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# On Metric Dimension of Hendecagonal Circular Ladder $H_n$

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Abstract. Let $\zeta = (V, E)$ be a $n$th order connected graph. If the distance vectors to the vertices in an ordered subset $G$ of vertices can uniquely identify each vertex of the graph $\zeta$, then the set $G$ is known as resolving set for the graph $\zeta$. The resolving set $G$ with smallest cardinality serves as the metric basis of graph $\zeta$ and the cardinality of this smallest resolving set serves as metric dimension for $\zeta$. In this article, two families of convex polytopes that are closely linked are demonstrated and it is found that the metric dimension is three for both the families.

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## 1. Introduction

The metric dimension was found to be applicable in a variety of science and technology sectors. Mathematics and computer science both have multiple uses for the ideas of metric dimension and resolving set. Metric basis and resolving set are now basic aspects in combinatorial chemistry and molecular topology. Let $\zeta = (V, E)$ be a connected, simple and undirected graph, where $V$ and $E$ are the sets of vertices and edges respectively. Let the distance between the vertices $m$ and $n$ be denoted by $d_\zeta(m, n)$ (or simply $d(m, n)$). The maximum degree and minimum degree of $\zeta$ are denoted by $\Delta(\zeta)$ and $\delta(\zeta)$, respectively.

A vertex $y$ is said to resolve two distinct vertices $t, g \in V(\zeta)$ if $d(y, t) \neq d(y, g)$. If each pair of vertices $t, g \in V(\zeta)$ with $t \neq g$ is resolved by some vertex $y \in F$, where $F$ is a subset of vertices in $V(\zeta)$, then $F$ is called as *resolving set* or *metric generator* for $\zeta$. A resolving set with lowest cardinality is called as metric basis for graph $\zeta$ and this minimum cardinality is known as *metric dimension* of graph $\zeta$ and is represented by $dim(\zeta)$. For an ordered subset of vertices $M = \{l_1, l_2, l_3, ..., l_j\}$ the coordinate/$j$-code/representation of a vertex $k$ in $V(\zeta)$ is

$$d(k|F) = (d(l_1, k), d(l_2, k), d(l_3, k), ..., d(l_j, k)).$$

In this regard, the metric generator for $\zeta$ is the set $M$, if $d(h|M) \neq d(u|M)$, for any two vertices $h, u \in V(\zeta)$ with $h \neq u$.

The metric generators were referred to as the locating sets by Slater[14], who also developed the concept of the metric dimension. On the other side, the identical concept of metric dimension was demonstrated by Harary and Melter[4], who referred to metric generators as resolving sets. Throughout this paper, we shall adopt resolving

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sets for metric generators and locating sets. In addition to these two[4, 11] significant pioneering publications, there are other studies on applications and theoretical aspects of this invariant that may be found in the literature[1, 3, 7, 8, 9].

In basic geometry, polytope is a geometrical object with flat sides. A special case of polytopes is convex polytope which is also a polytope with the additional property of convexity and contained in the n-dimensional space $\mathbb{R}^n$. Applications for convex polytopes can be found throughout many branches of mathematics and computer science. Some kinds of convex polytopes' metric dimensions have been taken into consideration in [5, 6, 10]. Sharma and Bhat[11, 12, 13] introduced some plane graphs with constant metric dimension. Next, we describe some of the fundamental characteristics of a new family of convex polytopes.

**Hendecagonal Circular Ladder:** The hendecagonal circular ladder (HCL), designated by $H_n$, is a convex polytope with radial symmetry with $6n$ vertices and $7n$ edges. There are $2n$ vertices having degree 3 and $4n$ vertices having degree 2 in it. The hendecagonal circular ladder is made up of $n$ faces with 11 sides each, one face with $n$ sides, and another face with $2n$ sides, as shown in Figure 1.

An open problem was brought forward by Imran et al.[5] that:

**Problem:** *Characterize the classes of radially symmetrical plane graphs $H$ obtained from $\Gamma$ by itroducing new edges in $\Gamma$ such that $dim(H) = dim(\Gamma)$ and $V(H) = V(\Gamma)$.*

We create a plane graph family, $H_n$, as indicated above, in an effort to partially deal with this problem. In an attempt to partially answer this problem, we construct a plane graph family, $H_n$ as defined above. Next, by introducing new edges to $H_n$ at various locations, we create a new family of convex polytopes called $H_n^*$ with a similar set of vertices. In this article, we computed the metric dimension for two classes of convex polytopes that share a common set of vertices and are closely connected.

## 2. Preliminaries

We discuss some fundamental ideas about the metric dimension of graphs in this section.

**Independent Set[2]:** A subset of vertices in a graph in which there is no pair of vertices that are adjacent is called as independent set.

**Independent resolving set[2]:** A subset B of vertices in a graph which is both independent and resolving is known as independent resolving set.

In this study, we focus on two convex polytopes for which we have $V(H_n) = V(H_n^*) = \{p_i, q_i, r_i, s_i, t_i, u_i : 1 \leq i \leq n\}$. The set of coordinates or metric codes for the vertices $p_i, q_i, r_i, s_i, t_i$ and $u_i$ is denoted by P,Q,R,S,T and U respectively, for the convex polytopes $H_n$ and $H_n^*$.

For those graphs with two metric dimensions, Khuller et al.[7] introduced the subsequent result:

**Theorem 2.1.** Let $B \subset V(\zeta)$ be the metric basis for the connected graph $\zeta$ with cardinality two i.e., $\mid B \mid = 2$ and let $B = \{w, e\}$. Then, the following are true:
(i) Between the vertices w and e, a shortest path P uniquely exists.
(ii) The valencies of the vertices w and e cannot exceed 3.
(iii) The valency of any other vertex on P cannot exceed 5.

## 3. Metric dimension of Hendecagonal Circular Ladder $H_n$

The structure of the new family of $H_n$ is discussed in this section. We examine some of its fundamental attributes and determine its metric dimension.

### The Graph of $H_n$:

The HCL $H_n$ can be obtained from from the Heptagonal Circular Ladder $\Gamma_n$ by placing $2n$ new vertices between the vertices $p_t$ and $q_t$ $(1 \leq t \leq n)$ in $\Gamma_n$. It contains $7n$ edges and $6n$ vertices (see Fig.1). The HCL's $H_n$ vertex set and edge set are separately portrayed by $V(H_n)$ and $E(H_n)$, where $V(H_n) = \{p_i, q_i, r_i, s_i, t_i, u_i : 1 \leq i \leq n\}$ and $E(H_n) = \{p_tq_t, q_tr_t, r_ts_t, s_tt_t, t_tu_t, u_tt_{t+1}, p_tp_{t+1} : 1 \leq t \leq n\}$.

We call the cycle that the vertices $\{p_i : 1 \leq i \leq n\}$ induced in graph $H_n$ as $p$-cycle, the cycle generated by the vertices $\{t_i, u_i : 1 \leq i \leq n\}$ in $H_n$ as the $tu$-cycle, the vertices $\{q_i : 1 \leq i \leq n\}$ in $H_n$ as inner vertices, the vertices $\{s_i : 1 \leq i \leq n\}$ in the graph $H_n$ as outer vertices. In the subsequent result, we look into the HCL $H_n$ graph's metric dimension.

**Theorem 3.1.** $dim(H_n) = 3$, where $n \geq 21$ is odd integer.

*Proof.* Since, $n$ is odd, so $n = 2w + 1$, where $w \geq 3$ is an integer. Let $F = \{p_2, p_{w+1}, p_n\} \subset V(H_n)$. To show that $F$ is a resolving set for HCL $H_n$, we assign metric codes to each vertex of $H_n$ with respect to the set $F$.
The metric co-ordinates for the vertices $\{p_l : 1 \leq j \leq n\}$ are given by

$$d(p_j|F) = \begin{cases} (1, w, 1), & j = 1; \\ (j-2, w-j+1, j), & 2 \leq j \leq w; \\ (j-2, j-w-1, 2w-j+1), & w+1 \leq j \leq w+2; \\ (2w-j+3, j-w-1, 2w-j+1), & w+3 \leq j \leq 2w+1. \end{cases}$$

The metric co-ordinates for the vertices $\{q_j : 1 \leq j \leq n\}$ are

$$d(q_j|F) = \begin{cases} (2, w+1, 2), & j = 1; \\ (j-1, w-j+2, j+1), & 2 \leq j \leq w; \\ (j-1, j-w, 2w-j+2), & w+1 \leq j \leq w+2; \\ (2w-j+4, j-w, 2w-j+2), & w+3 \leq j \leq 2w+1. \end{cases}$$

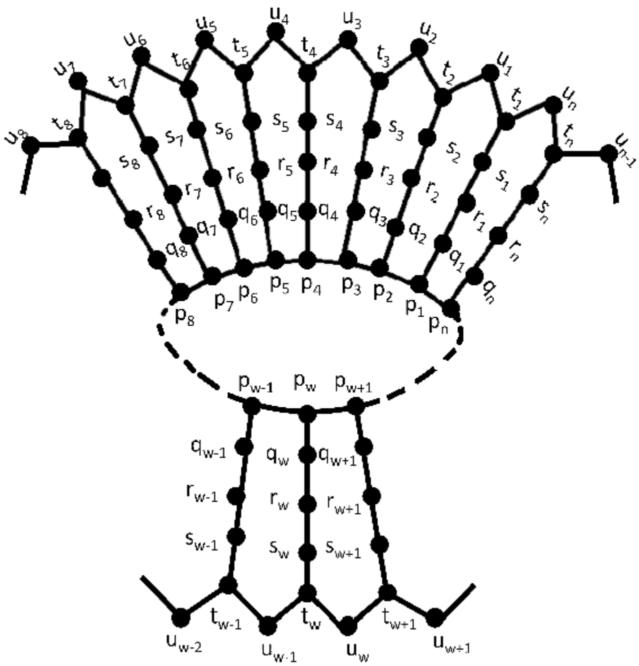

FIGURE 1. Hendecagonal Circular Ladder $H_n$

Next, the co-ordinates for the vertices $\{r_j : 1 \leq j \leq n\}$ are $d(r_j|F) = d(q_j|F)+(1,1,1)$ for $1 \leq j \leq n$, the co-ordinates for the vertices $\{s_j : 1 \leq j \leq n\}$ are $d(s_j|F) = d(q_j|F) + (2,2,2)$ for $1 \leq j \leq n$. Finally, the co-ordinates for the vertices of $tu$-cycle are $d(t_j|F) = d(q_j|F) + (3,3,3)$ for $1 \leq j \leq n$ and

$$d(u_j|F) = \begin{cases} (5, w+4, 6), & j = 1; \\ (j+3, w-j+5, j+5), & 2 \leq j \leq w; \\ (j+3, j-w+4, 2w-j+5), & w+1 \leq j \leq w+2; \\ (2w-j+7, j-w+4, 2w-j+5), & w+3 \leq j \leq 2w; \\ (6, w+5, 5), & j = 2w+1. \end{cases}$$

From the above codes, we have $P \cap Q \cap R \cap S \cap T \cap U = \phi$, so there does not exists two vertices in $H_n$ possessing the same metric co-ordinates, implying that $dim(H_n) \leq 3$. Now to complete the proof, we show that $dim(H_n) \geq 3$.

We prove that $dim(H_n) \geq 3$ by proving that there does not exists a resolving set $F$ with $| F |= 2$. Then, we have the following:

| Resolving set | Contradictions |
|---|---|
| $F=\{p_1,p_j\},\ p_j(2\le j\le n)$ | $d(q_1\|F)=d(p_n\|F)$, for $2\le j\le w$, and<br>$d(u_1\|F)=d(t_n\|F)$, for $j=w+1$, a contradiction. |
| $F=\{q_1,q_j\},\ q_j(2\le j\le n)$ | $d(q_n\|F)=d(p_{n-1}\|F)$, for $2\le j\le w-1$,<br>$d(t_2\|F)=d(r_{n-1}\|F)$, for $j=w$, and<br>$d(t_2\|F)=d(s_n\|F)$ for $j=w+1$, a contradiction. |
| $F=\{r_1,r_j\},\ r_j(2\le j\le n)$ | $d(q_n\|F)=d(p_{n-1}\|F)$, for $2\le j\le w-1$,<br>$d(u_2\|F)=d(r_n\|F)$, for $j=w$, and<br>$d(u_2\|F)=d(s_n\|F)$ for $j=w+1$, a contradiction. |
| $F=\{s_1,s_j\},\ s_j(2\le j\le n)$ | $d(q_n\|F)=d(p_{n-1}\|F)$, for $2\le j\le w-1$,<br>$d(r_2\|F)=d(p_{n-1}\|F)$, for $j=w$, and<br>$d(u_2\|F)=d(s_n\|F)$ for $j=w+1$, a contradiction. |
| $F=\{t_1,t_j\},\ s_j(2\le j\le n)$ | $d(q_{n-1}\|F)=d(p_{n-2}\|F)$, for $2\le j\le w-2$,<br>$d(s_3\|F)=d(p_n\|F)$, for $w-1\le j\le w$, and<br>$d(u_2\|F)=d(s_n\|F)$ for $j=w+1$, a contradiction. |
| $F=\{u_1,u_j\},\ u_j(2\le j\le n)$ | $d(q_{n-1}\|F)=d(p_{n-2}\|F)$, for $2\le j\le w-2$,<br>$d(r_3\|F)=d(p_1\|F)$, for $w-1\le j\le w$, and<br>$d(q_3\|F)=d(p_n\|F)$, for $j=w+1$, a contradiction. |
| $F=\{p_1,q_j\},\ q_j(1\le j\le n)$ | $d(q_n\|F)=d(p_{n-1}\|F)$, for $1\le j\le w-1$,<br>$d(u_1\|F)=d(s_{n-1}\|F)$, for $j=w$, and<br>$d(u_1\|F)=d(t_n\|F)$, for $j=w+1$, a contradiction. |
| $F=\{p_1,r_j\},\ r_j(1\le j\le n)$ | $d(q_n\|F)=d(p_{n-1}\|F)$, for $1\le j\le w-1$,<br>$d(u_1\|F)=d(s_{n-1}\|F)$ for $j=w$, and<br>$d(t_n\|F)=d(u_1\|F)$ for $j=w+1$, a contradiction. |
| $F=\{p_1,s_j\},\ s_j(1\le j\le n)$ | $d(q_n\|F)=d(p_{n-1}\|F)$, for $1\le j\le w-1$,<br>$d(t_1\|F)=d(s_n\|F)$ for $j=w$, and<br>$d(u_1\|F)=d(t_n\|F)$ for $j=w+1$, a contradiction. |
| $F=\{p_1,t_j\},\ t_j(1\le j\le n)$ | $d(q_{n-1}\|F)=d(p_{n-2}\|F)$, for $1\le j\le w-2$,<br>$d(q_n\|F)=d(p_{n-1}\|F)$, for $2\le j\le w-1$,<br>$d(t_n\|F)=d(u_n\|F)$, for $j=w$, and<br>$d(u_1\|F)=d(t_n\|F)$ for $j=w+1$, a contradiction. |
| $F=\{p_1,u_j\},\ u_j(1\le j\le n)$ | $d(q_{n-1}\|F)=d(p_{n-2}\|F)$, for $1\le j\le w-2$,<br>$d(q_n\|F)=d(p_{n-1}\|F)$, for $2\le j\le w-1$,<br>$d(u_n\|F)=d(t_n\|F)$ for $j=w$, and<br>$d(u_2\|F)=d(u_{n-1}\|F)$ for $j=w+1$, a contradiction. |
| $F=\{q_1,r_j\},\ r_j(1\le j\le n)$ | $d(q_n\|F)=d(p_{n-1}\|F)$, for $1\le j\le w-1$,<br>$d(t_2\|F)=d(r_{n-1}\|F)$, for $j=w$, and<br>$d(t_2\|F)=d(s_n\|F)$, for $j=w+1$, a contradiction. |
| $F=\{q_1,s_j\},\ s_j(1\le j\le n)$ | $d(q_n\|F)=d(p_{n-1}\|F)$, for $1\le j\le w-1$,<br>$d(t_2\|F)=d(r_{n-1}\|F)$, for $j=w$, and<br>$d(t_2\|F)=d(s_n\|F)$, for $j=w+1$, a contradiction. |
| $F=\{q_1,t_j\},\ t_j(1\le j\le n)$ | $d(q_{n-1}\|F)=d(p_{n-2}\|F)$, for $j=1$,<br>$d(q_n\|F)=d(p_{n-1}\|F)$, for $2\le j\le w-1$,<br>$d(t_2\|F)=d(r_{n-1}\|F)$, for $j=w$, and<br>$d(t_2\|F)=d(s_n\|F)$, for $j=w+1$, a contradiction. |

| Resolving set | Contradictions |
|---|---|
| $F=\{q_1,u_j\},\ u_j(1\leq j\leq n)$ | $d(q_{n-1}\vert F)=d(p_{n-2}\vert F)$, for $j=1$,<br>$d(q_n\vert F)=d(p_{n-1}\vert F)$, for $2\leq j\leq w-1$,<br>$d(s_2\vert F)=d(r_{n-1}\vert F)$, for $j=w$, and<br>$d(u_2\vert F)=d(u_{n-1}\vert F)$, for $j=w+1$, a contradiction. |
| $F=\{r_1,s_j\},\ s_j(1\leq j\leq n)$ | $d(q_n\vert F)=d(p_{n-1}\vert F)$, for $1\leq j\leq w-1$,<br>$d(s_2\vert F)=d(q_{n-1}\vert F)$, for $j=w$, and<br>$d(u_2\vert F)=d(s_n\vert F)$, for $j=w+1$ a contradiction. |
| $F=\{r_1,t_j\},\ t_j(1\leq j\leq n)$ | $d(q_n\vert F)=d(p_{n-1}\vert F)$, for $2\leq j\leq w-1$,<br>$d(s_2\vert F)=d(q_{n-1}\vert F)$, for $j=w$, and<br>$d(u_2\vert F)=d(s_n\vert F)$, for $j=w+1,1$, a contradiction. |
| $F=\{r_1,u_j\},\ u_j(1\leq j\leq n)$ | $d(q_{n-1}\vert F)=d(p_{n-2}\vert F)$, for $j=1$,<br>$d(q_n\vert F)=d(p_{n-1}\vert F)$, for $2\leq j\leq w-1$,<br>$d(r_2\vert F)=d(q_{n-1}\vert F)$, for $j=w$, and<br>$d(u_2\vert F)=d(u_{n-1}\vert F)$, for $j=w+1$, a contradiction. |
| $F=\{s_1,t_j\},\ t_j(1\leq j\leq n)$ | $d(q_n\vert F)=d(p_{n-1}\vert F)$, for $2\leq j\leq w-1$,<br>$d(r_2\vert F)=d(p_{n-1}\vert F)$ for $j=w$, and<br>$d(u_2\vert F)=d(s_n\vert F)$ for $j=w+1,1$, a contradiction. |
| $F=\{s_1,u_j\},\ u_j(1\leq j\leq n)$ | $d(q_{n-1}\vert F)=d(p_{n-2}\vert F)$, for $j=1$,<br>$d(q_n\vert F)=d(p_{n-1}\vert F)$, for $2\leq j\leq w-1$,<br>$d(s_3\vert F)=d(q_{n-1}\vert F)$, for $j=w$, and<br>$d(u_2\vert F)=d(u_{n-1}\vert F)$, for $j=w+1$, a contradiction. |
| $F=\{t_1,u_j\},\ u_j(1\leq j\leq n)$ | $d(q_{n-1}\vert F)=d(p_{n-2}\vert F)$, for $1\leq j\leq w-2$,<br>$d(p_n\vert F)=d(s_3\vert F)$, for $w-1\leq j\leq w$, and<br>$d(u_2\vert F)=d(u_{n-1}\vert F)$, for $j=w+1$, a contradiction. |

As a result, we draw the conclusion from the explanation above that no resolving set of cardinality two exists in $V(H_n)$ implying that $dim(H_n)=3$, which completes the proof. □

*Remark* 3.2. One can easily verify that the metric dimension of $H_n$, where $n$ is odd with $7\leq n\leq 19$, is also 3.

**Corollary 3.3.** The independent resolving number of HCL $H_n$ is 3, where $n\geq 21$ is odd integer.

## 4. Metric dimension of the convex polytope $H_n^*$

In this section, we start by talking about the structure of a new family of $H_n^*$ that we were able to derive from an HCL $H_n$. We investigate some of its fundamental properties and determine its metric dimension.

### The Graph of $H_n^*$:

The Convex polytope $H_n^*$ is obtained from the HCL $H_n$ by introducing $n$ new edges in the graph $H_n$ between the vertices $t_j$ and $t_{j+1}$ for $1\leq j\leq n$. It has $6n$ vertices and $8n$ edges. It consists of $n$ faces having 10-sides(see Figure 2). The vertex and edge

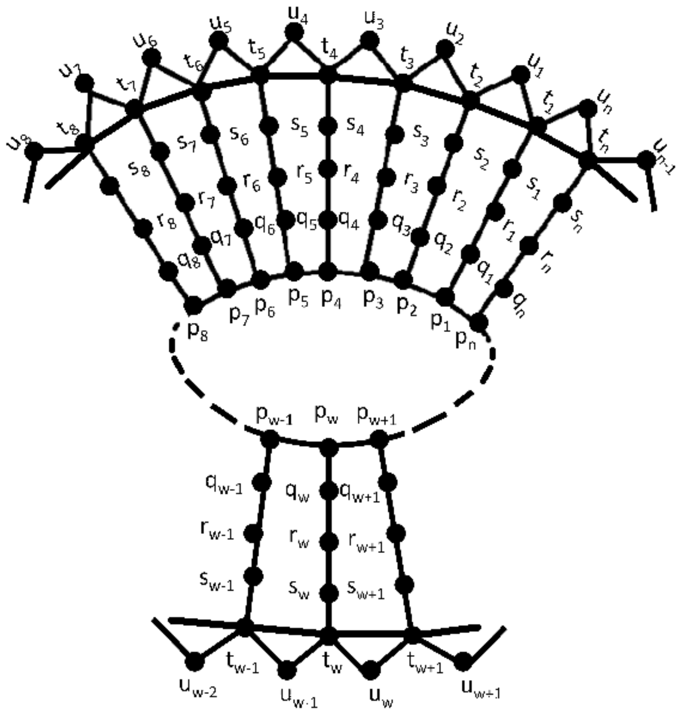

FIGURE 2. The Convex Polytope $H_n^*$

sets of $H_n^*$ are depicted separately by $V(H_n^*)$ and $E(H_n^*)$, where $V(H_n^*) = V(H_n)$ and $E(H_n^*) = E(H_n) \cup \{t_j t_{j+1} : 1 \leq j \leq n\}$.

The cycle generated by the vertices set $\{p_j : 1 \leq j \leq n\}$ in the graph $H_n^*$ as $p$-cycle, the cycle originated by the vertices $\{t_j, u_j : 1 \leq j \leq n\}$ in $H_n^*$ as $tu$-cycle, the vertices $\{q_j : 1 \leq j \leq n\}$ in the graph $H_n^*$ as inner vertices, the vertices $\{s_j : 1 \leq j \leq n\}$ in $H_n^*$ as outer vertices. We examine the metric dimension of the HCL $H_n^*$ graph in the ensuing finding.

**Theorem 4.1.** $dim(H_n^*) = 3$, where $n \geq 7$ is odd integer.

*Proof.* Since, $n$ is odd, so $n = 2w + 1$, where $w \geq 3$ is an integer. Let $F = \{p_2, p_{w+1}, p_n\} \subset V(H_n^*)$. Now, to show that $F$ is a resolving set for the convex polytope $H_n^*$, we give metric codes to each vertex of $H_n^*$ with respect to the set $F$.
The metric co-ordinates for the vertices $\{p_j : 1 \leq j \leq n\}$ are

$$d(p_j|F) = \begin{cases} (1, w, 1), & j = 1; \\ (j-2, w-j+1, j), & 2 \leq j \leq w; \\ (j-2, j-w-1, 2w-j+1), & w+1 \leq j \leq w+2; \\ (2w-j+3, j-w-1, 2w-j+1), & w+3 \leq j \leq 2w+1. \end{cases}$$

The metric co-ordinates for the vertices $\{q_j : 1 \leq j \leq n\}$ are

$$d(q_j|F) = \begin{cases} (2, w+1, 2), & j = 1; \\ (j-1, w-j+2, j+1), & 2 \leq j \leq w; \\ (j-1, j-w, 2w-j+2), & w+1 \leq j \leq w+2; \\ (2w-j+4, j-w, 2w-j+2), & w+3 \leq j \leq 2w+1. \end{cases}$$

Next, the co-ordinates for the vertices $\{r_j : 1 \leq j \leq n\}$ are $d(r_j|F) = d(q_j|F)+(1,1,1)$ for $1 \leq j \leq n$, the co-ordinates for the vertices $\{s_j : 1 \leq j \leq n\}$ are $d(s_j|F) = d(q_j|F) + (2, 2, 2)$ for $1 \leq j \leq n$. Finally, the co-ordinates for the vertices of $tu$-cycle are $d(t_j|F) = d(q_j|F) + (3, 3, 3)$ for $1 \leq j \leq n$ and

$$d(u_j|F) = \begin{cases} (5, w+4, 6), & j = 1; \\ (j+3, w-j+5, j+5), & 2 \leq j \leq w; \\ (j+3, j-w+4, 2w-j+5), & w+1 \leq j \leq w+2; \\ (2w-j+7, j-w+4, 2w-j+5), & w+3 \leq j \leq 2w; \\ (6, w+5, 5), & j = 2w+1. \end{cases}$$

From the above codes, we have $P \cap Q \cap R \cap S \cap T \cap U = \phi$, so we find that there does not exists a pair of vertices in $H_n^*$ possessing the same metric co-ordinates, implying that $dim(H_n^*) \leq 3$. Now to complete the proof, it is sufficient to prove that $dim(H_n^*) \geq 3$.

We prove that $dim(H_n^*) \geq 3$ by showing that no resolving set $F$ with $\mid F \mid= 2$ exists. Then, we have the following:

| Resolving set | Contradictions |
|---|---|
| $F = \{p_1, p_j\},\ p_j(2 \leq j \leq n)$ | $d(q_1|F) = d(p_n|F)$, for $2 \leq j \leq w$, and $d(u_1|F) = d(t_n|F)$ for $j = w+1$, a contradiction. |
| $F = \{q_1, q_j\},\ q_j(2 \leq j \leq n)$ | $d(q_n|F) = d(p_{n-1}|F)$, for $2 \leq j \leq w-1$, $d(t_n|F) = d(u_n|F)$, for $j = w$, and $d(u_1|F) = d(t_n|F)$ for $j = w+1$, a contradiction. |
| $F = \{r_1, r_j\},\ r_j(2 \leq j \leq n)$ | $d(t_n|F) = d(u_n|F)$, for $2 \leq j \leq w$, $d(u_1|F) = d(t_n|F)$ for $j = w+1$, a contradiction. |
| $F = \{s_1, s_j\},\ s_j(2 \leq j \leq n)$ | $d(t_n|F) = d(u_n|F)$, for $2 \leq j \leq w$, and $d(u_1|F) = d(t_n|F)$ for $j = w+1$, a contradiction. |
| $F = \{u_1, u_j\},\ u_j(2 \leq j \leq n)$ | $d(t_n|F) = d(u_n|F)$, for $2 \leq j \leq w$, and $d(u_2|F) = d(t_n|F)$ for $j = w+1$, a contradiction. |

| Resolving set | Contradictions |
|---|---|
| $F = \{p_1, q_j\},\ q_j(1 \leq j \leq n)$ | $d(q_n\|F) = d(p_{n-1}\|F)$, for $1 \leq j \leq w-1$, $d(t_n\|F) = d(u_n\|F)$, for $j = w$, and $d(u_1\|F) = d(t_n\|F)$, for $j = w+1$, a contradiction. |
| $F = \{p_1, r_j\},\ r_j(1 \leq j \leq n)$ | $d(t_n\|F) = d(u_n\|F)$, for $1 \leq j \leq w$, and $d(u_1\|F) = d(t_n\|F)$ for $j = w+1$, a contradiction. |
| $F = \{p_1, s_j\},\ s_j(1 \leq j \leq n)$ | $d(t_n\|F) = d(u_n\|F)$, for $1 \leq j \leq w$, and $d(u_1\|F) = d(t_n\|F)$ for $j = w+1$, a contradiction. |
| $F = \{p_1, u_j\},\ u_j(1 \leq j \leq n)$ | $d(t_n\|F) = d(u_n\|F)$, for $1 \leq j \leq w$, and $d(u_1\|F) = d(u_n\|F)$ for $j = w+1$, a contradiction. |
| $F = \{q_1, r_j\},\ r_j(1 \leq j \leq n)$ | $d(t_n\|F) = d(u_n\|F)$, for $1 \leq j \leq w$, and $d(u_1\|F) = d(t_n\|F)$ for $j = w+1$, a contradiction. |
| $F = \{q_1, s_j\},\ s_j(1 \leq j \leq n)$ | $d(t_n\|F) = d(u_n\|F)$, for $1 \leq j \leq w$, and $d(u_1\|F) = d(t_n\|F)$ for $j = w+1$, a contradiction. |
| $F = \{q_1, u_j\},\ u_j(1 \leq j \leq n)$ | $d(t_n\|F) = d(u_n\|F)$, for $1 \leq j \leq w$, and $d(u_1\|F) = d(u_n\|F)$ for $j = w+1$, a contradiction. |
| $F = \{r_1, s_j\},\ s_j(1 \leq j \leq n)$ | $d(t_n\|F) = d(u_n\|F)$, for $1 \leq j \leq w$, and $d(u_1\|F) = d(t_n\|F)$ for $j = w+1$, a contradiction. |
| $F = \{r_1, u_j\},\ u_j(1 \leq j \leq n)$ | $d(t_n\|F) = d(u_n\|F)$, for $1 \leq j \leq w$, and $d(u_1\|F) = d(u_n\|F)$ for $j = w+1$, a contradiction. |
| $F = \{s_1, u_j\},\ u_j(1 \leq j \leq n)$ | $d(t_n\|F) = d(u_n\|F)$, for $1 \leq j \leq w$, and $d(u_1\|F) = d(u_n\|F)$ for $j = w+1$, a contradiction. |

As a result, we draw the conclusion from the explanation above that there does not exists resolving set of cardinality two for $H_n^*$ implying that $dim(H_n^*) = 3$ which completes the proof. □

**Corollary 4.2.** The independent resolving number for the convex polytope $H_n^*$ is 3, where $n \geq 7$ is odd integer.

## 5. Conclusion

This study looks at the metric dimension of an HCL $H_n$ and a convex polytope $H_n^*$ for odd $n$. We have shown for these two structures that $V(H_n) = V(H_n^*)$ and $dim(H_n) = dim(H_n^*) = 3$ for odd $n \geq 7$ (a partly solution to the question outlined in [5]). We also proved that for odd integer $n \geq 7$, the resolving set for all of these families are independent.

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