Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 50(2), 2023, Pages 383–393, DOI: 10.52846/ami.v50i2.1720
ISSN: 1223-6934

Geometric Properties of the Generalized Wright-Bessel Functions

Gülfem Akin and Sevtap Sümer Eker

Abstract. In this article, we studied the geometric properties of generalized Wright-Bessel functions. For this purpose, we determined sufficient conditions for univalency, convexity, starlikeness and close-to-convexity of the generalized Wright-Bessel functions in the open unit disk.

2020 *Mathematics Subject Classification.* 33C10, 30C45, 30C55.
Key words and phrases. Generalized Wright-Bessel function; univalent; starlike; convex; close-to-convex.

1. Introduction

In recent years, the geometric properties of well-known special functions such as Bessel, Wright, Mittag-Leffler have been a subject that has been systematically discussed. The most researched geometric properties are starlikeness, convexity, and close-to-convexity. Raducanu [12], Bansal and Prajabat [13] worked on some geometric properties of Mittag-Leffler functions. The starlikeness and convexity conditions normalized Bessel functions were studied intensively by Ponnusamy and Baricz [16]. Prajabat [14] obtained some conditions for geometric properties of Wright functions. Bansal et al. [15] examined some geometric properties of τ-confluent hypergeometric functions. Recently, Eker and Ece investigated geometric properties of Rabotnov functions [18] and Miller-Ross functions [17]. Motivated by these works, we obtained sufficient conditions for close-to-convexity, univalency, starlikeness and convexity of the generalized Wright-Bessel functions.

In 1966, H.K. Pathak investigated the following special function, which is called the Generalized Wright-Bessel (-Lommel) function:

Definition 1.1. [1] The Generalized Wright-Bessel (-Lommel) function is defined as

$$\mathfrak{J}^{\mu}_{v,\lambda}(z) = \left(\frac{z}{2}\right)^{v+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{\Gamma(\lambda+k+1)\Gamma(v+\lambda+1+k\mu)}$$

where $\mu > 0$, $v, \lambda \in \mathbb{R}$.

In this function, with some special parameter choices, we can get some well known special functions, like Lommel function, Struve function, Bessel function and Wright function. For example, for $\mu = 1$ it contains Lommel and Struve functions and if we take $\lambda = 0$ and $\mu = 1$, we can get the familiar Bessel function. For more details we refer to [2]. (See also [3],[4]).

Received December 27, 2022. Accepted June 18, 2023.

The main aim of the present article is to determine geometric properties of Generalized Wright-Bessel function. For this, we need the following notation and definitions.

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Equivalently; if $f \in \mathcal{A}$, then it has the following Taylor-Maclaurin series representation:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ (z \in \mathbb{U}). \tag{1}$$

Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{A}$ is called starlike (with respect to the origin), denoted by $f \in \mathcal{S}^*$, if f is univalent in $\mathbb{U}$ and $f(\mathbb{U})$ is a starlike with respect to the origin. It is well-known that $f \in \mathcal{S}^*$ if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \ (z \in \mathbb{U}).$$

Furthermore, a function $f \in \mathcal{A}$ that maps $\mathbb{U}$ onto a convex domain is called convex function. We denote by $\mathcal{C}$ the class of all functions $f \in \mathcal{A}$ that are convex. $f \in \mathcal{C}$ if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ (z \in \mathbb{U}).$$

Next, a function $f \in \mathcal{A}$ is called close-to-convex, if the range $f(\mathbb{U})$ is close-to-convex, i.e. the complement of $f(\mathbb{U})$ can be written as the union of nonintersecting half-lines. We denote by $\mathcal{K}$ all close-to-convex functions. $f \in \mathcal{K}$ if and only if

$$Re\left(\frac{f'(z)}{g'(z)}\right) > 0, \ (z \in \mathbb{U}, \ g \in \mathcal{C}).$$

For these classes it is convenient to give the following chain of proper inclusions:

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}.$$

A univalent function f is in the class UCV of uniformly convex functions if for every circular arc γ contained in $\mathbb{U}$ with center $\xi \in \mathbb{U}$ the image arc $f(\gamma)$ is convex (see [6]). On the other hand, Rønning [7] defined the class of parabolic starlike functions S_p as follows:

$$S_p = \{f : \ f(z) = zF'(z), \ F \in UCV\}.$$

Given two functions $f, g \in \mathcal{A}$, where f is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the *Hadamard product* (or *convolution*) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \ , \ z \in \mathbb{U}.$$

For more details about the univalent functions theory we refer to [5], [6].

It is clear that the generalized Wright-Bessel function $\mathfrak{J}^{\mu}_{v,\lambda}(z)$ is not in the family $\mathcal{A}$. Thus, let us take into consideration the following normalization:

$$\begin{aligned}\mathbb{J}^{\mu}_{v,\lambda}(z) &= 2^{v+2\lambda}\Gamma(v+\lambda+1)\Gamma(\lambda+1)z^{1-\lambda-\frac{v}{2}}\mathfrak{J}^{\mu}_{v,\lambda}(\sqrt{z})\\ &= z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{4^{k-1}}\frac{\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)}z^{k}.\end{aligned}$$

To discuss the geometric properties of normalized generalized Wright-Bessel functions, here we define modified form:

$$\begin{aligned}J^{\mu}_{v,\lambda}(z) &= \frac{z}{1+z}*\mathbb{J}^{\mu}_{v,\lambda}(z)\\ &= z+\sum_{k=2}^{\infty}\frac{1}{4^{k-1}}\frac{\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)}z^{k}.\end{aligned} \tag{2}$$

In order to present our results we need the following interesting results.

Lemma 1.1 ([8]). *Let f define by (1) and suppose that*

$$1\geq 2a_2\geq\cdots\geq ka_k\geq\cdots\geq 0$$

or

$$1\leq 2a_2\leq\cdots\leq ka_k\leq\cdots\leq 2.$$

Then f is regular and univalent in $\mathbb{U}$.

Following the proof of Ozaki it can be proved that if a function f satisfies the conditions given in Lemma 1.1, then f is close-to-convex with respect to the convex function $-log(1-z)$.

Lemma 1.2 ([9]). *If the function $f\in\mathcal{A}$, satisfy $|(f(z)/z)-1|<1$ for each $z\in\mathbb{U}$, then f is univalent and starlike in $\mathbb{U}_{1/2}=\{z:\ |z|<1/2\}$.*

Lemma 1.3 ([10]). *If the function $f\in\mathcal{A}$, satisfy $|f'(z)-1|<1$ for each $z\in\mathbb{U}$, then f is convex in $\mathbb{U}_{1/2}$.*

Lemma 1.4 ([11]). *Assume that $f\in\mathcal{A}$. Then the following results hold true:*
(i) *If $\left|\frac{zf'(z)}{f(z)}-1\right|<\frac{1}{2}$, then $f\in S_p$.*
(ii) *If $\left|\frac{zf''(z)}{f'(z)}\right|<\frac{1}{2}$, then $f\in UCV$.*

2. Main results

Theorem 2.1. *Let $v,\mu>0$ and $\lambda\geq-1/2$. If $\lambda+v\geq 0,462$, then $J^{\mu}_{v,\lambda}(z)$ given in (2) is close-to-convex with respect to convex function $-log(1-z)$ and hence univalent in $\mathbb{U}$.*

Proof. Define

$$J^{\mu}_{v,\lambda}(z)=z+\sum_{k=2}^{\infty}A_k z^k$$

where

$$A_k=\frac{\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)}\quad k\geq 2\quad\text{and}\quad A_1=1. \tag{3}$$

We note that under the stated conditions $A_k > 0$ and $2A_2 \leq 1$ for all $k \geq 1$.

We will use Lemma 1.1 to prove that f is close-to-convex with respect to $-log(1-z)$. Therefore, we need to show that $\{kA_k\}$ is a decreasing sequence of positive real numbers. Using (3), we obtain

$$kA_k - (k+1)A_{k+1} = \frac{\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)}X(k)$$

where

$$X(k) = \frac{k}{\Gamma(v+\lambda+1+(k-1)\mu)} - \frac{k+1}{4(\lambda+k)\Gamma(v+\lambda+1+k\mu)}.$$

It is well known that the function $\Gamma(x)$ is increasing on (x_0, ∞) where $x_0 \approx 1.462$. Since under the hypotheses of our theorem

$$4k(\lambda+k)\Gamma(v+\lambda+1+k\mu) \geq (k+1)\Gamma(v+\lambda+1+(k-1)\mu),$$

we conclude that $kA_k - (k+1)A_{k+1} > 0$. This completes the proof of the Theorem 2.1. □

Example 2.1. The function $J^{\frac{1}{2}}_{1,-\frac{1}{2}}(z)$ is univalent in $\mathbb{U}$. (Figure 1). However, the function $J^{\frac{1}{2}}_{\frac{1}{10},-\frac{1}{2}}(z)$ is not univalent in $\mathbb{U}$. (Figure 2)

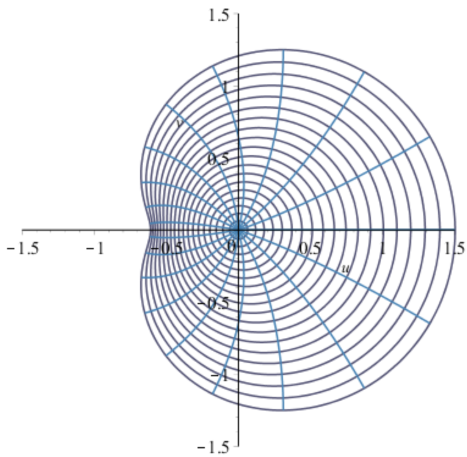

FIGURE 1. Mapping of $J^{\frac{1}{2}}_{1,-\frac{1}{2}}(z)$ over $\mathbb{U}$.

Theorem 2.2. *Let $v, \mu > 0$ and $\lambda \geq -1/2$. If $\lambda + v \geq 0,462$ and*

$$(\lambda+1)(v+\lambda+1)^{[|\mu|]} > \frac{2+\sqrt{2}}{4}$$

where $[|\mu|]$ denotes the greatest integer value of μ, then $J^{\mu}_{v,\lambda}(z)$ given in (2) is starlike in $\mathbb{U}$.

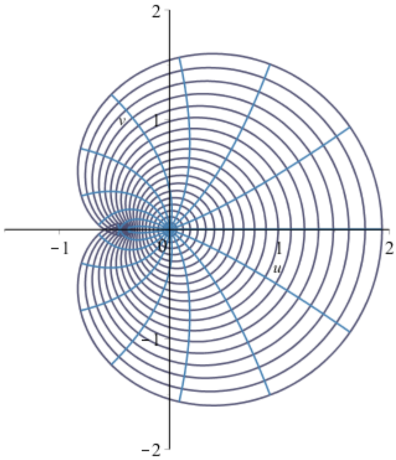

FIGURE 2. Mapping of $J^{\frac{1}{2}}_{\frac{1}{10},-\frac{1}{2}}(z)$ over $\mathbb{U}$.

Proof. Let $p(z)$ be the function defined by

$$p(z) = \frac{z(J^{\mu}_{v,\lambda})'(z)}{J^{\mu}_{v,\lambda}(z)}, \quad (z \in \mathbb{U}).$$

Since

$$\frac{J^{\mu}_{v,\lambda}(z)}{z} \neq 0, \quad (z \in \mathbb{U}),$$

the function p is analytic in $\mathbb{U}$ and $p(0) = 1$. To prove our theorem, we need to show that $Re\,(p(z)) > 0$, $z \in \mathbb{U}$. It is easy to show that, if $|p(z) - 1| < 1$, $z \in \mathbb{U}$, then $Re\,(p(z)) > 0$. For $\lambda \geq -1/2$ we have

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+k)} \leq \frac{1}{(\lambda+1)^{k-1}} \quad (k \in \mathbb{N}). \tag{4}$$

Furthermore, since

$$\Gamma(v+\lambda+1+(k-1)\mu) \geq \Gamma(v+\lambda+1+(k-1)[|\mu|])$$

where $\lambda + v \geq 0,462$ and $[|\mu|]$ denotes the greatest integer value of μ, we have

$$\frac{\Gamma(v+\lambda+1)}{\Gamma(v+\lambda+1+(k-1)\mu)} \leq \frac{1}{(v+\lambda+1)^{[|\mu|](k-1)}}. \tag{5}$$

Under the hypotheses of the theorem, using (2), (4) and (5), we can write

$$
\begin{aligned}
\left|(J^{\mu}_{v,\lambda})'(z)-\frac{J^{\mu}_{v,\lambda}(z)}{z}\right| &= \left|\sum_{k=2}^{\infty}\frac{(k-1)\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)}z^{k-1}\right| \\
&< \sum_{k=2}^{\infty}\frac{(k-1)\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)} \\
&< \sum_{k=2}^{\infty}\frac{(k-1)}{4^{k-1}}\frac{1}{(\lambda+1)^{k-1}}\frac{1}{(v+\lambda+1)^{[|\mu|](k-1)}} \qquad (6)\\
&= \sum_{k=2}^{\infty}\frac{k-1}{(4(\lambda+1)(v+\lambda+1)^{[|\mu|]})^{k-1}} \\
&= \frac{4(\lambda+1)(v+\lambda+1)^{[|\mu|]}}{(4(\lambda+1)(v+\lambda+1)^{[|\mu|]}-1)^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{J^{\mu}_{v,\lambda}(z)}{z}\right| &= \left|1+\sum_{k=2}^{\infty}\frac{\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)}z^{k-1}\right| \\
&> 1-\sum_{k=2}^{\infty}\frac{\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)} \\
&\geq 1-\sum_{k=2}^{\infty}\frac{1}{4^{k-1}(\lambda+1)^{k-1}}\frac{1}{(v+\lambda+1)^{(k-1)[|\mu|]}} \qquad (7)\\
&= 1-\sum_{k=2}^{\infty}\frac{1}{(4(\lambda+1)(v+\lambda+1)^{[|\mu|]})^{k-1}} \\
&= \frac{4(\lambda+1)(v+\lambda+1)^{[|\mu|]}-2}{4(\lambda+1)(v+\lambda+1)^{[|\mu|]}-1},
\end{aligned}
$$

for $z\in\mathbb{U}$. From (6) and (7), we get

$$
\begin{aligned}
|p(z)-1| &= \left|\frac{z(J^{\mu}_{v,\lambda})'(z)}{J^{\mu}_{v,\lambda}(z)}-1\right| = \left|\frac{(J^{\mu}_{v,\lambda})'(z)-\frac{J^{\mu}_{v,\lambda}(z)}{z}}{\frac{J^{\mu}_{v,\lambda}(z)}{z}}\right| \\
&< \frac{4(\lambda+1)(v+\lambda+1)^{[|\mu|]}}{(4(\lambda+1)(v+\lambda+1)^{[|\mu|]}-1)(4(\lambda+1)(v+\lambda+1)^{[|\mu|]}-2)}.
\end{aligned}
$$

Hence we deduce that $|p(z)-1|<1$ if

$$(\lambda+1)(v+\lambda+1)^{[|\mu|]} > \frac{2+\sqrt{2}}{4}.$$

This completes the proof of the Theorem 2.2. □

Example 2.2. The function $J^{\frac{14}{10}}_{1,-\frac{1}{2}}(z)$ is starlike in $\mathbb{U}$. (Figure 3)

Theorem 2.3. *Let $v,\mu>0$ and $\lambda\geq -1/2$. If $\lambda+v\geq 0,462$ and*

$$(\lambda+1)(v+\lambda+1)^{[|\mu|]} > 2$$

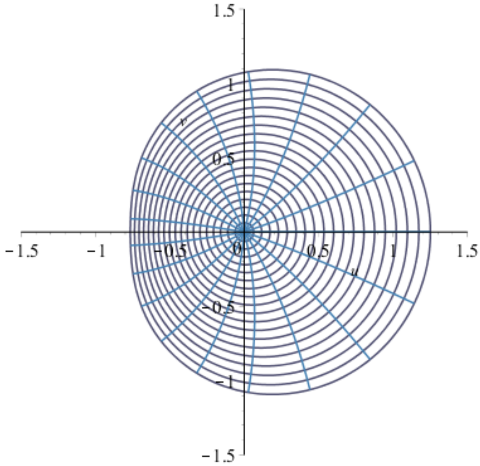

FIGURE 3. Mapping of $J^{\frac{14}{10}}_{1,-\frac{1}{2}}(z)$ over $\mathbb{U}$.

where $[|\mu|]$ *denotes the greatest integer value of* μ, *then* $J^{\mu}_{v,\lambda}(z)$ *given in (2) is convex in* $\mathbb{U}$.

Proof. Let $p(z)$ be the function defined by

$$p(z) = 1 + \frac{z(J^{\mu}_{v,\lambda})''(z)}{(J^{\mu}_{v,\lambda})'(z)}, \quad (z \in \mathbb{U}).$$

Then $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. To prove $J^{\mu}_{v,\lambda}(z)$ is convex in $\mathbb{U}$, we need to show that $|p(z) - 1| < 1$, $z \in \mathbb{U}$. For $z \in \mathbb{U}$, using (2), (4), (5) and the fact that $k(k-1) \leq 2^k$ for all $k \geq 1$, we get

$$\begin{aligned}
\left|z(J^{\mu}_{v,\lambda})''(z)\right| &= \left|\sum_{k=2}^{\infty} \frac{k(k-1)\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)} z^{k-1}\right| \\
&< \sum_{k=2}^{\infty} \frac{k(k-1)\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)} \\
&\leq \sum_{k=2}^{\infty} \frac{k(k-1)}{4^{k-1}(\lambda+1)^{k-1}(v+\lambda+1)^{(k-1)[|\mu|]}} \qquad (8) \\
&\leq 2\sum_{k=2}^{\infty} \frac{1}{(2(\lambda+1)(v+\lambda+1)^{[|\mu|]})^{k-1}} \\
&= \frac{2}{2(\lambda+1)(v+\lambda+1)^{[|\mu|]} - 1}.
\end{aligned}$$

Furthermore, using (2), (4), (5) and the fact that $k \leq 2^{k-1}$ for all $k \geq 1$, we get

$$\begin{aligned}
\left|(J^{\mu}_{v,\lambda})'(z)\right| &= \left|1+\sum_{k=2}^{\infty}\frac{k\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)}z^{k-1}\right| \\
&> 1-\sum_{k=2}^{\infty}\frac{k\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)} \\
&\geq 1-\sum_{k=2}^{\infty}\frac{k}{4^{k-1}(\lambda+1)^{k-1}(v+\lambda+1)^{(k-1)[|\mu|]}} \\
&\geq 1-\sum_{k=2}^{\infty}\frac{1}{(2(\lambda+1)(v+\lambda+1)^{[|\mu|]})^{k-1}} \\
&= \frac{2(\lambda+1)(v+\lambda+1)^{[|\mu|]}-2}{2(\lambda+1)(v+\lambda+1)^{[|\mu|]}-1},
\end{aligned} \tag{9}$$

under the given hypotheses. From (8) and (9), we obtain

$$\left|\frac{z(J^{\mu}_{v,\lambda})''(z)}{(J^{\mu}_{v,\lambda})'(z)}\right| < \frac{1}{(\lambda+1)(v+\lambda+1)^{[|\mu|]}-1},$$

and the last expression is less than 1 by our assumption. This completes the proof of the Theorem 2.3. □

Example 2.3. The function $J^{3}_{1,1}(z)$ is convex in $\mathbb{U}$. (Figure 4)

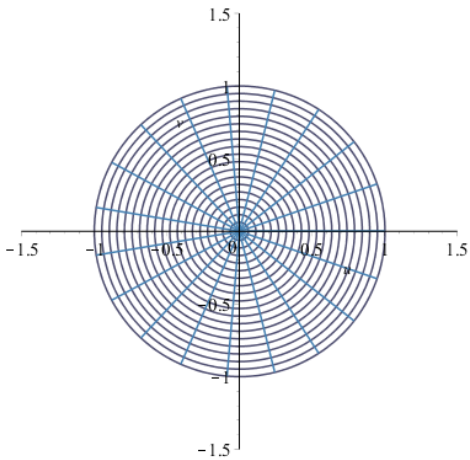

FIGURE 4. Mapping of $J^{3}_{1,1}(z)$ over $\mathbb{U}$.

Theorem 2.4. *Let $v, \mu > 0$ and $\lambda > -1/2$. If $\lambda + v \geq 0,462$, then $J^{\mu}_{v,\lambda}(z)$ given in (2) is univalent and starlike in $\mathbb{U}_{1/2}$.*

Proof. Under the hypotheses of the theorem, the straightforward calculation would yield

$$\begin{aligned}\left|\frac{J^{\mu}_{v,\lambda}(z)}{z}-1\right| &= \left|\sum_{k=2}^{\infty}\frac{\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)}z^{k-1}\right| \\ &< \sum_{k=2}^{\infty}\frac{1}{4^{k-1}(\lambda+1)^{(k-1)}(v+\lambda+1)^{(k-1)[|\mu|]}} \\ &= \sum_{k=2}^{\infty}\frac{1}{\left(4(\lambda+1)(v+\lambda+1)^{[|\mu|]}\right)^{(k-1)}} \\ &= \frac{1}{4(\lambda+1)(v+\lambda+1)^{[|\mu|]}-1}.\end{aligned}$$

In view of Lemma 1.2, $J^{\mu}_{v,\lambda}(z)$ is starlike in $\mathbb{U}_{1/2}$, if the last expression less than 1, or equivalently, if $(\lambda+1)(v+\lambda+1)^{[|\mu|]} > \frac{1}{2}$. But, this is already a consequence of the hypotheses of the theorem. This completes the proof of the Theorem 2.4. □

Example 2.4. The function $J^{1}_{2,\frac{1}{3}}(z)$ is starlike in $\mathbb{U}_{1/2}$. (Figure 5)

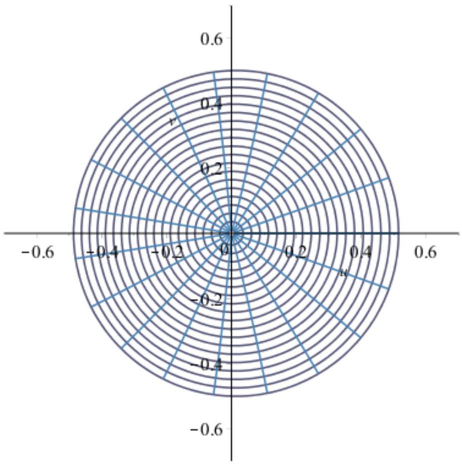

FIGURE 5. Mapping of $J^{1}_{2,\frac{1}{3}}(z)$ over $\mathbb{U}_{1/2}$.

Theorem 2.5. *Let $v,\mu>0$ and $\lambda \geq -1/2$. If $\lambda+v \geq 0,462$ and*

$$(\lambda+1)(v+\lambda+1)^{[|\mu|]} > \frac{2+\sqrt{2}}{4}$$

where $[|\mu|]$ denotes the greatest integer value of μ, then $J^{\mu}_{v,\lambda}(z)$ given in (2) is convex in $\mathbb{U}_{1/2}$.

Proof. Under the hypotheses of the theorem, the straightforward calculation would yield

$$\begin{aligned}\left|(J^{\mu}_{v,\lambda})'(z)-1\right| &= \left|\sum_{k=2}^{\infty}\frac{k\Gamma(\lambda+1)\Gamma(v+\lambda+1)}{4^{k-1}\Gamma(\lambda+k)\Gamma(v+\lambda+1+(k-1)\mu)}z^{k-1}\right|\\ &< \sum_{k=2}^{\infty}\frac{k}{4^{k-1}(\lambda+1)^{k-1}(v+\lambda+1)^{(k-1)[|\mu|]}}\\ &= \sum_{k=2}^{\infty}\frac{k}{(4(\lambda+1)(v+\lambda+1)^{[|\mu|]})^{(k-1)}}\\ &= \frac{8(\lambda+1)(v+\lambda+1)^{[|\mu|]}-1}{(4(\lambda+1)(v+\lambda+1)^{[|\mu|]}-1)^2}.\end{aligned}$$

In view of Lemma 1.3, $J^{\mu}_{v,\lambda}(z)$ is convex in $\mathbb{U}_{1/2}$, if

$$8(\lambda+1)(v+\lambda+1)^{[|\mu|]}-1<(4(\lambda+1)(v+\lambda+1)^{[|\mu|]}-1)^2.$$

The last inequality holds if

$$(\lambda+1)(v+\lambda+1)^{[|\mu|]}>\frac{1}{4}(2+\sqrt{2}).$$

This completes the proof of the Theorem 2.5. □

Example 2.5. The function $J^{1}_{2,1}(z)$ is convex in $\mathbb{U}_{1/2}$. (Figure 6)

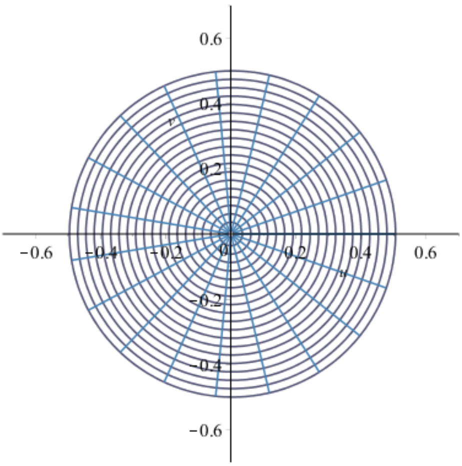

FIGURE 6. Mapping of $J^{1}_{2,1}(z)$ over $\mathbb{U}_{1/2}$.

The following results can be proved in a manner that is analogous to the proofs of the earlier results in this section. Therefore, we omit the details.

Theorem 2.6. *Let $v,\mu>0$ and $\lambda\geq -1/2$. If $\lambda+v\geq 0,462$ and*

$$(\lambda+1)(v+\lambda+1)^{[|\mu|]}>\frac{1}{8}(5+\sqrt{17})$$

where $[|\mu|]$ denotes the greatest integer value of μ, then the function $J^{\mu}_{v,\lambda}(z)$ given in (2) is belongs to the class of S_p.

Example 2.6. The function $J^{1}_{\frac{1}{2},1}(z)$ is in the class S_p.

Theorem 2.7. *Let $v,\mu>0$ and $\lambda\geq -1/2$. If $\lambda+v\geq 0,462$ and*

$$(\lambda+1)(v+\lambda+1)^{[|\mu|]}>3$$

where $[|\mu|]$ denotes the greatest integer value of μ, then the function $J^{\mu}_{v,\lambda}(z)$ given in (2) is belongs to the class of UCV.

Example 2.7. The function $J^{2}_{2,2}(z)$ is in the class UCV.

References

[1] R.S. Pathak, Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transformations, *Proc. Natl. Acad. Sci. India A* **36** (1966), no. 1, 81–86.
[2] V. Kiryakova, A Guide to Special Functions in Fractional Calculus, *Mathematics* **9** (2021), 106. DOI:10.3390/math9010106
[3] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer Monographs in Mathematics, Springer Verlag, Berlin-Heidelberg, 2014.
[4] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, UK, 1995.
[5] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer Verlag, New York, NY, USA, 1983.
[6] A.W. Goodman, *Univalent Functions*, Mariner Publishing Company, New York, USA, 1983.
[7] F. Ronning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Am. Math. Soc.* **118** (1993), 189–196.
[8] S. Ozaki, On the theory of multivalent functions, *Science Reports of the Tokyo Bunrika Daigaku, Section A* **2** (1935), no. 40, 167–188.
[9] T.H. MacGregor, The radius of univalence of certain analytic functions II, *Proc. Amer. Math. Soc.* **14** (1963), no. 3, 521–524.
[10] T.H. MacGregor, A class of univalent functions, *Proc. Amer. Math. Soc.* **15** (1964), 311–317.
[11] V. Ravichandran, On uniformly convex functions, *Ganita* **53** (2002), 117–124.
[12] D. Răducanu,Geometric properties of Mittag-Leffler functions, In: C. Flaut, Š. Hošková-Mayerová, D. Flaut (Eds.), *Models and Theories in Social Systems*, Studies in Systems, Decision and Control **179**, Springer, Cham, (2019), 403–415. DOI:10.1007/978-3-030-00084-4_22
[13] D. Bansal, J.K. Prajapat, Certain geometric properties of the Mittag-Leffler functions, *Complex Var. Elliptic Equ.* **61** (2016), no. 3, 338–350.
[14] J.K. Prajapat, Certain geometric properties of the Wright functions, *Integral Transforms Spec. Funct.* **26** (2015), no. 3, 203–212.
[15] D. Bansal, A. Soni, M.K. Soni, Geometric properties of τ-confluent hypergeometric function, *Anal.Math.Phys.* **10** (2020), 73. DOI:10.1007/s13324-020-00426-8
[16] S. Ponnusamy, A. Baricz, Starlikeness and convexity of generalized Bessel functions, *Integral Transform Spec Funct.* **21** (2010), no. 9, 641–653.
[17] S. Sümer Eker, S. Ece, Geometric Properties of the Miller-Ross Functions, *Iran J. Sci. Technol. Trans. Sci.* **46** (2022), 631–636. DOI:10.1007/s40995-022-01268-8
[18] S. Sümer Eker, S. Ece, Geometric Properties of Normalized Rabotnov Function, *Hacet. J. Math. Stat.* **51**, (2022), no. 5, 1248–1259. DOI:10.15672/hujms.980307

(Gülfem Akin) Institute of Natural and Applied Science, Dicle University, Diyarbakir, Türkey
E-mail address: gulfemy@hotmail.com

(Sevtap Sümer Eker) Department of Mathematics, Faculty of Science, Dicle University, Diyarbakir, Türkey
E-mail address: sevtaps@dicle.edu.tr