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Existence and Stability Results for Implicit Impulsive Convex Combined Caputo Fractional Differential Equations

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Abstract. This paper deals with the existence and uniqueness results for a class of impulsive implicit fractional initial value problems of the convex combined Caputo fractional derivative. The arguments are based on Banach's contraction principle, Schauder's and Mönch's fixed point theorems. We will also establish the Ulam stability and give some examples to illustrate our results.

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1. Introduction

In recent years, fractional calculus has shown to be a very useful method for dealing with the complexity structures encountered in a variety of fields. It is concerned with the extension of integer order differentiation and integration of a function to non-integer order. The reader is directed to the publications [1, 2, 3, 9, 10, 11, 20, 21], for more details. Numerous books and papers have recently appeared in which the authors discussed the existence, stability, and uniqueness of solutions for various problems with fractional differential equations and inclusions using various fractional derivatives and conditions. One may see the papers [25, 15, 19, 8], and the references therein. Several papers in the literature discuss the Ulam stabilities of various types of differential and integral equations, see [17, 22, 24, 16, 27, 30, 28, 29] and the references therein.

The theory of impulsive differential equations is essential in describing many phenomena, it has received too much attention in the literature. For more details, we recommend [7, 23, 14].

In [23], the authors established existence and uniqueness results to the following k-generalized ψ-Hilfer problem with nonlinear implicit fractional differential equation

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with impulses involving both retarded and advanced arguments:

$$\begin{cases} \left({}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{\theta_i^+}x\right)(\theta) = f\left(\theta, x^{\theta}(\cdot), \left({}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{\theta_i^+}x\right)(\theta)\right), \quad \theta \in J_i,\ i = 0,\dots,\beta, \\ \left(\mathcal{J}^{k(1-\xi),k;\psi}_{\theta_i^+}x\right)(\theta_i^+) = \left(\mathcal{J}^{k(1-\xi),k;\psi}_{\theta_{i-1}^+}x\right)(\theta_i^-) + L_i(x(\theta_i^-)); i = 1,\dots,\beta, \\ \alpha_1\left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+}x\right)(a^+) + \alpha_2\left(\mathcal{J}^{k(1-\xi),k;\psi}_{\theta_\beta^+}x\right)(b) = \alpha_3, \\ x(\theta) = \varpi(\theta), \quad \theta \in [a-\lambda, a],\ \lambda > 0, \\ x(\theta) = \tilde{\varpi}(\theta), \quad \theta \in \left[b, b+\tilde{\lambda}\right],\ \tilde{\lambda} > 0, \end{cases}$$

where ${}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{a+}$, $\mathcal{J}^{k(1-\xi),k;\psi}_{a+}$ are the k-generalized ψ-Hilfer fractional derivative of order $\vartheta \in (0,k)$ and type $r \in [0,1]$, and k-generalized ψ-fractional integral of order $k(1-\xi)$ respectively, where $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$, $k > 0$, $\varpi \in C([a-\lambda,a],\mathbb{R})$, $\tilde{\varpi} \in C\left(\left[b, b+\tilde{\lambda}\right],\mathbb{R}\right)$, $f : [a,b] \times PC_{\xi,k;\psi}\left(\left[-\lambda,\tilde{\lambda}\right]\right) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given appropriate function specified latter, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\alpha_1 + \alpha_2 \neq 0$, $J_i := (\theta_i, \theta_{i+1}]; i = 0,\dots,\beta$, $a = \theta_0 < \theta_1 < \dots < \theta_\beta < \theta_{\beta+1} = b < \infty$, $x(\theta_i^+) = \lim_{\epsilon\to 0^+} x(\theta_i+\epsilon)$ and $x(\theta_i^-) = \lim_{\epsilon\to 0^-} x(\theta_i+\epsilon)$ represent the right and left hand limits of $x(\theta)$ at $\theta = \theta_i$ and $L_i : \mathbb{R} \to \mathbb{R}; i = 1,\dots,\beta$, are given continuous functions. They based their arguments on the Banach contraction principle and Schauder's fixed point theorem.

In this article, we present the combined Caputo fractional derivative which is a convex combination of the left Caputo fractional derivative of order κ_1 and the right Caputo fractional derivative of order κ_2. The main feature of the convex combined Caputo fractional operator is that it is a two sided operator, this property plays a decisive role in the fractional modeling. See [4], for more information.

In this paper, we study the existence and stability results for the following implicit impulsive fractional problem:

$$ {}^{C}_{0}D^{\kappa_1,\kappa_2;\gamma}_{\varkappa}\varphi(\theta) = f(\theta, \varphi(\theta), {}^{C}_{0}D^{\kappa_1,\kappa_2;\gamma}_{\varkappa}\varphi(\theta)), \quad \theta \in (\theta_\jmath, \theta_{\jmath+1}], \jmath = 0,\dots,\beta, \tag{1}$$

$$\Delta\varphi|_{\theta_\jmath} = I_\jmath(\varphi(\theta_\jmath^-)), \qquad \jmath = 1,\dots,\beta, \tag{2}$$

$$\varphi(0) = \varphi_0, \tag{3}$$

where ${}^{C}_{0}D^{\kappa_1,\kappa_2;\gamma}_{\varkappa}$ represents the convex combined Caputo fractional derivative of order $(\kappa_1,\kappa_2) \in (0,1]$, $\gamma \in [0,1]$, $f : \Theta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, where $\Theta = [0,\varkappa]$. $I_\jmath : \mathbb{R} \to \mathbb{R}$, and $\varphi_0 \in \mathbb{R}$, $0 = \theta_0 < \theta_1 < \dots < \theta_\beta < \theta_{\beta+1} = \varkappa$.
Let $\Delta\varphi|_{\theta_\jmath} = \varphi(\theta_\jmath^+) - \varphi(\theta_\jmath^-)$, with $\varphi(\theta_\jmath^+) = \lim_{h\to 0^+}\varphi(\theta_\jmath+h)$ and $\varphi(\theta_\jmath^-) = \lim_{h\to 0^-}\varphi(\theta_\jmath+h)$ represent the right and left limits of φ_θ at $\theta = \theta_\jmath$ respectively.

This article is organized as follows: Section 2 represents some definitions and previous results. In Section 3, we present the existence results for the problem (1)-(3) that are based on Banach's contraction principle and Schauder's fixed point theorem. In section 4, we study an existence result of a problem similar to problem (1)-(3) in a Banach space. Moreover, the Ulam stability of this problem is discussed. Finally, we give some examples to show the applicability of our results.

2. Preliminaries

In this section, we introduce some notations, definitions and previous results which are used throughout this paper.

Let Ξ be a Banach space and consider $C(\Theta,\Xi)$, where $\Theta=[0,\varkappa]$, the Banach space of all continuous functions from Θ to Ξ with the norm

$$\|\varphi\|_\infty=\sup\{\|\varphi(\theta)\|:\theta\in\Theta\}.$$

Consider the following sets of functions:

$$PC(\Theta,\mathbb{R})=\Big\{\varphi:[0,\varkappa]\to\mathbb{R}:\varphi\in C((\theta_\jmath,\theta_{\jmath+1}],\mathbb{R}),\jmath=0,\dots,\beta,\\ \text{and there exist }\varphi(\theta_\jmath^-),\varphi(\theta_\jmath^+),\jmath=1,\dots,\beta,\text{ with}\varphi(\theta_\jmath^-)=\varphi(\theta_\jmath)\Big\}.$$

Let $PC([0,\varkappa],\mathbb{R})$ is a Banach space with the norm

$$\|\varphi\|_{PC}=\sup_{\theta\in\Theta}\|\varphi(\theta)\|.$$

$$PC_1(\Theta,\Xi)=\Big\{\varphi:[0,\varkappa]\to\Xi:\varphi\in C((\theta_\jmath,\theta_{\jmath+1}],\Xi),\jmath=0,\dots,\beta,\\ \text{and there exist }\varphi(\theta_\jmath^-),\varphi(\theta_\jmath^+),\jmath=1,\dots,\beta,\text{ with }\varphi(\theta_\jmath^-)=\varphi(\theta_\jmath)\Big\}.$$

$PC_1([0,\varkappa],\Xi)$ is a Banach space with the norm

$$\|\varphi\|_{PC_1}=\sup_{\theta\in\Theta}\|\varphi(\theta)\|.$$

Definition 2.1 ([13]). Let $\kappa_1>0$. The left and right Riemann-Liouville fractional integrals of a function $\varphi\in C(\Theta,\Xi)$ of order κ_1 are given respectively by

$${}_0I_\theta^{\kappa_1}\varphi(\theta)=\frac{1}{\Gamma(\kappa_1)}\int_0^\theta(\theta-\varrho)^{\kappa_1-1}\varphi(\varrho)d\varrho,$$

and

$${}_\theta I_\varkappa^{\kappa_1}\varphi(\theta)=\frac{1}{\Gamma(\kappa_1)}\int_\theta^\varkappa(\varrho-\theta)^{\kappa_1-1}\varphi(\varrho)d\varrho.$$

Definition 2.2 ([4, 26]). Let $\kappa_1,\kappa_2>0$. The combined Riemann fractional integral of a function $\varphi\in C(\Theta,\Xi)$ of order (κ_1,κ_2) is defined by

$${}_0I_\varkappa^{\kappa_1,\kappa_2}\varphi(\theta)={}_0I_\theta^{\kappa_1}\varphi(\theta)+{}_\theta I_\varkappa^{\kappa_2}\varphi(\theta),$$

where ${}_0I_\theta^{\kappa_1}$ and ${}_\theta I_\varkappa^{\kappa_2}$ are the left and right fractional integrals of Riemann-Liouville of order κ_1 and κ_2 respectively.

Definition 2.3 ([13]). Let $\kappa_1\in(n,n+1]$, $n\in\mathbb{N}_0$. The left and right Caputo fractional derivatives of a function $\varphi\in C^{n+1}(\Theta,\Xi)$ of order κ_1 are given respectively by

$${}_0^CD_\theta^{\kappa_1}\varphi(\theta)=\frac{1}{\Gamma(n+1-\kappa_1)}\int_0^\theta(\theta-\varrho)^{n-\kappa_1}\varphi^{(n+1)}(\varrho)d\varrho,$$

and

$$ {}^{C}_{\theta}D^{\kappa_1}_{\varkappa}\varphi(\theta) = \frac{(-1)^{n+1}}{\Gamma(n+1-\kappa_1)} \int_{\theta}^{\varkappa} (\varrho-\theta)^{n-\kappa_1}\varphi^{(n+1)}(\varrho)d\varrho. $$

Definition 2.4 ([4, 26])**.** Let $\kappa_1, \kappa_2 \in (n, n+1]$, $n \in \mathbb{N}_0$, $\gamma \in [0,1]$. The convex combined Caputo fractional derivative of a function $\varphi \in C^{n+1}(\Theta, \Xi)$ of order (κ_1, κ_2) is given by

$$ {}^{C}_{0}D^{\kappa_1,\kappa_2;\gamma}_{\varkappa}\varphi(\theta) = \gamma\, {}^{C}_{0}D^{\kappa_1}_{\theta}\varphi(\theta) + (-1)^{n+1}(1-\gamma){}^{C}_{\theta}D^{\kappa_2}_{\varkappa}\varphi(\theta), $$

where ${}^{C}_{0}D^{\kappa_1}_{\theta}$ is the left Caputo derivative and ${}^{C}_{\theta}D^{\kappa_2}_{\varkappa}$ is the right one.

Lemma 2.1 ([13])**.** *If $\xi \in C^{n+1}(\Theta, \Xi)$ and $\kappa_1, \kappa_2 \in (n, n+1]$, $n \in \mathbb{N}_0$, $\gamma \in [0,1]$, then we have*

$$ {}_{0}I^{\kappa_1}_{\theta}\; {}^{C}_{0}D^{\kappa_1}_{\theta}\xi(\theta) = \xi(\theta) - \sum_{\jmath=0}^{n} \frac{\xi^{(\jmath)}(0)}{\jmath!}\theta^{\jmath}, $$

and

$$ {}_{\theta}I^{\kappa_2}_{\varkappa}\; {}^{C}_{\theta}D^{\kappa_2}_{\varkappa}\xi(\theta) = (-1)^{n+1}\left[\xi(\theta) - \sum_{\jmath=0}^{n} \frac{(-1)^{\jmath}\xi^{(\jmath)}(\varkappa)}{\jmath!}(\varkappa-\theta)^{\jmath}\right]. $$

Consequently, we may have

$$ {}_{0}I^{\kappa_1,\kappa_2}_{\varkappa}\; {}^{C}_{0}D^{\kappa_1,\kappa_2;\gamma}_{\varkappa}\xi(\theta) = \gamma\; {}_{0}I^{\kappa_1}_{\theta}\; {}^{C}_{0}D^{\kappa_1}_{\theta}\xi(\theta) + (-1)^{n+1}(1-\gamma)\; {}_{\theta}I^{\kappa_2}_{\varkappa}\; {}^{C}_{\theta}D^{\kappa_2}_{\varkappa}\xi(\theta)). $$

In particular, if $0 < \kappa_1, \kappa_2 \leq 1$, then we obtain

$$ {}_{0}I^{\kappa_1,\kappa_2}_{\varkappa}\; {}^{C}_{0}D^{\kappa_1,\kappa_2;\gamma}_{\varkappa}\xi(\theta) = \xi(\theta) - \gamma\xi(0) - (1-\gamma)\xi(\varkappa). $$

Remark 2.1. If we take $\gamma = \frac{1}{2}$ and $\kappa_1 = \kappa_2$, the convex combined Caputo fractional derivative coincides with the Riesz-Caputo derivative.

2.1. Measure of Noncompactness.

Definition 2.5 ([6])**.** Let X be a Banach space and let Ω_X be the family of bounded subsets of X. The Kuratowski measure of noncompactness is the map $\zeta : \Omega_X \longrightarrow [0, \infty)$ defined by

$$ \zeta(\chi) = inf\left\{\varepsilon > 0 : \chi \subset \bigcup_{j=1}^{\beta} \chi_j, diam(\chi_j) \leq \varepsilon\right\}, $$

where $\chi \in \Omega_X$.

The map ζ satisfies the following properties:

- $\zeta(\chi) = 0 \Leftrightarrow \overline{\chi}$ is compact (χ is relatively compact);
- $\zeta(\chi) = \zeta(\overline{\chi})$;
- $\chi_1 \subset \chi_2 \Rightarrow \zeta(\chi_1) \leq \zeta(\chi_2)$;
- $\zeta(\chi_1 + \chi_2) \leq \zeta(\Omega_1) + \zeta(\Omega_2)$;
- $\zeta(c\chi) = |c|\zeta(\chi)$, $c \in \mathbb{R}$;
- $\zeta(conv\chi) = \zeta(\chi)$.

Lemma 2.2 ([12])**.** *Let $\Omega \subset PC_1(\Theta, X)$ be a bounded and equicontinuous set. Then,*

a) The function $\vartheta \to \zeta(\Omega(\vartheta))$ is continuous on Θ, and

$$ \zeta_c(\Omega) = \sup_{\vartheta\in\Theta} \zeta(\Omega(\vartheta)), $$

b) $\zeta\left(\int_0^{\varkappa}\varphi(\varrho)d\varrho : \varphi\in\Omega\right)\leq\int_0^{\varkappa}\zeta(\Omega(\varrho))d\varrho$, *where*

$$\Omega(\vartheta)=\{\varphi(\vartheta):\varphi\in\Omega\},\vartheta\in\Theta.$$

2.2. Some Fixed Point Theorems.

Theorem 2.3 (Banach's fixed point theorem [13]). *Let X be a Banach space and $\mathcal{H}: X\longrightarrow X$ a contraction, i.e. there exists $\jmath\in[0,1)$ such that*

$$\|\mathcal{H}(\xi_1)-\mathcal{H}(\xi_2)\|\leq\jmath\|\xi_1-\xi_2\|,\qquad \text{for all } \xi_1,\xi_2\in X.$$

Then $\mathcal{H}$ has a unique fixed point.

Theorem 2.4 (Schauder's fixed point Theorem [13]). *Let X be a Banach space, D a bounded, closed, convex subset of X, and $T: D\longrightarrow D$ a compact and continuous map. Then T has at least one fixed point in D.*

Theorem 2.5 (Mönch's fixed point theorem [18]). *Let D be a non-empty, closed, bounded and convex subset of a Banach space X such that $0\in D$ and let $\mathcal{H}: D\longrightarrow D$ be a continuous mapping. If the implication*

$$\Omega=\overline{conv}\mathcal{H}(\Omega)\text{ or }\Omega=\mathcal{H}(\Omega)\cup\{0\}\Rightarrow\zeta(\Omega)=0,$$

holds for every subset Ω of D, then $\mathcal{H}$ has at least one fixed point.

3. Existence Results

Consider the following fractional differential problem:

$$ {}_0^CD_{\varkappa}^{\kappa_1,\kappa_2;\gamma}\varphi(\theta)=\mu(\theta),\quad \text{for each } \theta\in(\theta_\jmath,\theta_{\jmath+1}], \jmath=0,\ldots,\beta, \tag{4}$$

$$\Delta\varphi|_{\theta_\jmath}=\tau_\jmath,\qquad \jmath=1,\ldots,\beta, \tag{5}$$

$$\varphi(0)=\varphi_0, \tag{6}$$

where $\mu:\Theta\to\mathbb{R}$ is a continuous function and $\tau_\jmath$ are real constants.

Lemma 3.1. *Let $\kappa_1,\kappa_2\in(0,1],\gamma\in[0,1]$, and $\mu:\Theta\to\mathbb{R}$ be continuous. Then, the problem (4)-(6) has a unique solution given by:*

$$\varphi(\theta)=\begin{cases}\varphi_0-\frac{1}{\Gamma(\kappa_2)}\int_0^{\theta_1}s^{\kappa_2-1}\mu(s)ds+\frac{1}{\Gamma(\kappa_1)}\int_0^{\theta}(\theta-s)^{\kappa_1-1}\mu(s)ds\\ +\frac{1}{\Gamma(\kappa_2)}\int_\theta^{\theta_1}(s-\theta)^{\kappa_2-1}\mu(s)ds, \qquad \text{if } \theta\in[0,\theta_1],\\ \varphi_0-\frac{1}{\Gamma(\kappa_2)}\sum_{l=1}^{\jmath+1}\int_{\theta_{l-1}}^{\theta_l}(s-\theta_{l-1})^{\kappa_2-1}\mu(s)ds\\ +\frac{1}{\Gamma(\kappa_1)}\sum_{l=1}^{\jmath}\int_{\theta_{l-1}}^{\theta_l}(\theta_l-s)^{\kappa_1-1}\mu(s)ds\\ +\frac{1}{\Gamma(\kappa_1)}\int_{\theta_\jmath}^{\theta}(\theta-s)^{\kappa_1-1}\mu(s)ds+\frac{1}{\Gamma(\kappa_2)}\int_\theta^{\theta_{\jmath+1}}(s-\theta)^{\kappa_2-1}\mu(s)ds\\ +\sum_{l=1}^{\jmath}\tau_l, \qquad \text{if } \theta\in(\theta_\jmath,\theta_{\jmath+1}].\end{cases} \tag{7}$$

Proof. Suppose that φ satisfies (4)-(6). Then, from Lemma 2.1, if $\theta \in [0, \theta_1]$, then

$$\begin{aligned}\varphi(\theta) &= \gamma\varphi(0) + (1-\gamma)\varphi(\theta_1^-) + {}_0I_{\theta_1}^{\kappa_1,\kappa_2}\mu(\theta) \\ &= \gamma\varphi_0 + (1-\gamma)\varphi(\theta_1^-) + \frac{1}{\Gamma(\kappa_1)}\int_0^\theta (\theta-s)^{\kappa_1-1}\mu(s)ds \\ &\quad + \frac{1}{\Gamma(\kappa_2)}\int_\theta^{\theta_1}(s-\theta)^{\kappa_2-1}\mu(s)ds.\end{aligned}$$

For $\theta = 0$, we have

$$(1-\gamma)\varphi(\theta_1^-) = (1-\gamma)\varphi_0 - \frac{1}{\Gamma(\kappa_2)}\int_0^{\theta_1} s^{\kappa_2-1}\mu(s)ds.$$

Thus,

$$\begin{aligned}\varphi(\theta) = \varphi_0 &- \frac{1}{\Gamma(\kappa_2)}\int_0^{\theta_1} s^{\kappa_2-1}\mu(s)ds + \frac{1}{\Gamma(\kappa_1)}\int_0^\theta(\theta-s)^{\kappa_1-1}\mu(s)ds \\ &+ \frac{1}{\Gamma(\kappa_2)}\int_\theta^{\theta_1}(s-\theta)^{\kappa_2-1}\mu(s)ds.\end{aligned}$$

If $\theta \in (\theta_1, \theta_2]$, then we have

$$\begin{aligned}\varphi(\theta) = \gamma\varphi(\theta_1^+) &+ (1-\gamma)\varphi(\theta_2^-) + \frac{1}{\Gamma(\kappa_1)}\int_{\theta_1}^\theta(\theta-s)^{\kappa_1-1}\mu(s)ds \\ &+ \frac{1}{\Gamma(\kappa_2)}\int_\theta^{\theta_2}(s-\theta)^{\kappa_2-1}\mu(s)ds.\end{aligned}$$

For $\theta = \theta_1$, we have

$$(1-\gamma)\varphi(\theta_2^-) = (1-\gamma)\varphi(\theta_1^+) - \frac{1}{\Gamma(\kappa_2)}\int_{\theta_1}^{\theta_2}(s-\theta_1)^{\kappa_2-1}\mu(s)ds,$$

and

$$\varphi(\theta_1^+) = \varphi(\theta_1^-) + \tau_1.$$

Then,

$$\begin{aligned}\varphi(\theta) = \varphi_0 &- \frac{1}{\Gamma(\kappa_2)}\int_0^{\theta_1} s^{\kappa_2-1}\mu(s)ds + \frac{1}{\Gamma(\kappa_1)}\int_0^{\theta_1}(\theta_1-s)^{\kappa_1-1}\mu(s)ds \\ &- \frac{1}{\Gamma(\kappa_2)}\int_{\theta_1}^{\theta_2}(s-\theta_1)^{\kappa_2-1}\mu(s)ds + \frac{1}{\Gamma(\kappa_1)}\int_{\theta_1}^\theta(\theta-s)^{\kappa_1-1}\mu(s)ds \\ &+ \frac{1}{\Gamma(\kappa_2)}\int_\theta^{\theta_2}(s-\theta)^{\kappa_2-1}\mu(s)ds + \tau_1.\end{aligned}$$

If $\theta \in (\theta_2, \theta_3]$, then we have

$$\begin{aligned}\varphi(\theta) = \gamma\varphi(\theta_2^+) &+ (1-\gamma)\varphi(\theta_3^-) + \frac{1}{\Gamma(\kappa_1)}\int_{\theta_2}^\theta(\theta-s)^{\kappa_1-1}\mu(s)ds \\ &+ \frac{1}{\Gamma(\kappa_2)}\int_\theta^{\theta_3}(s-\theta)^{\kappa_2-1}\mu(s)ds.\end{aligned}$$

For $\theta=\theta_2$, we have

$$(1-\gamma)\varphi(\theta_3^-)=(1-\gamma)\varphi(\theta_2^+)-\frac{1}{\Gamma(\kappa_2)}\int_{\theta_2}^{\theta_3}(s-\theta_2)^{\kappa_2-1}\mu(s)ds.$$

Thus,

$$\begin{aligned}\varphi(\theta)=\varphi_0&-\frac{1}{\Gamma(\kappa_2)}\int_0^{\theta_1}s^{\kappa_2-1}\mu(s)ds+\frac{1}{\Gamma(\kappa_1)}\int_0^{\theta_1}(\theta_1-s)^{\kappa_1-1}\mu(s)ds\\&-\frac{1}{\Gamma(\kappa_2)}\int_{\theta_1}^{\theta_2}(s-\theta_1)^{\kappa_2-1}\mu(s)ds+\frac{1}{\Gamma(\kappa_1)}\int_{\theta_1}^{\theta_2}(\theta_2-s)^{\kappa_1-1}\mu(s)ds\\&-\frac{1}{\Gamma(\kappa_2)}\int_{\theta_2}^{\theta_3}(s-\theta_2)^{\kappa_2-1}\mu(s)ds+\frac{1}{\Gamma(\kappa_1)}\int_{\theta_2}^{\theta}(\theta-s)^{\kappa_1-1}\mu(s)ds\\&+\frac{1}{\Gamma(\kappa_2)}\int_{\theta}^{\theta_3}(s-\theta)^{\kappa_2-1}\mu(s)ds+\tau_1+\tau_2.\end{aligned}$$

By repeating the same procedure, for $\theta\in(\theta_\jmath,\theta_{\jmath+1}]$, we get

$$\begin{aligned}\varphi(\theta)=\varphi_0-\frac{1}{\Gamma(\kappa_2)}\sum_{l=1}^{\jmath+1}\int_{\theta_{l-1}}^{\theta_l}(s-\theta_{l-1})^{\kappa_2-1}\mu(s)ds+\frac{1}{\Gamma(\kappa_1)}\sum_{l=1}^{\jmath}\int_{\theta_{l-1}}^{\theta_l}(\theta_l-s)^{\kappa_1-1}\mu(s)ds\\+\frac{1}{\Gamma(\kappa_1)}\int_{\theta_\jmath}^{\theta}(\theta-s)^{\kappa_1-1}\mu(s)ds+\frac{1}{\Gamma(\kappa_2)}\int_{\theta}^{\theta_{\jmath+1}}(s-\theta)^{\kappa_2-1}\mu(s)ds+\sum_{l=1}^{\jmath}\tau_l.\end{aligned}$$

Conversely, assume that φ satisfies (7). Since the Caputo derivative of constant is zero, if $\theta\in[0,\theta_1]$, we obtain ${}_0^CD_{\varkappa}^{\kappa_1,\kappa_2;\gamma}\varphi(\theta)=\mu(\theta)$, and if $\theta\in(\theta_\jmath,\theta_{\jmath+1}]$, $\jmath=0,\dots\beta$, we get ${}_0^CD_{\varkappa}^{\kappa_1,\kappa_2;\gamma}\varphi(\theta)=\mu(\theta)$. Also we can easily prove that $\Delta\varphi|_{\theta_\jmath}=\tau_\jmath$. □

Definition 3.1. By a solution of the problem (1)-(3) we mean a function $\varphi\in PC(\Theta,\mathbb{R})$ that satisfies the equation (1) and the conditions (2)-(3).

Lemma 3.2. *Let $f:\Theta\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Then the problem (1)-(3) is equivalent to the following integral equation:*

$$\varphi(\theta)=\begin{cases}\varphi_0-\dfrac{1}{\Gamma(\kappa_2)}\displaystyle\int_0^{\theta_1}s^{\kappa_2-1}f(s,\varphi(s),g(s))ds\\+\dfrac{1}{\Gamma(\kappa_1)}\displaystyle\int_0^{\theta}(\theta-s)^{\kappa_1-1}f(s,\varphi(s),g(s))ds\\+\dfrac{1}{\Gamma(\kappa_2)}\displaystyle\int_\theta^{\theta_1}(s-\theta)^{\kappa_2-1}f(s,\varphi(s),g(s))ds, \qquad \theta\in[0,\theta_1],\\ \varphi_0-\dfrac{1}{\Gamma(\kappa_2)}\displaystyle\sum_{l=1}^{\jmath+1}\int_{\theta_{l-1}}^{\theta_l}(s-\theta_{l-1})^{\kappa_2-1}f(s,\varphi(s),g(s))ds\\+\dfrac{1}{\Gamma(\kappa_1)}\displaystyle\sum_{l=1}^{\jmath}\int_{\theta_{l-1}}^{\theta_l}(\theta_l-s)^{\kappa_1-1}f(s,\varphi(s),g(s))ds\\+\dfrac{1}{\Gamma(\kappa_1)}\displaystyle\int_{\theta_\jmath}^{\theta}(\theta-s)^{\kappa_1-1}f(s,\varphi(s),g(s))ds\\+\dfrac{1}{\Gamma(\kappa_2)}\displaystyle\int_\theta^{\theta_{\jmath+1}}(s-\theta)^{\kappa_2-1}f(s,\varphi(s),g(s))ds+\sum_{l=1}^{\jmath}I_l(\varphi(\theta_l^-)),\ \theta\in(\theta_\jmath,\theta_{\jmath+1}],\end{cases}$$

where $g \in C(\Theta, \mathbb{R})$ *satisfies the following functional equation*

$$g(\theta) = f(\theta, \varphi(\theta), g(\theta)).$$

We are now in a position to prove the existence result of the problem (1)-(3) based on the Banach's contraction principle.

Let us put the following conditions:

($B1$) The function $f : \Theta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

($B2$) There exist constants $\lambda > 0$ and $0 < L < 1$ such that

$$|f(\theta, \xi, \delta) - f(\theta, \bar{\xi}, \bar{\delta})| \le \lambda |\xi - \bar{\xi}| + L|\delta - \bar{\delta}|,$$

for any $\xi, \bar{\xi}, \delta, \bar{\delta} \in \mathbb{R}$ and $\theta \in \Theta$.

($B3$) There exists constant $C > 0$ such that

$$|I_\jmath(\delta) - I_\jmath(\bar{\delta})| \le C|\delta - \bar{\delta}|,$$

for any $\delta, \bar{\delta} \in \mathbb{R}$ and $\jmath = 1, \dots, \beta$.

Theorem 3.3. *Assume that the assumptions* $(B1)$-$(B3)$ *hold. If*

$$\frac{\lambda \varkappa^{\kappa_2}(\beta + 2)}{(1 - L)\Gamma(\kappa_2 + 1)} + \frac{\lambda \varkappa^{\kappa_1}(\beta + 1)}{(1 - L)\Gamma(\kappa_1 + 1)} + C\beta < 1,$$

then the problem (1)-(3) *has a unique solution on* Θ.

Proof. Consider the operator $\aleph : PC(\Theta, \mathbb{R}) \longrightarrow PC(\Theta, \mathbb{R})$ defined by:

$$\begin{aligned}
\aleph\varphi(\theta) = \varphi_0 &- \frac{1}{\Gamma(\kappa_2)} \sum_{0<\theta_{\jmath+1}<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s - \theta_{\jmath-1})^{\kappa_2 - 1} g(s) ds \\
&+ \frac{1}{\Gamma(\kappa_1)} \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1 - 1} g(s) ds + \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1 - 1} g(s) ds \\
&+ \frac{1}{\Gamma(\kappa_2)} \int_{\theta}^{\theta_{\jmath+1}} (s - \theta)^{\kappa_2 - 1} g(s) ds + \sum_{0<\theta_\jmath<\theta} I_\jmath(\varphi(\theta_\jmath^-)).
\end{aligned}$$

Clearly the fixed points of the operator $\aleph$ are solutions of the problem (1)-(3). Let $\varphi, z \in PC(\Theta, \mathbb{R})$ and $\theta \in \Theta$. Then

$$\begin{aligned}
|\aleph\varphi(\theta) - \aleph z(\theta)| \le\ & \frac{1}{\Gamma(\kappa_2)} \sum_{0<\theta_{\jmath+1}<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s - \theta_{\jmath-1})^{\kappa_2 - 1} |g(s) - h(s)| ds \\
&+ \frac{1}{\Gamma(\kappa_1)} \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1 - 1} |g(s) - h(s)| ds \\
&+ \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1 - 1} |g(s) - h(s)| ds \\
&+ \frac{1}{\Gamma(\kappa_2)} \int_{\theta}^{\theta_{\jmath+1}} (s - \theta)^{\kappa_2 - 1} |g(s) - h(s)| ds \\
&+ \sum_{0<\theta_\jmath<\theta} |I_\jmath(\varphi(\theta_\jmath^-)) - I_\jmath(z(\theta_\jmath^-))|,
\end{aligned}$$

where g and h are two functions verifying the functional equations:

$$g(\theta) = f(\theta, \varphi(\theta), g(\theta)),$$
$$h(\theta) = f(\theta, z(\theta), h(\theta)).$$

Then, by $(B2)$, we have

$$\begin{aligned} |g(\theta) - h(\theta)| &= |f(\theta, \varphi(\theta), g(\theta)) - f(\theta, z(\theta), h(\theta))| \\ &\leq \lambda|\varphi(\theta) - z(\theta)| + L|g(\theta) - h(\theta)|, \end{aligned}$$

which implies

$$|g(\theta) - h(\theta)| \leq \frac{\lambda}{1-L}|\varphi(\theta) - z(\theta)|.$$

Thus,

$$\begin{aligned} |\aleph\varphi(\theta) - \aleph z(\theta)| \leq & \frac{\lambda}{(1-L)\Gamma(\kappa_2)} \sum_{0<\theta_{\jmath+1}<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s-\theta_{\jmath-1})^{\kappa_2-1}|\varphi(\theta) - z(\theta)|\, ds \\ & + \frac{\lambda}{(1-L)\Gamma(\kappa_1)} \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1-1}|\varphi(\theta) - z(\theta)|\, ds \\ & + \frac{\lambda}{(1-L)\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1-1}|\varphi(\theta) - z(\theta)|\, ds \\ & + \frac{\lambda}{(1-L)\Gamma(\kappa_2)} \int_{\theta}^{\theta_{\jmath+1}} (s-\theta)^{\kappa_2-1}|\varphi(\theta) - z(\theta)|\, ds \\ & + \sum_{\jmath=1}^{\beta} C|\varphi(\theta) - z(\theta)| \\ \leq & \left[\frac{\lambda\varkappa^{\kappa_2}(\beta+2)}{(1-L)\Gamma(\kappa_2+1)} + \frac{\lambda\varkappa^{\kappa_1}(\beta+1)}{(1-L)\Gamma(\kappa_1+1)} + C\beta\right]\|\varphi - z\|_{PC}. \end{aligned}$$

Consequently, by the Banach's contraction principle, the operator $\aleph$ has a unique fixed point which is solution of the fractional problem (1)-(3). □

Remark 3.1. Let us put

$$q_1(\theta) = |f(\theta,0,0)|,\ \lambda = q_2^*,\ L = q_3^*,\ C = p_1^*,\ p_2^* = \max_{\jmath=1,\dots,\beta}|I_\jmath(0)|.$$

Then, the hypothesis $(B2)$ implies that

$$|f(\theta,\xi,\delta)| \leq q_1(\theta) + q_2^*|\xi| + q_3^*|\delta|,$$

for $\theta \in \Theta$, $\xi \in \mathbb{R}$, $\delta \in \mathbb{R}$ and $q_1 \in C(\Theta, \mathbb{R}_+)$, such that

$$q_1^* = \sup_{\theta\in\Theta} q_1(\theta).$$

And, from hypothesis $(B3)$, we have

$$|I_\jmath(\delta)| \leq p_1^*|\xi| + p_2^*,$$

for each $\delta \in \mathbb{R}$, $\jmath = 1, \dots, \beta$.

Theorem 3.4. *Assume that the hypotheses* $(B1)$-$(B3)$ *hold. If*

$$\frac{q_2^* \varkappa^{\kappa_2}(\beta+2)}{(1-q_3^*)\Gamma(\kappa_2+1)} + \frac{q_2^* \varkappa^{\kappa_1}(\beta+1)}{(1-q_3^*)\Gamma(\kappa_1+1)} + \beta p_1^* < 1,$$

then the impulsive implicit fractional problem (1)-(3) *has at least one solution.*

Proof. In this proof, we will use Schauder's fixed point theorem. The proof will be given in several steps.

Step 1: The operator $\aleph : PC(\Theta, \mathbb{R}) \longrightarrow PC(\Theta, \mathbb{R})$ is continuous.
Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence such that $\varphi_n \longrightarrow \varphi$ in $PC(\Theta, \mathbb{R})$. Then, for each $\theta \in \Theta$, we have

$$\begin{aligned} |\aleph\varphi_n(\theta) - \aleph\varphi(\theta)| \leq & \frac{1}{\Gamma(\kappa_2)} \sum_{0<\theta_{\jmath+1}<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s-\theta_{\jmath-1})^{\kappa_2-1} |g_n(s) - g(s)| ds \\ & + \frac{1}{\Gamma(\kappa_1)} \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1-1} |g_n(s) - g(s)| ds \\ & + \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1-1} |g_n(s) - g(s)| ds \\ & + \frac{1}{\Gamma(\kappa_2)} \int_{\theta}^{\theta_{\jmath+1}} (s-\theta)^{\kappa_2-1} |g_n(s) - g(s)| ds \\ & + \sum_{0<\theta_\jmath<\theta} |I_\jmath(\varphi_n(\theta_\jmath^-)) - I_\jmath(\varphi(\theta_\jmath^-))|. \end{aligned}$$

By $(B2)$, we have

$$\begin{aligned} |g_n(\theta) - g(\theta)| &= |f(\theta, \varphi(\theta), g_n(\theta)) - f(\theta, \varphi(\theta), g(\theta))| \\ &\leq \lambda |\varphi_n(\theta) - \varphi(\theta)| + L|g_n(\theta) - g(\theta)|. \end{aligned}$$

Then,

$$|g_n(\theta) - g(\theta)| \leq \frac{\lambda}{1-L} |\varphi_n(\theta) - \varphi(\theta)|.$$

Thus,

$$\begin{aligned} |\aleph\varphi_n(\theta) - \aleph\varphi(\theta)| \leq & \frac{\lambda}{(1-L)\Gamma(\kappa_2)} \sum_{0<\theta_{\jmath+1}<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s-\theta_{\jmath-1})^{\kappa_2-1} |\varphi_n(s) - \varphi(s)| ds \\ & + \frac{\lambda}{(1-L)\Gamma(\kappa_1)} \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1-1} |\varphi_n(s) - \varphi(s)| ds \\ & + \frac{\lambda}{(1-L)\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1-1} |\varphi_n(s) - \varphi(s)| ds \\ & + \frac{\lambda}{(1-L)\Gamma(\kappa_2)} \int_{\theta}^{\theta_{\jmath+1}} (s-\theta)^{\kappa_2-1} |\varphi_n(s) - \varphi(s)| ds \\ & + \sum_{\jmath=1}^{\beta} C|\varphi_n(\theta) - z(\theta)|. \end{aligned}$$

By applying the Lebesgue dominated convergence theorem, we get

$$|\aleph\varphi_n(\theta) - \aleph\varphi(\theta)| \longrightarrow 0 \quad as\ n \longrightarrow \infty.$$

Hence,

$$\|\aleph\varphi_n - \aleph\varphi\|_{PC} \longrightarrow 0 \quad as\ n \longrightarrow \infty,$$

which implies that $\aleph$ is continuous.

Let $R > 0$ such that

$$R \geq \frac{|\varphi_0| + \frac{q_1^* \varkappa^{\kappa_2}(\beta+2)}{(1-q_3^*)\Gamma(\kappa_2+1)} + \frac{q_1^* \varkappa^{\kappa_1}(\beta+1)}{(1-q_3^*)\Gamma(\kappa_1+1)} + \beta p_2^*}{1 - \frac{q_2^* \varkappa^{\kappa_2}(\beta+2)}{(1-q_3^*)\Gamma(\kappa_2+1)} - \frac{q_2^* \varkappa^{\kappa_1}(\beta+1)}{(1-q_3^*)\Gamma(\kappa_1+1)} - \beta p_1^*}.$$

Define the ball

$$D_R = \{\varphi \in PC(\Theta, \mathbb{R}) : \|\varphi\|_{PC} \leq R\}.$$

Step 2: $\aleph(D_R) \subset D_R$.
Let $\varphi \in D_R$ and $\theta \in \Theta$. Then,

$$\begin{aligned}
|\aleph\varphi(\theta)| \leq{}& |\varphi_0| + \frac{1}{\Gamma(\kappa_2)} \sum_{0<\theta_{\jmath+1}<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s-\theta_{\jmath-1})^{\kappa_2-1}|g(s)|ds \\
&+ \frac{1}{\Gamma(\kappa_1)} \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1-1}|g(s)|ds + \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1-1}|g(s)|ds \\
&+ \frac{1}{\Gamma(\kappa_2)} \int_{\theta}^{\theta_{\jmath+1}} (s-\theta)^{\kappa_2-1}|g(s)|ds + \sum_{0<\theta_\jmath<\theta} |I_\jmath(\varphi(\theta_\jmath^-))|.
\end{aligned}$$

From hypothesis $(B2)$, we have

$$\begin{aligned}
|g(\theta)| &= |f(\theta, \varphi(\theta), g(\theta))| \\
&\leq q_1(\theta) + q_2^*\|\varphi\|_{PC} + q_3^*|g(\theta)| \\
&\leq q_1^* + q_2^* R + q_3^*|g(\theta)|.
\end{aligned}$$

Then,

$$|g(\theta)| \leq \frac{q_1^* + q_2^* R}{1 - q_3^*}.$$

Thus,

$$\begin{aligned}
|\aleph\varphi(\theta)| \leq{}& |\varphi_0| + \frac{\varkappa^{\kappa_2}(q_1^* + q_2^* R)(\beta+2)}{(1-q_3^*)\Gamma(\kappa_2+1)} + \frac{\varkappa^{\kappa_1}(q_1^* + q_2^* R)(\beta+1)}{(1-q_3^*)\Gamma(\kappa_1+1)} + \beta(p_1^* R + p_2^*) \\
\leq{}& R.
\end{aligned}$$

Hence, $\aleph(D_R) \subset D_R$.

Step 3: $\aleph(D_R)$ is equicontinuous.
Let $\theta_1, \theta_2 \in \Theta$, where $\theta_1 < \theta_2$ and $\varphi \in D_R$. Then,

$$|\aleph\varphi(\theta_2) - \aleph\varphi(\theta_1)| = \Bigg| \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta_2} (\theta_2 - s)^{\kappa_1-1} g(s)ds + \frac{1}{\Gamma(\kappa_2)} \int_{\theta_2}^{\theta_{\jmath+1}} (s-\theta_2)^{\kappa_2-1} g(s)ds$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta_1} (\theta_1 - s)^{\kappa_1 - 1} g(s) ds - \frac{1}{\Gamma(\kappa_2)} \int_{\theta_1}^{\theta_{\jmath+1}} (s - \theta_1)^{\kappa_2 - 1} g(s) ds \\
&+ \sum_{\theta_1 < \theta_\jmath < \theta_2} I_\jmath(\varphi(\theta_\jmath^-)) \Bigg| \\
\leq & \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta_1} [(\theta_2 - s)^{\kappa_1 - 1} - (\theta_1 - s)^{\kappa_1 - 1}] |g(s)| ds \\
&+ \frac{1}{\Gamma(\kappa_1)} \int_{\theta_1}^{\theta_2} (\theta_2 - s)^{\kappa_1 - 1} |g(s)| ds + \frac{1}{\Gamma(\kappa_2)} \int_{\theta_1}^{\theta_2} (s - \theta_1)^{\kappa_2 - 1} |g(s)| ds \\
&+ \frac{1}{\Gamma(\kappa_2)} \int_{\theta_2}^{\theta_{\jmath+1}} [(s - \theta_2)^{\kappa_2 - 1} - (s - \theta_1)^{\kappa_2 - 1}] |g(s)| ds \\
&+ \sum_{\theta_1 < \theta_\jmath < \theta_2} |I_\jmath(\varphi(\theta_\jmath^-))| \\
\leq & \frac{(q_1^* + q_2^* R)}{(1 - q_3^*)\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta_1} [(\theta_2 - s)^{\kappa_1 - 1} - (\theta_1 - s)^{\kappa_1 - 1}] ds \\
&+ \frac{(q_1^* + q_2^* R)(\theta_2 - \theta_1)^{\kappa_1}}{(1 - q_3^*)\Gamma(\kappa_1 + 1)} + \frac{(q_1^* + q_2^* R)(\theta_2 - \theta_1)^{\kappa_2}}{(1 - q_3^*)\Gamma(\kappa_2 + 1)} \\
&+ \frac{(q_1^* + q_2^* R)}{(1 - q_3^*)\Gamma(\kappa_2)} \int_{\theta_2}^{\theta_{\jmath+1}} [(s - \theta_2)^{\kappa_2 - 1} - (s - \theta_1)^{\kappa_2 - 1}] ds \\
&+ \sum_{\theta_1 < \theta_\jmath < \theta_2} p_1^* R + p_2^*.
\end{aligned}
$$

As $\theta_1 \longrightarrow \theta_2$, the right-hand side of the preceding inequality tend to zero, then $\aleph(D_R)$ is equicontinuous. According to the three steps and Arzela-Ascoli theorem, we deduce that the operator $\aleph$ has at least a fixed point which is solution of the problem (1)-(3). □

4. An Example

Consider the following impulsive problem:

$$
{}_0^C D_1^{\frac{1}{2},\frac{1}{3};\frac{1}{4}} \varphi(\theta) = \frac{7 + |\varphi(\theta)| + \left| {}_0^C D_1^{\frac{1}{2},\frac{1}{3};\frac{1}{4}} \varphi(\theta) \right|}{5\sqrt{\pi} e^{\theta + \frac{1}{3}} \left(1 + |\varphi(\theta)| + \left| {}_0^C D_1^{\frac{1}{2},\frac{1}{3};\frac{1}{4}} \varphi(\theta) \right| \right)}, \quad \theta \in \Theta_0 \cup \Theta_1, \tag{8}
$$

$$
\Delta\varphi|_{\theta=\frac{1}{2}} = \frac{1}{9} \ln \left(2 + e + \varphi\left(\frac{1}{2}^- \right) \right), \tag{9}
$$

$$
\varphi(0) = 1, \tag{10}
$$

where $\Theta_0 = [0, \frac{1}{2}], \Theta_1 = (\frac{1}{2}, 1]$.
Set

$$
f(\theta, \xi, \delta) = \frac{7 + |\xi| + |\delta|}{5\sqrt{\pi} e^{\theta + \frac{1}{3}} (1 + |\xi| + |\delta|)}, \quad \theta \in [0, 1], \ \xi \in \mathbb{R} \text{ and } \delta \in \mathbb{R}.
$$

Obviously f is a continuous function, the the hypothesis $(B1)$ is satisfied. And, for each $\xi, \bar{\xi}, \delta, \bar{\delta} \in \mathbb{R}$ and $\theta \in [0,1]$, we have

$$|f(\theta,\xi,\delta) - f(\theta,\bar{\xi},\bar{\delta})| \leq \frac{1}{5\sqrt{\pi}e^{\frac{1}{3}}}[|\xi - \bar{\xi}| + |\delta - \bar{\delta}|].$$

Then, the hypothesis $(B2)$ is satisfied with $\lambda = \gamma = \frac{1}{5\sqrt{\pi}e^{\frac{1}{3}}}$. Let

$$I_1(\delta) = \frac{1}{9}\ln(2 + e + \delta), \qquad \delta \in \mathbb{R}^+.$$

Then, for $\delta, \bar{\delta} \in \mathbb{R}^+$, we have

$$\begin{aligned} |I_1(\delta) - I_1(\bar{\delta})| &= \left|\frac{1}{9}\ln(2+e+\delta) - \frac{1}{10}\ln(2+e+\bar{\delta})\right| \\ &\leq \frac{1}{9}|\delta - \bar{\delta}|. \end{aligned}$$

Thus, $(B3)$ is met with $C = \frac{1}{9}$. Also, we have

$$\begin{aligned} \frac{\lambda\varkappa^{\kappa_2}(\beta+2)}{(1-L)\Gamma(\kappa_2+1)} + \frac{\lambda\varkappa^{\kappa_1}(\beta+1)}{(1-L)\Gamma(\kappa_1+1)} + C\beta &= \frac{3}{(5\sqrt{\pi}e^{\frac{1}{3}} - 1)\Gamma(\frac{4}{3})} \\ &\quad + \frac{2}{(5\sqrt{\pi}e^{\frac{1}{3}} - 1)\Gamma(\frac{3}{2})} + \frac{1}{9} \\ &\leq 1, \end{aligned}$$

for $\varkappa = 1$, $\beta = 1$ and $C = \frac{1}{9}$. It follows from Theorem 3.3 that the problem (8)-(10) has a unique solution on $[0,1]$.

5. Impulsive Implicite Problem in Banach Spaces

This section is devoted to the study of existence and stability of a problem similar to problem (1)-(3) in a Banach space. Consider the following problem:

$${}_0^C D_{\varkappa}^{\kappa_1,\kappa_2;\gamma}\varphi(\theta) = f(\theta, \varphi(\theta), {}_0^C D_{\varkappa}^{\kappa_1,\kappa_2;\gamma}\varphi(\theta)), \quad \textit{for each } \theta \in (\theta_\jmath, \theta_{\jmath+1}], \jmath = 0,\dots,\beta, \tag{11}$$

$$\Delta\varphi|_{\theta_\jmath} = I_\jmath(\varphi(\theta_\jmath^-)), \qquad \jmath = 1,\dots,\beta, \tag{12}$$

$$\varphi(0) = \varphi_0, \tag{13}$$

where ${}_0^C D_{\varkappa}^{\kappa_1,\kappa_2;\gamma}$ is the convex combined Caputo fractional derivative of order $(\kappa_1,\kappa_2) \in (0,1]$, $\gamma \in [0,1]$, $f : \Theta \times \Xi \times \Xi \to \Xi$ is a given function, where $\Theta = \Theta$. $I_\jmath : \Xi \to \Xi$, and $\varphi_0 \in \Xi$, $0 = \theta_0 < \theta_1 < \dots < \theta_\beta < \theta_{\beta+1} = \varkappa$. Let $\Delta\varphi|_{\theta_\jmath} = \varphi(\theta_\jmath^+) - \varphi(\theta_\jmath^-)$, with $\varphi(\theta_\jmath^+) = \lim_{h\to 0^+} \varphi(\theta_\jmath + h)$ and $\varphi(\theta_\jmath^-) = \lim_{h\to 0^-} \varphi(\theta_\jmath + h)$ represent the right and left limits of φ_θ at $\theta = \theta_\jmath$ respectively.

Definition 5.1. By a solution of the problem (11)-(13) we mean a function $\varphi \in PC_1(\Theta, \Xi)$ that satisfies the equation (11) and the conditions (12)-(13).

Lemma 5.1. *Let $f : \Theta \times \Xi \times \Xi \to \Xi$ be a continuous function. Then the problem (11)-(13) is equivalent to the following integral equation:*

$$\varphi(\theta) = \begin{cases} \varphi_0 - \frac{1}{\Gamma(\kappa_2)} \int_0^{\theta_1} s^{\kappa_2 - 1} f(s, \varphi(s), g(s)) ds \\ + \frac{1}{\Gamma(\kappa_1)} \int_0^{\theta} (\theta - s)^{\kappa_1 - 1} f(s, \varphi(s), g(s)) ds \\ + \frac{1}{\Gamma(\kappa_2)} \int_\theta^{\theta_1} (s - \theta)^{\kappa_2 - 1} f(s, \varphi(s), g(s)) ds, \qquad \theta \in [0, \theta_1], \\ \varphi_0 - \frac{1}{\Gamma(\kappa_2)} \sum_{l=1}^{\jmath+1} \int_{\theta_{l-1}}^{\theta_l} (s - \theta_{l-1})^{\kappa_2 - 1} f(s, \varphi(s), g(s)) ds \\ + \frac{1}{\Gamma(\kappa_1)} \sum_{l=1}^{\jmath} \int_{\theta_{l-1}}^{\theta_l} (\theta_l - s)^{\kappa_1 - 1} f(s, \varphi(s), g(s)) ds \\ + \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1 - 1} f(s, \varphi(s), g(s)) ds \\ + \frac{1}{\Gamma(\kappa_2)} \int_\theta^{\theta_{\jmath+1}} (s - \theta)^{\kappa_2 - 1} f(s, \varphi(s), g(s)) ds + \sum_{l=1}^{\jmath} I_l(\varphi(\theta_l^-)), \ \theta \in (\theta_\jmath, \theta_{\jmath+1}]. \end{cases}$$

Let us set the following assumptions:

($B4$) The function $f : \Theta \times \Xi \times \Xi \to \Xi$ is continuous.

($B5$) There exist constants $\lambda > 0$ and $0 < L < 1$ such that

$$\|f(\theta, \xi, \delta) - f(\theta, \bar{\xi}, \bar{\delta})\| \leq \lambda \|\xi - \bar{\xi}\| + L \|\delta - \bar{\delta}\|,$$

for any $\xi, \bar{\xi} \in \Xi$, $\delta, \bar{\delta} \in \Xi$ and $\theta \in \Theta$.

($B6$) There exists constant $C > 0$ such that

$$\|I_\jmath(\delta) - I_\jmath(\bar{\delta})\| \leq C \|\delta - \bar{\delta}\|,$$

for any $\delta, \bar{\delta} \in \Xi$ and $\jmath = 1, \dots \beta$.

($B7$) For each $\theta \in \Theta$ and bounded sets $B_1, B_2 \subseteq \Xi$, we have

$$\zeta(f(\theta, B_1, B_2)) \leq \lambda \zeta(B_1) + L \zeta(B_2).$$

($B8$) For each $\theta \in \Theta$ and bounded set $B_2 \subseteq \Xi$, $\jmath = 1, \dots, \beta$, we have

$$\zeta(I_\jmath(B_2)) \leq C \zeta(B_2).$$

Remark 5.1 ([5])**.** It is worth noting that the hypotheses ($B5$) and ($B7$) are equivalent as well as the hypothesis ($B6$) and ($B8$).

Remark 5.2. Let us put

$$q_1(\theta) = \|f(\theta, 0, 0)\|, \ \lambda = q_2^*, \ L = q_3^*, \ C = p_1^*, \ p_2^* = \max_{\jmath = 1, \dots, \beta} \|I_\jmath(0)\|.$$

Then the assumption ($B5$) implies that

$$\|f(\theta, \xi, \delta)\| \leq q_1(\theta) + q_2^* \|\xi\| + q_3^* \|\delta\|,$$

for $\theta \in \Theta$, $\xi \in \Xi$, $\delta \in \Xi$ and $q_1 \in C(\Theta, \mathbb{R}_+)$, such that

$$q_1^* = \sup_{\theta \in \Theta} q_1(\theta).$$

And from hypothesis $(B6)$, we have

$$\|I_{\jmath}(\delta)\| \leq p_1^*\|\xi\| + p_2^*,$$

for each $\delta \in \Xi$, $\jmath = 1, \dots, \beta$.

Theorem 5.2. *Assume* $(B4)$-$(B6)$ *are verified. If*

$$\frac{q_2^* \varkappa^{\kappa_2}(\beta+2)}{(1-q_3^*)\Gamma(\kappa_2+1)} + \frac{q_2^* \varkappa^{\kappa_1}(\beta+1)}{(1-q_3^*)\Gamma(\kappa_1+1)} + \beta p_1^* < 1,$$

then the problem (11)-(13) *has at least one solution.*

To prove the existence of solution of the problem (11)-(13), we will use the concept of measure of noncompactness and Mönch's fixed point theorem.

Proof. Transform problem (11)-(13) into a fixed point problem.
The proof will be given in several steps.

Step 1: The operator $\aleph : PC_1(\Theta, \Xi) \longrightarrow PC_1(\Theta, \Xi)$ is continuous.
Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence such that $\varphi_n \longrightarrow \varphi$ in $PC_1(\Theta, \Xi)$. Then for each $\theta \in \Theta$, we have

$$\begin{aligned}
\|\aleph\varphi_n(\theta) - \aleph\varphi(\theta)\| \leq & \frac{1}{\Gamma(\kappa_2)} \sum_{0<\theta_{\jmath+1}<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s-\theta_{\jmath-1})^{\kappa_2-1}\|g_n(s)-g(s)\|ds \\
& + \frac{1}{\Gamma(\kappa_1)} \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1-1}\|g_n(s)-g(s)\|ds \\
& + \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1-1}\|g_n(s)-g(s)\|ds \\
& + \frac{1}{\Gamma(\kappa_2)} \int_{\theta}^{\theta_{\jmath+1}} (s-\theta)^{\kappa_2-1}\|g_n(s)-g(s)\|ds \\
& + \sum_{0<\theta_\jmath<\theta} \|I_\jmath(\varphi_n(\theta_\jmath^-)) - I_\jmath(\varphi(\theta_\jmath^-))\|.
\end{aligned}$$

By $(B5)$, we have

$$\begin{aligned}
\|g_n(\theta) - g(\theta)\| &= \|f(\theta, \varphi(\theta), g_n(\theta)) - f(\theta, \varphi(\theta), g(\theta))\| \\
&\leq \lambda\|\varphi_n(\theta) - \varphi(\theta)\| + L\|g_n(\theta) - g(\theta)\|.
\end{aligned}$$

Then,

$$\|g_n(\theta) - g(\theta)\| \leq \frac{\lambda}{1-L}\|\varphi_n(\theta) - \varphi(\theta)\|.$$

Thus,

$$\begin{aligned}
\|\aleph\varphi_n(\theta) - \aleph\varphi(\theta)\| \leq & \frac{\lambda}{(1-L)\Gamma(\kappa_2)} \sum_{0<\theta_{\jmath+1}<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s-\theta_{\jmath-1})^{\kappa_2-1}\|\varphi_n(s)-\varphi(s)\|ds \\
& + \frac{\lambda}{(1-L)\Gamma(\kappa_1)} \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1-1}\|\varphi_n(s)-\varphi(s)\|ds
\end{aligned}$$

$$+\frac{\lambda}{(1-L)\Gamma(\kappa_1)}\int_{\theta_\jmath}^{\theta}(\theta-s)^{\kappa_1-1}\|\varphi_n(s)-\varphi(s)\|ds$$
$$+\frac{\lambda}{(1-L)\Gamma(\kappa_2)}\int_{\theta}^{\theta_{\jmath+1}}(s-\theta)^{\kappa_2-1}\|\varphi_n(s)-\varphi(s)\|ds$$
$$+\sum_{\jmath=1}^{\beta}C\|\varphi_n(\theta)-z(\theta)\|.$$

By applying the Lebesgue dominated convergence theorem, we get

$$\|\aleph\varphi_n(\theta)-\aleph\varphi(\theta)\|\longrightarrow 0 \quad as\ n\longrightarrow\infty.$$

Hence,

$$\|\aleph\varphi_n-\aleph\varphi\|_{PC}\longrightarrow 0 \quad as\ n\longrightarrow\infty,$$

which implies that $\aleph$ is continuous.

Let $R>0$ such that

$$R\geq\frac{\|\varphi_0\|+\frac{q_1^*\varkappa^{\kappa_2}(\beta+2)}{(1-q_3^*)\Gamma(\kappa_2+1)}+\frac{q_1^*\varkappa^{\kappa_1}(\beta+1)}{(1-q_3^*)\Gamma(\kappa_1+1)}+\beta p_2^*}{1-\frac{q_2^*\varkappa^{\kappa_2}(\beta+2)}{(1-q_3^*)\Gamma(\kappa_2+1)}-\frac{q_2^*\varkappa^{\kappa_1}(\beta+1)}{(1-q_3^*)\Gamma(\kappa_1+1)}-\beta p_1^*}.$$

Define the ball

$$D_R=\{\varphi\in PC_1(\Theta,\Xi):\|\varphi\|_{PC_1}\leq R\}.$$

Step 2: $\aleph(D_R)\subset D_R$.
Let $\varphi\in D_R$ and $\theta\in\Theta$. Then

$$\|\aleph\varphi(\theta)\|\leq\|\varphi_0\|+\frac{1}{\Gamma(\kappa_2)}\sum_{0<\theta_{\jmath+1}<\theta}\int_{\theta_{\jmath-1}}^{\theta_\jmath}(s-\theta_{\jmath-1})^{\kappa_2-1}\|g(s)\|ds$$
$$+\frac{1}{\Gamma(\kappa_1)}\sum_{0<\theta_\jmath<\theta}\int_{\theta_{\jmath-1}}^{\theta_\jmath}(\theta_\jmath-s)^{\kappa_1-1}\|g(s)\|ds+\frac{1}{\Gamma(\kappa_1)}\int_{\theta_\jmath}^{\theta}(\theta-s)^{\kappa_1-1}\|g(s)\|ds$$
$$+\frac{1}{\Gamma(\kappa_2)}\int_{\theta}^{\theta_{\jmath+1}}(s-\theta)^{\kappa_2-1}\|g(s)\|ds+\sum_{0<\theta_\jmath<\theta}\|I_\jmath(\varphi(\theta_\jmath^-))\|.$$

From $(B5)$, we have

$$\begin{aligned}\|g(\theta)\|&=\|f(\theta,\varphi(\theta),g(\theta))\|\\&\leq q_1(\theta)+q_2^*\|\varphi\|_{PC_1}+q_3^*\|g(\theta)\|\\&\leq q_1^*+q_2^*R+q_3^*\|g(\theta)\|.\end{aligned}$$

Then,

$$\|g(\theta)\|\leq\frac{q_1^*+q_2^*R}{1-q_3^*}.$$

Thus,

$$\begin{aligned}\|\aleph\varphi(\theta)\|&\leq\|\varphi_0\|+\frac{\varkappa^{\kappa_2}(q_1^*+q_2^*R)(\beta+2)}{(1-q_3^*)\Gamma(\kappa_2+1)}+\frac{\varkappa^{\kappa_1}(q_1^*+q_2^*R)(\beta+1)}{(1-q_3^*)\Gamma(\kappa_1+1)}+\beta(p_1^*R+p_2^*)\\&\leq R.\end{aligned}$$

Hence, $\aleph(D_R) \subset D_R$.

Step 3: $\aleph(D_R)$ is equicontinuous.
Let $\theta_1, \theta_2 \in \Theta$, where $\theta_1 < \theta_2$ and $\varphi \in D_R$. Then,

$$\begin{aligned}
\|\aleph\varphi(\theta_2) - \aleph\varphi(\theta_1)\| &\leq \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta_1} [(\theta_2 - s)^{\kappa_1 - 1} - (\theta_1 - s)^{\kappa_1 - 1}] \|g(s)\| ds \\
&\quad + \frac{1}{\Gamma(\kappa_1)} \int_{\theta_1}^{\theta_2} (\theta_2 - s)^{\kappa_1 - 1} \|g(s)\| ds \\
&\quad + \frac{1}{\Gamma(\kappa_2)} \int_{\theta_1}^{\theta_2} (s - \theta_1)^{\kappa_2 - 1} \|g(s)\| ds \\
&\quad + \frac{1}{\Gamma(\kappa_2)} \int_{\theta_2}^{\theta_{\jmath+1}} [(s - \theta_2)^{\kappa_2 - 1} - (s - \theta_1)^{\kappa_2 - 1}] \|g(s)\| ds \\
&\quad + \sum_{\theta_1 < \theta_\jmath < \theta_2} \|I_\jmath(\varphi(\theta_\jmath^-))\| \\
&\leq \frac{(q_1^* + q_2^* R)}{(1 - q_3^*)\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta_1} [(\theta_2 - s)^{\kappa_1 - 1} - (\theta_1 - s)^{\kappa_1 - 1}] ds \\
&\quad + \frac{(q_1^* + q_2^* R)(\theta_2 - \theta_1)^{\kappa_1}}{(1 - q_3^*)\Gamma(\kappa_1 + 1)} \\
&\quad + \frac{(q_1^* + q_2^* R)(\theta_2 - \theta_1)^{\kappa_2}}{(1 - q_3^*)\Gamma(\kappa_2 + 1)} \\
&\quad + \frac{(q_1^* + q_2^* R)}{(1 - q_3^*)\Gamma(\kappa_2)} \int_{\theta_2}^{\theta_{\jmath+1}} [(s - \theta_2)^{\kappa_2 - 1} - (s - \theta_1)^{\kappa_2 - 1}] ds \\
&\quad + \sum_{\theta_1 < \theta_\jmath < \theta_2} p_1^* R + p_2^*.
\end{aligned}$$

As $\theta_1 \longrightarrow \theta_2$, the right-hand side of the inequality above tend to zero, then $\aleph(D_R)$ is equicontinuous.

Step 4: The implication of Mönch's theorem.
Let B be a subset of D_R such that $B \subset \aleph(B) \cup \{0\}$. Therefore, the function $\theta \longrightarrow b(\theta) = \zeta(B(\theta))$ is continuous on Θ. Then, for $\theta \in \Theta$, we have

$$\begin{aligned}
b(\theta) &= \zeta(B(\theta)) \\
&= \zeta\big\{\aleph\varphi(\theta), \quad \varphi \in B\big\} \\
&= \zeta\Bigg\{\varphi_0 - \frac{1}{\Gamma(\kappa_2)} \sum_{0 < \theta_{\jmath+1} < \theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s - \theta_{\jmath-1})^{\kappa_2 - 1} g(s) ds \\
&\quad + \frac{1}{\Gamma(\kappa_1)} \sum_{0 < \theta_\jmath < \theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1 - 1} g(s) ds + \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1 - 1} g(s) ds \\
&\quad + \frac{1}{\Gamma(\kappa_2)} \int_{\theta}^{\theta_{\jmath+1}} (s - \theta)^{\kappa_2 - 1} g(s) ds + \sum_{0 < \theta_\jmath < \theta} I_\jmath(\varphi(\theta_\jmath^-)), \quad \varphi \in B\Bigg\}
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\kappa_2)} \sum_{0<\theta_{\jmath+1}<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s-\theta_{\jmath-1})^{\kappa_2-1}\{\zeta(g(s))ds, \quad \varphi \in B\}$$
$$+ \frac{1}{\Gamma(\kappa_1)} \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1-1}\{\zeta(g(s))ds, \quad \varphi \in B\}$$
$$+ \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1-1}\{\zeta(g(s))ds, \quad \varphi \in B\}$$
$$+ \frac{1}{\Gamma(\kappa_2)} \int_{\theta}^{\theta_{\jmath+1}} (s-\theta)^{\kappa_2-1}\{\zeta(g(s))ds, \quad \varphi \in B\} + \sum_{0<\theta_\jmath<\theta} \{\zeta(I_\jmath(\varphi(\theta_\jmath^-))), \quad \varphi \in B\}.$$

From $(B7)$, we have

$$\zeta(g(\theta)) = \zeta(f(\theta, \varphi(\theta), g(\theta)) \\ \leq \lambda\zeta(\varphi(\theta)) + L\zeta(g(\theta)).$$

Thus,

$$\zeta(g(\theta)) \leq \frac{\lambda}{1-L}\zeta(\varphi(\theta)).$$

Also, we have for each $\theta \in \Theta$ and $\jmath = 1, \ldots, \beta$,

$$\sum_{0<\theta_\jmath<\theta} \zeta(I_\jmath(\varphi(\theta_\jmath^-))) \leq \beta C\zeta(\varphi(\theta)).$$

Then,

$$b(\theta) = \zeta(B(\theta))$$
$$\leq \frac{\lambda(\beta+1)}{(1-L)\Gamma(\kappa_2)} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s-\theta_{\jmath-1})^{\kappa_2-1}\{\zeta(\varphi(s))ds, \quad \varphi \in B\}$$
$$+ \frac{\lambda\beta}{(1-L)\Gamma(\kappa_1)} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1-1}\{\zeta(\varphi(s))ds, \quad \varphi \in B\}$$
$$+ \frac{\lambda}{(1-L)\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1-1}\{\zeta(\varphi(s))ds, \quad \varphi \in B\}$$
$$+ \frac{\lambda}{(1-L)\Gamma(\kappa_2)} \int_{\theta}^{\theta_{\jmath+1}} (s-\theta)^{\kappa_2-1}\{\zeta(\varphi(s))ds, \quad \varphi \in B\}$$
$$+ \beta C\{\zeta(\varphi(s))ds, \quad \varphi \in B\}$$
$$\leq \left[\frac{\lambda\varkappa^{\kappa_2}(\beta+2)}{(1-L)\Gamma(\kappa_2+1)} + \frac{\lambda\varkappa^{\kappa_1}(\beta+1)}{(1-L)\Gamma(\kappa_1+1)} + \beta C\right]\zeta_c(B).$$

Therefore,

$$\zeta_c(B) \leq \left[\frac{\lambda\varkappa^{\kappa_2}(\beta+2)}{(1-L)\Gamma(\kappa_2+1)} + \frac{\lambda\varkappa^{\kappa_1}(\beta+1)}{(1-L)\Gamma(\kappa_1+1)} + \beta C\right]\zeta_c(B).$$

And, by Remark 5.2, we have

$$\zeta_c(B) \leq \left[\frac{q_2^*\varkappa^{\kappa_2}(\beta+2)}{(1-q_3^*)\Gamma(\kappa_2+1)} + \frac{q_2^*\varkappa^{\kappa_1}(\beta+1)}{(1-q_3^*)\Gamma(\kappa_1+1)} + \beta p_1^*\right]\zeta_c(B),$$

which implies that $\zeta_c(B) = 0$. Then we deduce that the operator $\aleph$ has a fixed point that is the solution of the problem (11)-(13), according to Mönch's fixed point theorem. □

6. Ulam-Hyers Stability

In this section, we will establish the Ulam stability for the problem (11)-(13).

Definition 6.1 ([1, 21]). Problem (11)-(13) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $\varphi \in PC_1(\Theta, \Xi)$ of the inequality

$$\begin{cases} \left\| {}_0^C D_{\varkappa}^{\kappa_1,\kappa_2;\gamma}\varphi(\theta) - f\left(\theta, \varphi(\theta),\ {}_0^C D_{\varkappa}^{\kappa_1,\kappa_2;\gamma}\varphi(\theta)\right)\right\| < \varepsilon, & \theta \in (\theta_\jmath, \theta_{\jmath+1}], \jmath = 0, \dots, \beta, \\ \left\| \Delta\varphi|_{\theta_\jmath} - I_\jmath(\varphi(\theta_\jmath^-))\right\| \leq \varepsilon, & \jmath = 1, \dots, \beta, \end{cases} \tag{14}$$

there exists a solution $\bar{\varphi} \in PC_1(\Theta, \Xi)$ of the problem (11)-(13) with

$$\|\varphi(\theta) - \bar{\varphi}(\theta)\| < C_f \varepsilon, \quad \theta \in \Theta.$$

Definition 6.2 ([1, 21]). Problem (11)-(13) is generalized Ulam-Hyers stable if there exists $\phi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\phi_f(0) = 0$ such that for each solution $\varphi \in PC_1(\Theta, \Xi)$ of the inequality (14) there exists a solution $\bar{\varphi} \in PC_1(\Theta, \Xi)$ of the problem (11)-(13) with

$$\|\varphi(\theta) - \bar{\varphi}(\theta)\| < \phi_f \varepsilon, \quad \theta \in \Theta.$$

Remark 6.1. A function $\varphi \in PC_1(\Theta, \Xi)$ is a solution of the inequality (14) if and only if there exist a function $\sigma \in PC_1(\Theta, \Xi)$ and a sequence $\sigma_\jmath; \jmath = 1, \dots, \beta$ (which depend on φ), such that
(1) $\|\sigma(\theta)\| \leq \varepsilon, \quad \theta \in (\theta_\jmath, \theta_{\jmath+1}], \jmath = 0, \dots, \beta$ and $\|\sigma_\jmath\| \leq \varepsilon\psi, \jmath = 1, \dots, \beta$,
(2) ${}_0^C D_{\varkappa}^{\kappa_1,\kappa_2;\gamma}\varphi(\theta) = f(\theta, \varphi(\theta),\ {}_0^C D_{\varkappa}^{\kappa_1,\kappa_2;\gamma}\varphi(\theta)) + \sigma(\theta), \quad \theta \in (\theta_\jmath, \theta_{\jmath+1}], \jmath = 0, \dots, \beta$,
(3) $\Delta\varphi|_{\theta_\jmath} = I_\jmath(\varphi(\theta_\jmath^-)) + \sigma_\jmath, \jmath = 1, \dots, \beta$.

Lemma 6.1. *The solution of the following perturbed problem*

$$\begin{gathered} {}_0^C D_{\varkappa}^{\kappa_1,\kappa_2;\gamma}\varphi(\theta) = f\left(\theta, \varphi(\theta),\ {}_0^C D_{\varkappa}^{\kappa_1,\kappa_2;\gamma}\varphi(\theta)\right) + \ell(\theta), \quad \theta \in (\theta_\jmath, \theta_{\jmath+1}], \jmath = 0, \dots, \beta, \\ \Delta\varphi|_{\theta=\theta_\jmath} = I_\jmath(\varphi(\theta_\jmath^-)) + \sigma_\jmath, \quad \jmath = 1, \dots, \beta, \\ \varphi(0) = \varphi_0, \end{gathered}$$

is given by

$$\begin{aligned} \varphi(\theta) = \varphi_0 &- \frac{1}{\Gamma(\kappa_2)} \sum_{0<\theta_{\jmath+1}<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s - \theta_{\jmath-1})^{\kappa_2 - 1} g(s) ds \\ &+ \frac{1}{\Gamma(\kappa_1)} \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\theta_\jmath - s)^{\kappa_1 - 1} g(s) ds + \frac{1}{\Gamma(\kappa_1)} \int_{\theta_\jmath}^{\theta} (\theta - s)^{\kappa_1 - 1} g(s) ds \\ &+ \frac{1}{\Gamma(\kappa_2)} \int_{\theta}^{\theta_{\jmath+1}} (s - \theta)^{\kappa_2 - 1} g(s) ds + \sum_{0<\theta_\jmath<\theta} I_\jmath(\varphi(\theta_\jmath^-)) \\ &- \frac{1}{\Gamma(\kappa_2)} \sum_{0<\theta_{\jmath+1}<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (s - \theta_{\jmath-1})^{\kappa_2 - 1} \ell(s) ds \end{aligned}$$

$$+\frac{1}{\Gamma(\kappa_1)}\sum_{0<\theta_\jmath<\theta}\int_{\theta_{\jmath-1}}^{\theta_\jmath}(\theta_\jmath-s)^{\kappa_1-1}\ell(s)ds+\frac{1}{\Gamma(\kappa_1)}\int_{\theta_\jmath}^{\theta}(\theta-s)^{\kappa_1-1}\ell(s)ds$$
$$+\frac{1}{\Gamma(\kappa_2)}\int_{\theta}^{\theta_{\jmath+1}}(s-\theta)^{\kappa_2-1}\ell(s)ds+\sum_{0<\theta_\jmath<\theta}\sigma_\jmath$$

Moreover, the solution satisfies the following inequality

$$\begin{aligned}\Bigg\|\varphi(\theta) &- \Bigg[\varphi_0-\frac{1}{\Gamma(\kappa_2)}\sum_{0<\theta_{\jmath+1}<\theta}\int_{\theta_{\jmath-1}}^{\theta_\jmath}(s-\theta_{\jmath-1})^{\kappa_2-1}g(s)ds\\
&+ \frac{1}{\Gamma(\kappa_1)}\sum_{0<\theta_\jmath<\theta}\int_{\theta_{\jmath-1}}^{\theta_\jmath}(\theta_\jmath-s)^{\kappa_1-1}g(s)ds\\
&+ \frac{1}{\Gamma(\kappa_1)}\int_{\theta_\jmath}^{\theta}(\theta-s)^{\kappa_1-1}g(s)ds\\
&+ \frac{1}{\Gamma(\kappa_2)}\int_{\theta}^{\theta_{\jmath+1}}(s-\theta)^{\kappa_2-1}g(s)ds+\sum_{0<\theta_\jmath<\theta}I_\jmath(\varphi(\theta_\jmath^-))\Bigg]\Bigg\|\\
&\le \left[\frac{\varkappa^{\kappa_2}(\beta+2)}{\Gamma(\kappa_2+1)}+\frac{\varkappa^{\kappa_1}(\beta+1)}{\Gamma(\kappa_1+1)}+\beta\psi\right]\varepsilon.\end{aligned}$$

Theorem 6.2. *Assume that* $(B4)$*-*$(B6)$ *hold. If*

$$\frac{\lambda\varkappa^{\kappa_2}(\beta+2)}{(1-L)\Gamma(\kappa_2+1)}+\frac{\lambda\varkappa^{\kappa_1}(\beta+1)}{(1-L)\Gamma(\kappa_1+1)}+C\beta<1,$$

then the problem (11)*-*(13) *is Ulam-Hyers stable.*

Proof. Let $\varphi\in PC_1(\Theta,\Xi)$ be a solution of the inequality (14) and $\bar\varphi\in PC_1(\Theta,\Xi)$ the solution of the problem (11)-(13), then

$$\begin{aligned}\|\varphi(\theta)-\bar\varphi(\theta)\| \le& \left[\frac{\varkappa^{\kappa_2}(\beta+2)}{\Gamma(\kappa_2+1)}+\frac{\varkappa^{\kappa_1}(\beta+1)}{\Gamma(\kappa_1+1)}+\beta\psi\right]\varepsilon\\
&+\frac{1}{\Gamma(\kappa_2)}\sum_{0<\theta_{\jmath+1}<\theta}\int_{\theta_{\jmath-1}}^{\theta_\jmath}(s-\theta_{\jmath-1})^{\kappa_2-1}\|g(s)-h(s)\|ds\\
&+\frac{1}{\Gamma(\kappa_1)}\sum_{0<\theta_\jmath<\theta}\int_{\theta_{\jmath-1}}^{\theta_\jmath}(\theta_\jmath-s)^{\kappa_1-1}\|g(s)-h(s)\|ds\\
&+\frac{1}{\Gamma(\kappa_1)}\int_{\theta_\jmath}^{\theta}(\theta-s)^{\kappa_1-1}\|g(s)-h(s)\|ds\\
&+\frac{1}{\Gamma(\kappa_2)}\int_{\theta}^{\theta_{\jmath+1}}(s-\theta)^{\kappa_2-1}\|g(s)-h(s)\|ds\\
&+\sum_{0<\theta_\jmath<\theta}\|I_\jmath(\varphi(\theta_\jmath^-))-I_\jmath(\bar\varphi(\theta_\jmath^-))\|.\end{aligned}$$

By hypothesis $(B5)$, we have

$$\|g(\theta)-h(\theta)\|\le\lambda\|\varphi-\bar\varphi\|_{PC_1}+L\|g(\theta)-h(\theta)\|.$$

Then,

$$\|g(\theta)-h(\theta)\| \le \frac{\lambda}{1-L}\|\varphi-\bar{\varphi}\|_{PC_1}.$$

Thus,

$$\begin{aligned}\|\varphi(\theta)-\bar{\varphi}(\theta)\| \le &\left[\frac{\varkappa^{\kappa_2}(\beta+2)}{\Gamma(\kappa_2+1)}+\frac{\varkappa^{\kappa_1}(\beta+1)}{\Gamma(\kappa_1+1)}+\beta\psi\right]\varepsilon\\ &+\left[\frac{\lambda\varkappa^{\kappa_2}(\beta+2)}{(1-L)\Gamma(\kappa_2+1)}+\frac{\lambda\varkappa^{\kappa_1}(\beta+1)}{(1-L)\Gamma(\kappa_1+1)}+C\beta\right]\|\varphi-\bar{\varphi}\|_{PC_1}:=C_f\varepsilon.\end{aligned}$$

Consequently, the problem (11)-(13) is Ulam-Hyers stable.
If we take $\phi_f(\varepsilon)=C_f\varepsilon$ and $\phi_f(0)=0$ then we get the generalized Ulam-Hyers stability of the problem (11)-(13). □

7. An Example

Set

$$\Xi = l^1 = \left\{\varphi=(\varphi_1,\varphi_2,\dots,\varphi_n,\dots), \sum_{n=1}^{\infty}|\varphi_n|<\infty\right\}.$$

Ξ is a Banach space with the norm $\|\varphi\| = \sum_{n=1}^{\infty}|\varphi_n|$.

Consider the following impulsive problem:

$${}_0^C D_1^{\frac{1}{2},\frac{1}{2};\frac{1}{3}}\varphi_n(\theta) = \frac{|\varphi_n(\theta)|+\left|\left({}_0^C D_1^{\frac{1}{2},\frac{1}{2};\frac{1}{3}}\varphi_n(\theta)\right)\right|}{3e^{\theta+1}\left(1+\|\varphi(\theta)\|+\left\|{}_0^C D_1^{\frac{1}{2},\frac{1}{3};\frac{1}{4}}\varphi(\theta)\right\|\right)}, \quad for\ \theta\in\Theta_0\cup\Theta_1, \tag{15}$$

$$\Delta\varphi|_{\theta=\frac{1}{3}} = \frac{\varphi\left(\frac{1}{3}^-\right)}{\varphi\left(\frac{1}{3}^-\right)+17}, \tag{16}$$

$$\varphi_n(0)=1, \tag{17}$$

where $\Theta_0=[0,\frac{1}{3}], \Theta_1=(\frac{1}{3},1]$.
Set

$$f(\theta,\xi,\delta)=\frac{\|\xi\|+\|\delta\|+\cos(\theta)}{3e^{\theta+1}(1+\|\xi\|+\|\delta\|)}, \quad \theta\in[0,1],\ \xi\in\Xi \text{ and } \delta\in\Xi.$$

Clearly f is a continuous function, the condition $(B4)$ is verified. For any $\xi,\bar{\xi}\in\Xi$, $\delta,\bar{\delta}\in\Xi$ and $\theta\in[0,1]$, we have

$$\|f(\theta,\xi,\delta)-f(\theta,\bar{\xi},\bar{\delta})\| \le \frac{1}{3e}[\|\xi-\bar{\xi}\|+\|\delta-\bar{\delta}\|].$$

Then, the assumption $(B5)$ is satisfied with $\lambda=\gamma=\frac{1}{3e}$. Also we have

$$\|f(\theta,\xi,\delta)\| \le \frac{\cos(\theta)}{3e^{\theta+1}}+\frac{1}{3e}[\|\xi\|+\|\delta\|].$$

So $q_1(\theta) = \dfrac{\cos(\theta)}{3e^{\theta+1}}$, $q_2^* = q_3^* = \frac{1}{3e}$. Let

$$I_1(\delta) = \frac{\delta}{\delta + 17}, \qquad \delta \in \Xi.$$

Then, for $\delta, \bar{\delta} \in \Xi$, we have

$$\|I_1(\delta)\| = \frac{1}{17}\|\delta\| + 1.$$

Thus, $p_1^* = \frac{1}{17}$, $p_2^* = 1$, And as

$$\begin{aligned} \frac{q_2^* \varkappa^{\kappa_2}(\beta+2)}{(1-q_3^*)\Gamma(\kappa_2+1)} + \frac{q_2^* \varkappa^{\kappa_1}(\beta+1)}{(1-q_3^*)\Gamma(\kappa_1+1)} + \beta p_1^* &= \frac{3}{(3e-1)\Gamma(\frac{3}{2})} \\ &+ \frac{2}{(3e-1)\Gamma(\frac{3}{2})} + \frac{1}{17} \\ &\leq 1. \end{aligned}$$

Thus, by Theorem 5.2, the problem has at least one solution. Moreover

$$\begin{aligned} \frac{\lambda \varkappa^{\kappa_2}(\beta+2)}{(1-L)\Gamma(\kappa_2+1)} + \frac{\lambda \varkappa^{\kappa_1}(\beta+1)}{(1-L)\Gamma(\kappa_1+1)} + C\beta &= \frac{3}{(3e-1)\Gamma(\frac{3}{2})} \\ &+ \frac{2}{(3e-1)\Gamma(\frac{3}{2})} + \frac{1}{17} \\ &\leq 1. \end{aligned}$$

Then, Theorem 6.2 assures that the problem (15)-(17) is Ulam-Hyers stable.

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