$lacunary statistical boundedness of order β for sequences of fuzzy numbers in this section.

Definition 2.1. Let (X_k) be a fuzzy sequence, $\theta = (k_r)$ be a lacunary sequence, f be a unbounded modulus function and β be a real number with $0 < \beta \leq 1$. A sequence $X = (X_k)$ is said to be $\Delta^{f}-$lacunary statistically Cauchy sequence of order β according to f provided that for every $\varepsilon > 0$ there exists a subsequence $X_{k_r'}$ of X such that $k_r' \in I_r$ for each r, $\lim \Delta X_{k_r'} = X_0$, and for every $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{f\left(h_r^{\beta}\right)} . f\left(\left|\left\{k \in I_r : d\left(\Delta X_k, \Delta X_{k_r'}\right) \geq \varepsilon\right\}\right|\right) = 0$$

$SC_{\theta}^{f,\beta}(\Delta_F)$ will denote the set of all $\Delta^{f}-$lacunary statistically Cauchy sequence of order β with respect to a modulus function f.

Definition 2.2. Let (X_k) be a fuzzy sequence, $\theta = (k_r)$ be a lacunary sequence, f be a unbounded modulus function and β be a real number with $0 < \beta \leq 1$. A sequence (X_k) is said to be $\Delta^{f}-$lacunary statistically bounded above of order β with respect to a modulus function f if there exists a fuzzy number X_0 satisfying

$$\lim_{r\to\infty} \frac{1}{f\left(h_r^{\beta}\right)} . f\left(\left|\left\{k \in I_r : \Delta X_k > X_0\right\} \cup \left\{k \in I_r : \Delta X_k \nsim X_0\right\}\right|\right) = 0 \qquad (1)$$

and a sequence (X_k) is said to be $\Delta^{f}-$lacunary statistically bounded below of order β with respect to a modulus function f if there exists a fuzzy number X_0 satisfying

$$\lim_{r\to\infty} \frac{1}{f\left(h_r^{\beta}\right)} . f\left(\left|\left\{k \in I_r : \Delta X_k < X_0\right\} \cup \left\{k \in I_r : \Delta X_k \nsim X_0\right\}\right|\right) = 0 \qquad (2)$$

where $I_r = (k_{r-1}, k_r]$. In the above limits, the elements of the set $\{k \in \mathbb{N} : \Delta X_k \nsim X_0\}$ added to the sets $\{k \in I_r : \Delta X_k > X_0\}$ and $\{k \in I_r : \Delta X_k < X_0\}$ since $L(\mathbb{R})$ has not property of partial order relation.

It is called Δ^f-lacunary statistically bounded of order β with respect to a modulus function f if the limits (1) and (2) hold and $S_\theta^{f,\beta}(\Delta_F, b)$ will denote the set of all Δ^f-lacunary statistically bounded sequences of order β with respect to a modulus function f.

An example related to Δ^f-lacunary statistically boundedness of order β is given below:

Example 2.3. Take a fuzzy sequence (X_k) which has membership function as follows

$$X_k(x) = \begin{cases} \left.\begin{array}{cc} 3x-3k+1, & \text{for } x \in \left[k-\frac{1}{3}, k\right] \\ -3x+3k+1, & \text{for } x \in \left[k, k+\frac{1}{3}\right] \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k = n^4, \ (n = 1, 2, ...) \\ \left.\begin{array}{cc} 3x-2, & \text{for } x \in \left[\frac{2}{3}, 1\right] \\ -3x+4, & \text{for } x \in \left[1, \frac{4}{3}\right] \\ 0, & \text{otherwise} \end{array}\right\} & = X_0, \ \text{otherwise} \end{cases}$$

We take unbounded modulus function $f(x) = x$ and so $\alpha-$level sets of sequences (X_k) and (ΔX_k) are

$$[X_k]^\alpha = \begin{cases} [k + \left(\frac{\alpha-1}{3}\right), k - \left(\frac{\alpha-1}{3}\right)], & \text{if } k = n^4 \\ [\frac{\alpha+2}{3}, \frac{-\alpha+4}{3}], & \text{if } k \neq n^4 \end{cases}$$

and

$$[\triangle X_k]^\alpha = \begin{cases} [k + \left(\frac{2\alpha-5}{3}\right), k - \left(\frac{2\alpha+1}{3}\right)], & \text{if } k = n^4 \\ [-k + \left(\frac{2\alpha-2}{3}\right), -k - \left(\frac{2\alpha-2}{3}\right)], & \text{if } k+1 = n^4 \\ [\frac{2\alpha-2}{3}, \frac{-2\alpha+2}{3}], & \text{otherwise} \end{cases}$$

Hence, we can write for $\beta > \frac{1}{4}$

$$\begin{aligned} &\delta^{f,\beta}\left(\{k \in I_r : \Delta X_k > X_0\} \cup \{k \in I_r : \Delta X_k \nsim X_0\}\right) \\ &= \delta^{f,\beta}\left(\{16, 81, 256, ...\} \cup \{\emptyset\}\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} &\delta^{f,\beta}\left(\{k \in I_r : \Delta X_k < X_0\} \cup \{k \in I_r : \Delta X_k \nsim X_0\}\right) \\ &= \delta^{f,\beta}\left(\{\emptyset\}\right) \\ &= 0 \end{aligned}$$

Eventually, $(X_k) \in S_\theta^{f,\beta}(\Delta_F, b)$, but (X_k) is not Δ^f-bounded for lacunary sequence $\theta = (2^r)$ with respect to modulus function $f(x) = x$. (See Fig. 1).

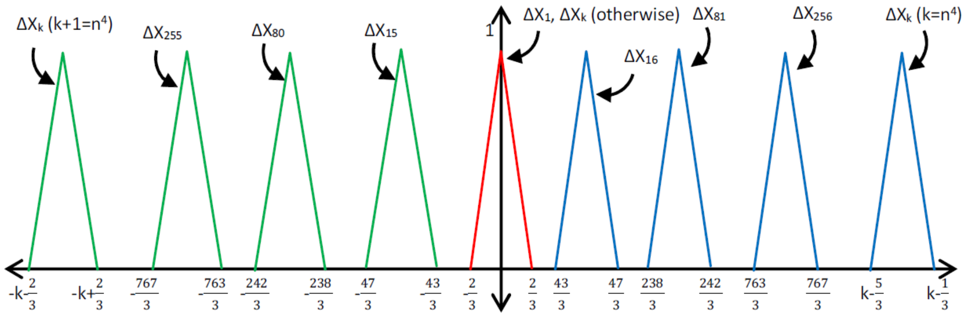

Fig. 1: The terms of fuzzy difference sequences (ΔX_k)

The following three propositions don't have proofs because they are obtained using known operations.

Proposition 2.4. Let (X_k) be a fuzzy sequence, $\theta = (k_r)$ be a lacunary sequence, f be a unbounded modulus function and β be a real number with $0 < \beta \leq 1$. Then $S^f(\Delta_F) \subset S_\theta^{f,\beta}(\Delta_F, b)$ and inclusion is strict.

Proposition 2.5. Let $X = (X_k)$ be a sequence of fuzzy numbers, f be an unbounded modulus function and $0 < \beta \leq 1$ be given. Then $S_\theta^{f,\beta}(\Delta_F) \subset SC_\theta^{f,\beta}(\Delta_F)$ and inclusion is strict.

Proposition 2.6. Let $X = (X_k)$ be a sequence of fuzzy numbers, f be a modulus function and $0 < \beta \leq 1$ be given. Then $SC_\theta^{f,\beta}(\Delta_F) \subset S_\theta^{f,\beta}(\Delta_F, b)$ and inclusion is strict.

Theorem 2.7. Let $0 < \beta \leq 1$ be given and f be an unbounded modulus function. Then, every Δ^f-bounded sequence of fuzzy numbers is Δ^f-lacunary statistically bounded of order β, but the opposite does not always hold.

Proof. The first half of the proof is straightforward. Take modulus function $f(x) = x^p$ for $0 < p \leq 1$ and lacunary sequence $\theta = (2^r)$ for second half of the proof and define a fuzzy sequence $X = (X_k)$ which has following membership function:

$$X_k(x) = \begin{cases} \left.\begin{array}{cc} \frac{x}{3} - \left(\frac{k-3}{3}\right), & \text{for } x \in [k-3, k] \\ -\frac{x}{3} + \left(\frac{k+3}{3}\right), & \text{for } x \in [k, k+3] \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k = n^3,\ n = 1, 2, ... \\ \left.\begin{array}{cc} \frac{x}{3}, & \text{for } x \in [0, 3] \\ -\frac{x}{3} + 2, & \text{for } x \in [3, 6] \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k \neq n^3 \end{cases}$$

After routine operations, $\alpha-$level sets and membership functions of (X_k) and (ΔX_k) can be found as follows:

$$[X_k]^\alpha = \begin{cases} [k + 3\alpha - 3, k - 3\alpha + 3], & \text{if } k = n^3 \\ [3\alpha, -3\alpha + 6], & \text{if } k \neq n^3 \end{cases}$$

and

$$[\triangle X_k]^\alpha = \begin{cases} [k+6\alpha-9, k-6\alpha+3], & \text{if } k=n^3 \\ [-k+6\alpha-4, -k-6\alpha+8], & \text{if } k+1=n^3 \\ [6\alpha-6, -6\alpha+6], & \text{otherwise} \end{cases}$$

$$\triangle X_k(x) = \begin{cases} \left.\begin{array}{ll} \frac{1}{6}(x-k+9), & k-9 \le x \le k-3 \\ \frac{1}{6}(-x+k+3), & k-3 \le x \le k+3 \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k=n^3 \\ \left.\begin{array}{ll} \frac{1}{6}(x+k+4), & -k-4 \le x \le -k+2 \\ \frac{1}{6}(-x-k+8), & -k+2 \le x \le -k+8 \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k+1=n^3 \\ \begin{array}{ll} \frac{1}{6}(x+6), & -6 \le x \le 0 \\ \frac{1}{6}(-x+6), & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{array} & \text{if } k \ne n^3 \end{cases}$$

Hence, we can write

$$\begin{aligned} &\delta^{f,\beta}\left(\{k \in I_r : \Delta X_k > X_0\} \cup \{k \in I_r : \Delta X_k \nsim X_0\}\right) \\ &= \delta^{f,\beta}\left(\{8, 27, 64, \ldots\} \cup \{\emptyset\}\right) \\ &= \sqrt[3]{n} \end{aligned}$$

and

$$\begin{aligned} &\frac{\delta^{f,\beta}\left(\{k \in I_r : \Delta X_k > X_0\} \cup \{k \in I_r : \Delta X_k \nsim X_0\}\right)}{f\left(n^\beta\right)} \\ &\le \frac{\left(\sqrt[3]{n}\right)^p}{\left[n^\beta\right]^p} \to 0, (n \to \infty) \end{aligned}$$

for $\beta > \frac{1}{3}$. Then, it can be seen that $X = (X_k) \in S_\theta^{f,\beta}(\Delta_F, b)$, but sequence (X_k) is not Δ^f-bounded for $\theta = (2^r)$ (see Fig. 2).

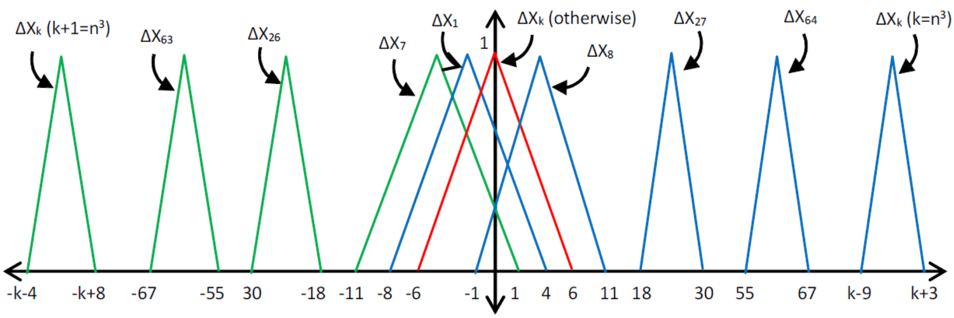

Fig. 2: The terms of fuzzy difference sequences (ΔXk)

Corollary 2.8. If a sequence of fuzzy numbers is Δ^f-bounded with respect to an unbounded modulus function f, then it is Δ^f-lacunary statistically bounded with respect to same function f, but the converse does not hold.

Theorem 2.9. Let $X = (X_k)$ be a sequence of fuzzy numbers, f be an unbounded modulus function and $0 < \beta \le 1$ be given. Then $S_\theta^{f,\beta}(\Delta_F) \subset S_\theta^{f,\beta}(\Delta_F, b)$ with respect to same f, but the opposite is not always true.

Proof. Proposition 2.5 and Proposition 2.6 help us to make first part of the proof. Let $\theta = (2^r)$ be given and take modulus function $f(x) = x$ to show that the reverse is not true. Take fuzzy sequence $X = (X_k)$ which has membership function as follows:

$$X_k(x) = \begin{cases} \left.\begin{array}{cc} x, & 0 \le x \le 1 \\ -x+2, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{array}\right\} & \text{, if } k \text{ is odd} \\ \left.\begin{array}{cc} x-2, & 2 \le x \le 3 \\ -x+4, & 3 \le x \le 4 \\ 0, & \text{otherwise} \end{array}\right\} & \text{, if } k \text{ is even} \end{cases}$$

and the $\alpha-$level sets of sequences (X_k) and (ΔX_k) are

$$[X_k]^\alpha = \begin{cases} [\alpha, -\alpha+2], & \text{if } k \text{ is odd} \\ [\alpha+2, -\alpha+4], & \text{if } k \text{ is even} \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} [2\alpha-4, -2\alpha], & \text{if } k \text{ is odd} \\ [2\alpha, -2\alpha+4], & \text{if } k \text{ is even} \end{cases}$$

Then $X \in S_\theta^{f,\beta}(\Delta_F, b)$, but $X \notin S_\theta^{f,\beta}(\Delta_F)$ for $f(x) = x$, $\beta = 1$ and $\theta = (2^r)$. (See Fig. 3)

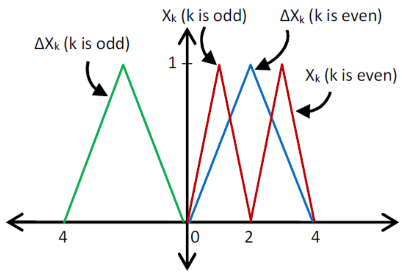

Fig. 3: The terms of fuzzy difference sequences (X_k) and (ΔX_k)

Corollary 2.10. Let $X = (X_k)$ be a sequence of fuzzy numbers, f be an unbounded modulus function and $0 < \beta \le 1$ be given. Then $S_\theta^f(\Delta_F) \subset S_\theta^f(\Delta_F, b)$ with respect to same f and inclusion is strict.

Theorem 2.11. *i)* $S_\theta^{f,\beta}(\Delta_F, b)$ is not symmetric,

ii) $S_\theta^{f,\beta}(\Delta_F, b)$ is normal and hence monotone,

Proof. i) Take a fuzzy sequence $X = (X_k)$ which has membership function as follows:

$$X_k(x) = \begin{cases} \left.\begin{array}{cc} x-k-2, & x \in [k+2, k+3] \\ -x+k+4, & x \in [k+3, k+4] \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k = n^3 \\ 0, \quad \text{if } k \ne n^3 \end{cases}$$

Take unbounded modulus function $f(x) = x$. After routin operations, $\alpha-$level sets and membership functions of (X_k) and (ΔX_k) can be found as follows:

$$[X_k]^\alpha = \begin{cases} [k+\alpha+2, k-\alpha+4], & \text{if } k = n^3 \\ [0,0], & \text{if } k \ne n^3 \end{cases}$$

and

$$[\triangle X_k]^\alpha = \begin{cases} [k+\alpha+2, k-\alpha+4], & \text{if } k = n^3 \\ [-k+\alpha-5, -k-\alpha-3], & \text{if } k+1 = n^3 \\ [0,0], & \text{otherwise} \end{cases}$$

$$\triangle X_k(x) = \begin{cases} \left.\begin{array}{cc} x-k-2, & k+2 \le x \le k+3 \\ -x+k+4, & k+3 \le x \le k+4 \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k = n^3 \\ \left.\begin{array}{cc} x+k+5, & -k-5 \le x \le -k-4 \\ -x-k-3, & -k-4 \le x \le -k-3 \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k+1 = n^3 \\ 0, & \text{if } k \ne n^3 \end{cases}$$

Sequence $X = (X_k)$ belongs to the set $S_\theta^{f,\beta}(\Delta_F, b)$ for $\beta > \frac{1}{3}$, $\theta = (2^r)$ and $f(x) = x$ Let $Y = (Y_k)$ be a rearrangement of (X_k), which is defined as follows:

$$(Y_k) = (X_1, X_2, X_8, X_3, X_{27}, X_4, X_{64}, X_5, X_{125}, X_6, X_{216}, X_7, ...)$$

and so difference sequence is

$$(\Delta Y_k) = (X_1 - X_2, X_2 - X_8, X_8 - X_3, X_3 - X_{27}, X_{27} - X_4, ...).$$

It is obvious that we can express

$$\delta_\theta^{f,\beta}(\{k \in I_r : \Delta Y_k > X_0\} \cup \{k \in I_r : \Delta Y_k \nsim X_0\}) \ne 0$$

in the special case $\beta = 1$, $\theta = (2^r)$ and $f(x) = x$ so $(Y_k) \notin S_\theta^{f,\beta}(\Delta_F, b)$, i.e. sequence class $S_\theta^{f,\beta}(\Delta_F, b)$ is not symmetric.

ii) Let $X = (X_k) \in S_\theta^{f,\beta}(\Delta_F, b)$ and $Y = (Y_k)$ be a sequence such that $d(\Delta Y_k, \bar{0}) \le d(\Delta X_k, \bar{0})$ for all $k \in \mathbb{N}$. Since $X \in S_\theta^{f,\beta}(\Delta_F, b)$ there exists a fuzzy number X_0 satisfying

$$\delta_\theta^{f,\beta}(\{k \in I_r : \Delta X_k > X_0\} \cup \{k \in I_r : \Delta X_k \nsim X_0\}) = 0$$

Clearly $(Y_k) \in S_\theta^{f,\beta}(\Delta_F, b)$ as

$$\begin{aligned} &\{k \in I_r : \Delta Y_k > X_0\} \cup \{k \in I_r : \Delta Y_k \nsim X_0\} \\ &\quad \subset \{k \in I_r : \Delta X_k > X_0\} \cup \{k \in I_r : \Delta X_k \nsim X_0\} \end{aligned}$$

Since any normal space is monotone, $S_\theta^{f,\beta}(\Delta_F, b)$ must also be monoton (See Fig. 4).

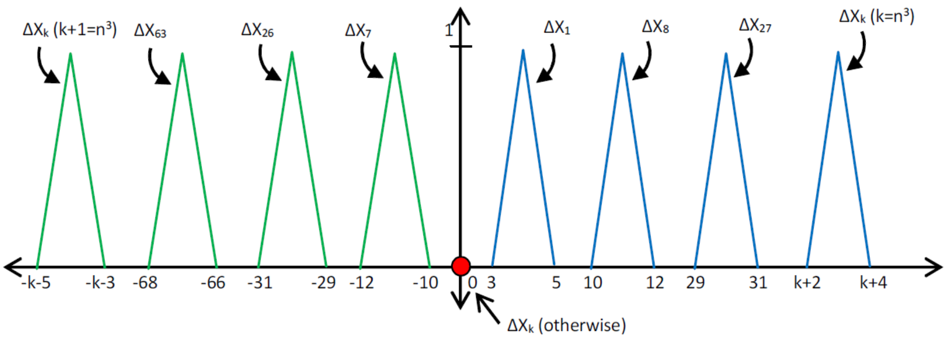

Fig. 4: The terms of fuzzy difference sequences (ΔX_k)

Theorem 2.12. Let f be an unbounded modulus function, $0<\beta\leq 1$, $X=(X_k)$ be a fuzzy sequence and $\theta=(k_r)$ be a lacunary sequence. If $\liminf_r q_r > 1$ and $\lim_{t\to\infty}\frac{f(t)^\beta}{t^\beta}>0$, then $S^{f,\beta}(\Delta_F,b)\subset S_\theta^{f,\beta}(\Delta_F,b)$.

Proof. Let $\liminf_r q_r>1$, hence we can find a number $\delta>0$ satisfying inequalities $q_r\geq 1+\delta$ and $\left(\frac{h_r}{k_r}\right)^\beta\geq\left(\frac{\delta}{1+\delta}\right)^\beta$ whenever r is large enough. Consider the sequence $(X_k)\in S^{f,\beta}(\Delta_F,b)$, then we have

$$\begin{aligned}
&\frac{1}{f(k_r)^\beta}f\left(\left|\{k\leq k_r:\Delta X_k>X_0\}\cup\{k\leq k_r:\Delta X_k\nsim X_0\}\right|\right)\\
&\geq\frac{1}{f(k_r)^\beta}f\left(\left|\{k\leq k_r:\Delta X_k>X_0\}\cup\{k\leq k_r:\Delta X_k\nsim X_0\}\right|\right)\\
&=\frac{f(h_r)^\beta}{f(k_r)^\beta}\frac{1}{f(h_r)^\beta}.f\left(\left|\{k\leq k_r:\Delta X_k>X_0\}\cup\{k\leq k_r:\Delta X_k\nsim X_0\}\right|\right)\\
&=\frac{f(h_r)^\beta}{h_r^\beta}.\frac{k_r^\beta}{f(k_r)^\beta}.\frac{h_r^\beta}{k_r^\beta}.\frac{f\left(\left|\{k\leq k_r:\Delta X_k>X_0\}\cup\{k\leq k_r:\Delta X_k\nsim X_0\}\right|\right)}{f(h_r)^\beta}\\
&\geq\frac{f(h_r)^\beta}{h_r^\beta}.\frac{k_r^\beta}{f(k_r)^\beta}.\left(\frac{\delta}{1+\delta}\right)^\beta.\frac{f\left(\left|\{k\leq k_r:\Delta X_k>X_0\}\cup\{k\leq k_r:\Delta X_k\nsim X_0\}\right|\right)}{f(h_r)^\beta}
\end{aligned}$$

for a fuzzy number X_0 and sufficiently large r. This completes the proof, i.e. $X_k\in S_\theta^{f,\beta}(\Delta_F,b)$.

Theorem 2.13. Let f be an unbounded modulus function, $0<\beta\leq 1$, $X=(X_k)$ be a fuzzy sequence and $\theta=(k_r)$ be a lacunary sequence. If $\limsup_r q_r<\infty$, then $S_\theta^{f,\beta}(\Delta_F,b)\subset S^f(\Delta_F,b)$.

Proof. Omitted.

Remark 2.14. For fuzzy sequences, we know that every subsequence of Δ^f−statistical bounded sequence with respect to unbounded modulus function f is Δ^f−statistical bounded. This property is not valid for Δ^f−lacunary statistically boundedness of order β as seen in the example below.

Example 2.15. Let $\theta=(k_r)$ be a lacunary sequence, with $\liminf_r q_r>1$, consider unbounded modulus function $f(x)=x$ and take a fuzzy sequence $X=(X_k)$ which has membership function as follows:

$$X_k(x)=\begin{cases}\left.\begin{array}{cc}x-k+3, & k-3\leq x\leq k-2\\ -x+k-1, & k-2\leq x\leq k-1\\ 0, & \text{otherwise}\end{array}\right\} & \text{if } k=n^4\\ 0,\quad \text{if } k\neq n^4\end{cases}$$

We can calculate the α−level sets of sequence (X_k) and (ΔX_k) as

$$[X_k]^\alpha=\begin{cases}[k+\alpha-3,k-\alpha-1], & \text{if } k=n^4\\ [0,0], & \text{if } k\neq n^4\end{cases}$$

and

$$[\triangle X_k]^\alpha=\begin{cases}[k+\alpha-3,k-\alpha-1], & \text{if } k=n^4\\ [-k+\alpha,-k-\alpha+2], & \text{if } k+1=n^4\\ [0,0], & \text{otherwise}\end{cases}$$

Then $(X_k) \in S^{f,\beta}(\Delta_F, b) \subset S_\theta^{f,\beta}(\Delta_F, b)$ since $\liminf_r q_r > 1$, that is, $X = (X_k)$ is Δ^f-lacunary statistically bounded for $\beta > \frac{1}{4}$. Moreover, let's start by defining a fuzzy subsequence (Y_k) of (X_k) by

$$Y_k(x) = \begin{cases} x - k^2 + 3, & \text{if } k^2 - 3 \le x \le k^2 - 2 \\ -x + k^2 - 1, & \text{if } k^2 - 2 \le x \le k^2 - 1 \\ 0, & \text{otherwise} \end{cases}$$

The $\alpha-$level sets of subsequence (Y_k) and (ΔY_k) are

$$[Y_k]^\alpha = [k^2 + \alpha - 3, k^2 - \alpha - 1]$$
$$[\Delta Y_k]^\alpha = [-2k + 2\alpha - 3, -2k - 2\alpha + 1].$$

It is easy to see that subsequence (Y_k) is not Δ^f-lacunary statistically bounded with respect to same modulus function. (See Fig. 5)

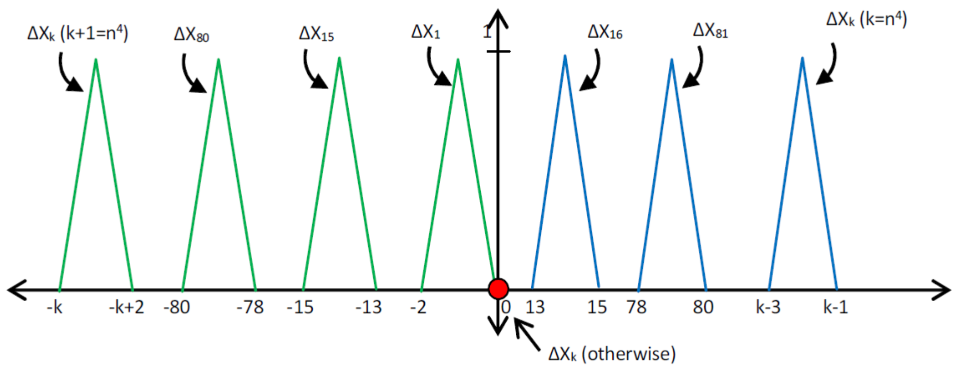

Fig. 5: The terms of fuzzy difference sequences (ΔX_k)

Theorem 2.16. Let $0 < \beta \le 1$, $X = (X_k)$ be a sequence of fuzzy numbers, f be an unbounded modulus function and $\theta = (k_r)$ be a lacunary sequence. If

$$\lim_{r\to\infty} \inf \frac{f(h_r)^\beta}{f(k_r)} \tag{3}$$

then $S^f(\Delta_F, b) \subset S^{f,\beta}(\Delta_F, b)$.

Proof. For a fuzzy number X_0, we have

$$\{k \le k_r : \Delta X_k > X_0\} \cup \{k \le k_r : \Delta X_k \nsim X_0\} \supset \{k \in I_r : \Delta X_k > X_0\}$$
$$\cup \{k \in I_r : \Delta X_k \nsim X_0\}$$

Therefore,

$$\frac{1}{f(k_r)}.|\{k \le k_r : \Delta X_k > X_0\} \cup \{k \le k_r : \Delta X_k \nsim X_0\}|$$
$$\ge \frac{1}{f(k_r)}.|\{k \in I_r : \Delta X_k > X_0\} \cup \{k \in I_r : \Delta X_k \nsim X_0\}|$$
$$= \frac{f(h_r)^\beta}{f(k_r)}.\frac{1}{f(h_r)^\beta}.|\{k \in I_r : \Delta X_k > X_0\} \cup \{k \in I_r : \Delta X_k \nsim X_0\}|$$

Taking limit as $r \to \infty$ and using (3), we get

$$X_k \in S^f(\Delta_F, b) \Longrightarrow X_k \in S^{f,\beta}(\Delta_F, b).$$

3. Conclusion

Following the presentation by Matloka [20] of the notion of a fuzzy sequence and the presentation by Nuray and Savaş [23] of the idea of statistical convergence in fuzzy sequences, a great number of research were conducted on the topic, and a connection was made with the theory of summability. Now in this paper, we generalized the study of Altinok *et al* [4] using difference operator according a modulus function f. Additionally, we gave some inclusion theorems with the help of many examples and figures, and presented various inclusion relations between this sequence class and other ones as a means of filling in the gaps that currently exist.

References

[1] A. Aizpuru, M.C. Listan-Garcia, F. Rambla-Barreno, Density by moduli and statistical convergence, *Quaest. Math.* **37** (2014), 525–530.

[2] K.E. Akbas, M. Isik, On asymptotically λ−statistical equivalent sequences of order α in probability, *Filomat* **34** (2020), no. 13, 4359–4365.

[3] H. Altınok, M. Mursaleen, Δ−Statistical boundedness for sequences of fuzzy numbers, *Taiwanese J. Math.* **15** (2011), no. 5, 2081–2093.

[4] H. Altinok, M. Et, Y. Altin, Lacunary statistical boundedness of order β for sequences of fuzzy numbers, *J. Intell. Fuzzy Syst.* **35** (2018), 2383–2390.

[5] S. Aytar, S. Pehlivan, Statistically monotonic and statistically bounded sequences of fuzzy numbers, *Inform. Sci.* **176** (2006), no. 6, 734–744.

[6] V.K. Bhardwaj, S. Dhawan, f−statistical convergence of order α and strong Cesaro summability of order α with respect to a modulus, *J. Inequal. Appl.* **332** (2015), 14 pp.

[7] R. Çolak, Statistical convergence of order α, In: *Modern Methods in Analysis and Its Applications*, Anamaya Pub, New Delhi, India, 121–129, 2010.

[8] R. Çolak, Ç.A. Bektaş, λ−Statistical convergence of order α, *Acta Math. Sin. Engl. Ser.* **31** (2011), no. 3, 953–959.

[9] J. Connor, M. Ganichev, V.A. Kadets, Characterization of Banach spaces with separable duals via weak statistical convergence, *J Math. Anal. Appl.* **244** (2000), 251–261.

[10] O. Duman, M.K. Khan, C.A. Orhan, Statistical convergence of approximating operators, *Math. Inequal. Appl.* **6** (2003), 689–699.

[11] M. Et, Generalized Cesàro difference sequence spaces of non-absolute type involving lacunary sequences, *Appl. Math. Comput.* **219** (2013), no. 17, 9372–9376.

[12] M. Et, R. Çolak, On some generalized difference sequence spaces, *Soochow J. Math.* **21** (1995), no. 4, 377–386.

[13] M. Et, S.A. Mohiuddine, H. Şengül, On lacunary statistical boundedness of order α, *Facta Universitatis Ser Math. Inform.* **31** (2016), no. 3, 707–716.

[14] M. Et, H. Şengül, Some Cesaro-type summability spaces of order α and lacunary statistical convergence of order α, *Filomat* **28** (2014), no. 8, 1593–1602.

[15] H. Fast, Sur la convergence statistique, *Colloq. Math.* **2** (1951), 241–244.

[16] J. Fridy, C. Orhan, Lacunary statistical convergence, *Pacific J. Math.* **160** (1993), 43–51.

[17] A.D. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.* **32** (2002), no. 1, 129–138.

[18] M. Işık, K.E. Akbaş, On λ−statistical convergence of order α in probability, *J. Inequal. Spec. Funct.* **8** (2017), no. 4, 57–64.

[19] H. Kızmaz, On certain sequence spaces, *Canad. Math. Bull.* **24** (1981), no. 2, 169–176.

[20] M. Matloka, Sequences of fuzzy numbers, *Busefal* **28** (1986), no. 13, 28–37.

[21] H.I. Miller, A measure theoretical subsequence characterization of statistical convergence, *Trans. Amer. Math. Soc.* **347** (1995), 1811–1819.

[22] H. Nakano, Concave modulars, *J. Math. Soc. Japan* **5** (1953), 29–49.
[23] F. Nuray, E. Savaş, Statistical convergence of sequences of fuzzy real numbers, *Math. Slovaca* **45** (1995), no. 3, 269–273.
[24] B. Sarma, On a class of sequences of fuzzy numbers defined by modulus function, *Inter. J. Sci. Tech.* **2** (2007), no. 1, 25–28.
[25] H. Şengül, M. Et, On $I-$lacunary statistical convergence of order α of sequences of sets, *Filomat* **31** (2017), no. 8, 2403–2412.
[26] I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* **66** (1959), 361-375.
[27] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloquium Mathematicum* **2** (1951), 73–74.
[28] L.A. Zadeh, Fuzzy sets, *Inf. Control* **8** (1965), 338–353.
[29] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, Cambridge, 1979.

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