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Existence results for a class of inequality problems with p-Laplacian

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Abstract. We are concerned with existence results for a class of inequality problems having the general form:

$$\text{Find } u \in K \text{ such that } F^0(u; v-u) + \int_\Omega |\nabla u|^{p-2} \nabla u \nabla (v-u) \geq 0, \ \forall v \in K,$$

where F^0 denotes the generalized directional derivative of a locally Lipschitz function $F : W^{1,p}(\Omega) \to \mathbb{R}$, and K is some closed, convex subset of $W^{1,p}(\Omega)$.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with smooth boundary $\Gamma = \partial\Omega$ and W be a nonzero closed subspace of the Sobolev space $W^{1,p}(\Omega)$, with $p \in (1, \infty)$. Given be a constant $\alpha \geq 0$ and denoting by W^* the dual space of W, we define the operator $T_\alpha : W \to W^*$ by

$$\langle T_\alpha(u), v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + \alpha \int_\Omega |u|^{p-2} uv, \ \ \forall u, v \in W. \tag{1.1}$$

The first term in the right hand side of the equality (1.1) is the minus p-Laplacian operator $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ when it is associated with some homogeneous boundary conditions such as e.g., the classical Dirichlet ($W = W_0^{1,p}(\Omega)$) or Neumann ($W = W^{1,p}(\Omega)$).

We consider a measurable function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, satisfying the growth condition

$$|g(x, s)| \leq c_1 |s|^{q-1} + c_2 \text{ for a.e. } x \in \Omega, \forall s \in \mathbb{R}, \tag{1.2}$$

where $c_1, c_2 \geq 0$ are constants, $1 < q < p^*$, and

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ \infty & \text{if } p \geq N. \end{cases}$$

Let $G : \Omega \times \mathbb{R} \to \mathbb{R}$ be the primitive of g, i.e.,

$$G(x, s) = \int_0^s g(x, t) dt \quad \text{for a.e. } x \in \Omega, \ \forall s \in \mathbb{R}. \tag{1.3}$$

The functional $\mathcal{G} : \ L^q(\Omega) \to \mathbb{R}$ given by

$$\mathcal{G}(u) = -\int_\Omega G(x, u), \ \forall u \in L^q(\Omega) \tag{1.4}$$

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is Lipschitz continuous on the bounded subsets of $L^q(\Omega)$ (see e.g. Chang [2]) and, by the continuity of the embedding $W \subset L^q(\Omega)$, we have that $\mathcal{G}\big|_W$ is locally Lipschitz on W endowed with the induced norm from $W^{1,p}(\Omega)$.

For a measurable set $\omega \subset \overline{\Omega}$ we define

$$K = K(W,\omega) := \{u \in W : \ u(x) \geq 0 \text{ for a.e. } x \in \omega\} \tag{1.5}$$

and we formulate the problem:

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ (\mathcal{G}\big|_W)^0(u; v-u) + \langle T_\alpha(u), v-u\rangle \geq 0, \quad \forall v \in K, \end{cases} \tag{1.6}$$

where $(\mathcal{G}\big|_W)^0(u;w)$ stands for the generalized directional derivative of $\mathcal{G}\big|_W$ (in the sense of Clarke [3]) at $u \in W$ in the direction $w \in W$. It is worth to point out that each solution of problem (1.6) also solves the hemivariational inequality:

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ \displaystyle\int_\Omega (-G)^0(x,u;v-u) + \langle T_\alpha(u), v-u\rangle \geq 0, \quad \forall v \in K, \end{cases} \tag{1.7}$$

where $(-G)^0(x,u;w)$ denotes the generalized directional derivative of the locally Lipschitz function $-G(x,\cdot)$ at $u(x)$ in the direction $w(x)$. At its turn, if g is Carathéodory, (1.7) becomes the variational inequality:

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ \displaystyle\int_\Omega (-g)(x,u)(v-u) + \langle T_\alpha(u), v-u\rangle \geq 0, \quad \forall v \in K, \end{cases} \tag{1.8}$$

The purpose of this paper is to obtain sufficient conditions ensuring the existence of solutions for problem (1.6). Our approach is a variational one and it relies upon abstract results from [7]. We obtain the existence of solutions in the coercive case as well as the existence of nontrivial solutions when the corresponding Euler-Lagrange functional has a mountain-pass geometry. So, we extend results from paper [7] stated for some particular choices of W to the general case when W is an arbitrary closed subspace of $W^{1,p}(\Omega)$. Existence of mountain-pass type solutions for problems of type (1.8) and (1.7) were obtained in earlier papers by Szulkin (Theorem 5.1 in [14]), respectively Motreanu and Panagiotopoulos (Section 3.5 in [11]) in the case $p = 2$, $W = W_0^{1,2}(\Omega)$ and $\omega = \Omega$. Our result (Theorem 3.2 in Section 3) is in this direction and more specifically, we extend the celebrated condition (p_5) of Ambrosetti and Rabinowitz [1] (condition (p_4) in [12]) to the general problem (1.6). In this respect we generalize different theorems in the smooth and nonsmooth variational analysis [1], [12], [2], [14], [5], [6], [7], [13].

The rest of the paper is organized as follows. Some notions and abstract results from [7] and [11] are presented in Section 2. The existence results for problem (1.6) are proved in Section 3. In Section 4 we give examples of applications to differential inclusions with p-Laplacian.

2. Preliminaries

In this section we list some notions and results which will be used in establishing the existence of solutions for problem (1.6). For the proofs we shall refer the reader to [7] and [11].

Let X be a real Banach space and X^* its dual. Recall, *the generalized directional derivative* of a locally Lipschitz function $F : X \to I\!R$ at $u \in X$ in the direction $v \in X$ is defined by

$$F^0(u;v) = \limsup_{w \to u, t \searrow 0} \frac{F(w+tv) - F(w)}{t}.$$

The generalized gradient (in the sense of Clarke [3]) of F at $u \in X$ is the subset of X^* given by

$$\partial F(u) = \{\eta \in X^* : \ F^0(u;v) \geq \langle \eta, v \rangle, \ \ \forall v \in X\},$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X.

The following abstract functional framework is assumed.

(f_1) $(X, \|\cdot\|)$ is a real reflexive Banach space, compactly embedded in the real Banach space $(Z, \|\cdot\|_Z)$

(f_2) C is a nonempty, closed and convex subset of X;

(f_3) $\varphi : X \to I\!R$ is Gâteaux differentiable and convex;

(f_4) $\mathcal{F} : Z \to I\!R$ is locally Lipschitz.

Note that $\mathcal{F}\big|_X$ is locally Lipschitz on X by virtue of (f_4) and (f_1). We consider the following inequality problem:

$$\begin{cases} \text{Find } u \in C \text{ such that} \\ (\mathcal{F}\big|_X)^0(u; v-u) + \langle d\varphi(u), v-u \rangle \geq 0, \ \ \forall v \in C. \end{cases} \tag{2.1}$$

The approach for problem (2.1) is a variational one and it relies upon the use of the energy functional $\Phi : X \to (-\infty, +\infty]$, defined by

$$\Phi = \mathcal{F}\big|_X + \varphi + I_C, \tag{2.2}$$

where I_C stands for the indicator function of the set C. An element $u \in C$ is called *critical point* of the functional Φ if the inequality below holds

$$(\mathcal{F}\big|_X)^0(u; v-u) + \varphi(v) - \varphi(u) \geq 0, \ \forall \ v \in C.$$

Proposition 2.1. *(Proposition 3.9 in* [7]*). If $u \in X$ is a critical point of Φ then u is a solution of problem* (2.1)*.*

Theorem 2.1. *(Theorem 3.10 in* [7]*). If the functional Φ is coercive on X, i.e.,*

$$\Phi(v) \to +\infty, \ \ \textit{as } \|v\| \to \infty,$$

then it is bounded from below and attains its infimum at some $u \in X$ and u is a critical point of Φ.

The functional Φ is said *to satisfy the Palais-Smale condition* if every sequence $\{u_n\} \subset X$ for which $\{\Phi(u_n)\}$ is bounded and

$$(\mathcal{F}\big|_X)^0(u_n; v-u_n) + \varphi(v) - \varphi(u_n) \geq -\varepsilon_n \|v - u_n\|, \ \ \forall v \in C,$$

for a sequence $\{\varepsilon_n\} \subset I\!R^+$ with $\varepsilon_n \to 0$, contains a strongly convergent subsequence in X. For the proof of the following result we refer the reader to the nonsmooth version of the Mountain Pass Theorem stated in Corollary 3.2 from [11] (also see Theorem 2.2 in [6] and Theorem 2.3 in [7]).

Theorem 2.2. *If Φ satisfies the Palais-Smale condition and there exist a number $\rho > 0$ and a point $e \in X$ with $\|e\| > \rho$ such that*

$$\inf_{\|v\|=\rho} \Phi(v) > \Phi(0) \geq \Phi(e), \tag{2.3}$$

then Φ has a nontrivial critical point.

To check that Φ satisfies the Palais-Smale condition the lemma below provides an useful tool. This is an easy consequence of Lemma 3.5 in [7] (also see Theorem 3.11 in [7]).

Lemma 2.1. *Assume that:*

(i) $d\varphi$ satisfies condition (S_+) on C, i.e., if $\{u_n\}$ is a sequence in C, provided $u_n \to u$, weakly in X, and

$$\limsup_{n\to\infty} \langle d\varphi(u_n), u_n - u\rangle \leq 0$$

then $u_n \to u$, strongly in X;

(ii) there are constants a_0, $a_1 > 0$, $b \in I\!\!R$ and $\sigma > 1$ such that:

$$\varphi(u) - a_1\langle d\varphi(u), u\rangle \geq a_0\|u\|^\sigma, \quad \forall u \in C, \tag{2.4}$$

$$\mathcal{F}(u) - a_1(\mathcal{F}\big|_X)^0(u; u) \geq b, \quad \forall u \in C. \tag{2.5}$$

Then Φ satisfies the Palais-Smale condition.

3. Existence results for problem (1.6)

The space $W \subset W^{1,p}(\Omega)$ is endowed with the norm

$$\|v\|_\eta = \left(\int_\Omega |\nabla v|^p + \eta \int_\Omega |v|^p\right)^{\frac{1}{p}}, \quad \forall v \in W,$$

where $\eta > 0$ is a constant. The results from the previous section will be applied by taking $X = W$, $\|\cdot\| = \|\cdot\|_\eta$, $Z = L^q(\Omega)$, $\|\cdot\|_Z = \|\cdot\|_{L^q} :=$ the usual norm on $L^q(\Omega)$, $C = K$ in (1.5), $\varphi = \varphi_\alpha : W \to I\!\!R$ defined by

$$\varphi_\alpha(v) = \frac{1}{p}\left(\int_\Omega |\nabla v|^p + \alpha \int_\Omega |v|^p\right), \quad \forall v \in W \tag{3.1}$$

and $\mathcal{F} = \mathcal{G}$ given by (1.4). It is worth noticing that, as $1 < q < p^*$, by virtue of Rellich-Kondrachov theorem the embedding $W \subset L^q(\Omega)$ is compact. Also, standard arguments show that the convex functional φ_α is continuously differentiable on W and its differential is T_α in (1.1), i.e.,

$$\langle d\varphi_\alpha(u), v\rangle = \langle T_\alpha(u), v\rangle, \quad \forall u, v \in W. \tag{3.2}$$

It is obvious that with the above choices problem (2.1) becomes (1.6). Moreover, by (2.2) we have $\Phi = \Phi_1$ with Φ_1 given by

$$\Phi_1 = \mathcal{G}\big|_W + \varphi_\alpha + I_K. \tag{3.3}$$

Now, using an idea from [10], we introduce the constant

$$\lambda_1 = \lambda_1(K, \alpha) := \alpha + \inf\left\{\frac{\|\nabla u\|^p_{L^p}}{\|u\|^p_{L^p}} \,:\, u \in K \setminus \{0\}\right\}, \tag{3.4}$$

for $\alpha \geq 0$. It should be noticed that $\lambda_1 = +\infty$ iff $K = \{0\}$ and in this case $u = 0$ is the unique solution of problem (1.6). Also, $\lambda_1(K, 0)$ can be either equal to 0 (e.g., if $W = W^{1,p}(\Omega)$) or > 0 (e.g., if $W = W^{1,p}_0(\Omega)$). We shall need to invoke the hypothesis

(H_{λ_1}) $\lambda_1 \in (0, +\infty)$.

Proposition 3.1. *If* (H_{λ_1}) *holds true then*

$$2^{-\frac{1}{p}}\|u\|_{\lambda_1} \le (\|\nabla u\|^p_{L^p} + \alpha\|u\|^p_{L^p})^{\frac{1}{p}} \le \|u\|_{\lambda_1}, \ \ \forall u \in K. \tag{3.5}$$

Proof. It is straightforward from (3.4). □

Theorem 3.1. *Assume* (1.2) *and* $\lambda_1 \in [0, +\infty)$ *together with*

(i) $\limsup_{s\to-\infty} \frac{pG(x,s)}{|s|^p} < \lambda_1$ *uniformly for a.e.* $x \in \Omega \setminus \omega$;

(ii) $\limsup_{s\to+\infty} \frac{pG(x,s)}{s^p} < \lambda_1$ *uniformly for a.e.* $x \in \Omega$.

Then problem (1.6) *has at least one solution.*

Proof. By Theorem 2.1 and Proposition 2.1 it suffices to show that the functional Φ_1 in (3.3) is coercive on $(W, \|\cdot\|_\eta)$, with some $\eta > 0$. From (i) and (ii) there are numbers $\sigma > 0$ and $s_0 > 0$ such that

$$G(x,s) \le \frac{\lambda_1 - \sigma}{p}|s|^p \ \text{ for a.e. } x \in \Omega \setminus \omega, \ \forall s < -s_0 \tag{3.6}$$

and

$$G(x,s) \le \frac{\lambda_1 - \sigma}{p}s^p \ \text{ for a.e. } x \in \Omega, \ \forall s > s_0. \tag{3.7}$$

If $\lambda_1 > 0$ we shall assume that $\sigma \in (0, \lambda_1)$. From (1.2) the primitive G satisfies

$$|G(x,s)| \le \frac{c_1}{q}|s|^q + c_2|s| \quad \text{for a.e. } x \in \Omega, \ \forall s \in \mathbb{R} \tag{3.8}$$

showing that there is a constant $k = k(s_0)$ such that

$$|G(x,s)| \le k \ \text{ for a.e. } x \in \Omega, \ \forall s \in [-s_0, s_0]. \tag{3.9}$$

Then, using (3.6) and (3.9) we can estimate

$$G(x,s) \le \frac{\lambda_1 - \sigma}{p}|s|^p + \tilde{k} \ \text{ for a.e. } x \in \Omega \setminus \omega, \ \forall s < 0, \tag{3.10}$$

where

$$\tilde{k} = \frac{|\lambda_1 - \sigma|}{p}s_0^p + k.$$

Similarly, from (3.7) and (3.9) one obtains

$$G(x,s) \le \frac{\lambda_1 - \sigma}{p}s^p + \tilde{k} \ \text{ for a.e. } x \in \Omega, \ \forall s \ge 0. \tag{3.11}$$

For $u \in K$, denote

$$\Omega_-(u) := \{x \in \Omega : u(x) < 0\}, \quad \Omega_+(u) := \Omega \setminus \Omega_-(u). \tag{3.12}$$

Noticing that $\Omega_-(u) \subset \Omega \setminus \omega$, by (3.10) we have

$$\int_{\Omega_-(u)} G(x,u) \le \frac{\lambda_1 - \sigma}{p}\int_{\Omega_-(u)} |u|^p + \tilde{k}|\Omega|, \tag{3.13}$$

where $|\Omega|$ stands for the measure of Ω. On the other hand by (3.11) we get

$$\int_{\Omega_+(u)} G(x,u) \le \frac{\lambda_1 - \sigma}{p}\int_{\Omega_+(u)} |u|^p + \tilde{k}|\Omega|. \tag{3.14}$$

From (3.13) and (3.14) one obtains the inequality

$$\int_\Omega G(x,u) \le \frac{\lambda_1 - \sigma}{p}\|u\|_{L^p}^p + 2\tilde{k}|\Omega|$$

which, taking into account (3.3), (1.4) and (3.1), yields

$$\Phi_1(u) \ge \frac{1}{p}\left(\|\nabla u\|_{L^p}^p + (\sigma - \lambda_1 + \alpha)\|u\|_{L^p}^p\right) - 2\tilde{k}|\Omega|, \ \ \forall u \in K. \tag{3.15}$$

If $\lambda_1 = 0$, we infer

$$\Phi_1(u) \ge \frac{1}{p}\|u\|_\sigma^p - 2\tilde{k}|\Omega|, \ \ \forall u \in K, \tag{3.16}$$

while in the case $\lambda_1 > 0$, using (3.15), (3.4) and (3.5), we estimate Φ_1 as follows

$$\begin{aligned}\Phi_1(u) &\ge \frac{1}{p}\left(\|\nabla u\|_{L^p}^p + \alpha\|u\|_{L^p}^p + (\sigma - \lambda_1)\frac{\|\nabla u\|_{L^p}^p + \alpha\|u\|_{L^p}^p}{\lambda_1}\right) - 2\tilde{k}|\Omega| \\ &\ge \frac{\sigma}{2p\lambda_1}\|u\|_{\lambda_1}^p - 2\tilde{k}|\Omega|, \ \ \forall u \in K. \end{aligned} \tag{3.17}$$

By virtue of (3.16) and (3.17) in both cases there are positive constants η and k_0 such that

$$\Phi_1(u) \ge k_0\|u\|_\eta^p - 2\tilde{k}|\Omega|, \ \ \forall u \in K,$$

showing that

$$\Phi_1(u) \to +\infty, \ \ \text{as } \|u\|_\eta \to \infty.$$

□

In the sequel we are concerned with existence of nontrivial solutions for problem (1.6). In order to apply Theorem 2.2 we have to ensure a mountain-pass geometry for Φ_1 in (3.3). The hypothesis (H_{λ_1}) will be assumed and W will be considered with the norm $\|\cdot\|_{\lambda_1}$.

Proposition 3.2. *If (H_{λ_1}) holds true then $d\varphi_\alpha$ satisfies condition (S_+) on K, i.e., if $\{u_n\}$ is a sequence in K, provided $u_n \to u$, weakly in W, and*

$$\limsup_{n\to\infty}\langle d\varphi_\alpha(u_n), u_n - u\rangle \le 0 \tag{3.18}$$

then $u_n \to u$, strongly in W.

Proof. Let φ_{λ_1} be defined by (3.1) with λ_1 instead of α, i.e.,

$$\varphi_{\lambda_1}(v) = \frac{1}{p}\|v\|_{\lambda_1}^p, \ \ \forall v \in W.$$

Clearly, one has

$$\varphi_{\lambda_1}(v) = \varphi_\alpha(v) + \frac{\lambda_1 - \alpha}{p}\|v\|_{L^p}^p, \ \ \forall v \in W,$$

hence,

$$\langle d\varphi_{\lambda_1}(v), w\rangle = \langle d\varphi_\alpha(v), w\rangle + (\lambda_1 - \alpha)\int_\Omega |v|^{p-2}vw, \ \ \forall v, w \in W. \tag{3.19}$$

Let $\{u_n\}$ be a sequence in K such that $u_n \to u$, weakly in W and (3.18) holds true. Taking into account the compact embedding $W \subset L^p(\Omega)$, we have that $u_n \to u$, strongly in $L^p(\Omega)$, which implies

$$\left|\int_\Omega |u_n|^{p-2}u_n(u_n - u)\right| \le \|u_n\|_{L^p}^{\frac{p}{p'}}\|u_n - u\|_{L^p} \to 0, \quad \text{as } n \to \infty. \tag{3.20}$$

From (3.18), (3.19) and (3.20) it follows

$$\limsup_{n\to\infty}\langle d\varphi_{\lambda_1}(u_n), u_n - u\rangle \le 0. \tag{3.20}$$

As $d\varphi_{\lambda_1}$ is the duality mapping on $(W, \|\cdot\|_{\lambda_1})$, corresponding to the gauge function $t \mapsto t^{p-1}$ and because W with the norm $\|\cdot\|_{\lambda_1}$ is uniformly convex, $d\varphi_{\lambda_1}$ satisfies condition (S_+) on W (see [5]). This together with (3.20) show that $u_n \to u$, strongly in W. □

For a.e. $x \in \Omega$ and all $s \in I\!R$, we put

$$\underline{g}(x,s) = \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-s|<\delta} g(x,t),$$

$$\overline{g}(x,s) = \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|t-s|<\delta} g(x,t).$$

The following condition will be invoked below:

$$\underline{g} \text{ and } \overline{g} \text{ are } N- \text{ measurable} \tag{3.21}$$

(recall, a function $h : \Omega \times I\!R \to I\!R$ is called *N-measurable* if $h(\cdot, u(\cdot)) : \Omega \to I\!R$ is measurable, whenever $u : \Omega \to I\!R$ is measurable).

Proposition 3.3. *Assume (H_{λ_1}), (1.2) and (3.21). If there are numbers $\theta > p$, $s_0 > 0$ such that*

$$\theta G(x,s) \le s g(x,s) \tag{3.22}$$

for a.e. $x \in \Omega \setminus \omega$, $\forall s \le -s_0$ and for a.e $x \in \Omega$, $\forall s \ge s_0$, then Φ_1 satisfies the Palais-Smale condition.

Proof. Using (3.22) we derive

$$G(x,s) \le \frac{s}{\theta}\overline{g}(x,s) \quad \text{for a.e. } x \in \Omega\setminus\omega,\ \forall s < -s_0, \tag{3.23}$$

$$G(x,s) \le \frac{s}{\theta}\underline{g}(x,s) \quad \text{for a.e. } x \in \Omega,\ \forall s > s_0. \tag{3.24}$$

Under the assumptions (1.2) and (3.21), for $u \in L^q(\Omega)$ it holds (see Theorem 2.1 in [2]):

$$w \in \partial(-\mathcal{G})(u) \Longrightarrow w(x) \in \big[\underline{g}(x,u(x)), \overline{g}(x,u(x))\big] \quad \text{for a.e. } x \in \Omega. \tag{3.25}$$

From (3.9), (3.23), (3.24), (1.2) and (3.25), for arbitrary $u \in K$ and $w \in \partial(-\mathcal{G})(u)$, we obtain

$$\begin{aligned}
-\mathcal{G}(u) = \int_\Omega G(x,u) &= \int_{[u<-s_0]} G(x,u) + \int_{[u>s_0]} G(x,u) + \int_{[|u|\le s_0]} G(x,u)\\
&\le \frac{1}{\theta}\left[\int_{[u<-s_0]} u\overline{g}(x,u) + \int_{[u>s_0]} u\underline{g}(x,u)\right] + k|\Omega|\\
&\le \frac{1}{\theta}\left[\int_{[u<-s_0]} uw + \int_{[u>s_0]} uw\right] + k|\Omega|\\
&= \frac{1}{\theta}\left[\int_\Omega uw - \int_{[|u|\le s_0]} uw\right] + k|\Omega| \le \frac{1}{\theta}\int_\Omega uw + \tilde{k},
\end{aligned}$$

with $\tilde{k} = \tilde{k}(s_0)$ a constant. As $\partial(-\mathcal{G})(u) = -\partial\mathcal{G}(u)$, it follows

$$\mathcal{G}(u) \ge \frac{1}{\theta}\int_\Omega uw - \tilde{k}, \quad \forall w \in \partial\mathcal{G}(u). \tag{3.26}$$

Then, using (3.26), taking into account that $\mathcal{G}^0(u;u) \geq (\mathcal{G}|_W)^0(u;u)$ and by virtue of the equality (see, e.g., Proposition 1.4 in [11]):

$$\mathcal{G}^0(u;u) = \max\left\{\int_\Omega uw \ : \ w \in \partial\mathcal{G}(u)\right\}$$

we deduce

$$\mathcal{G}(u) - \frac{1}{\theta}(\mathcal{G}|_W)^0(u;u) \geq -\tilde{k}, \ \ \forall u \in K. \tag{3.27}$$

Also, by (3.5) we get

$$\varphi_\alpha(u) - \frac{1}{\theta}\langle d\varphi_\alpha(u), u\rangle \geq \frac{1}{2}\left(\frac{1}{p} - \frac{1}{\theta}\right)\|u\|^p_{\lambda_1}, \ \ \forall u \in K. \tag{3.28}$$

Viewing Proposition 3.2 and the estimates (3.27), (3.28), Lemma 2.1 applies with

$$\sigma = p\ , \quad a_0 = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{\theta}\right), \quad a_1 = \frac{1}{\theta}\ , \quad b = -\tilde{k}$$

and the proof is complete. □

Theorem 3.2. *Assume* (H_{λ_1}), (1.2) *and* (3.21), *together with*

(*i*) $\limsup\limits_{s \nearrow 0} \dfrac{pG(x,s)}{|s|^p} < \lambda_1$ *uniformly for a.e.* $x \in \Omega \setminus \omega$,

(*ii*) $\limsup\limits_{s \searrow 0} \dfrac{pG(x,s)}{s^p} < \lambda_1$ *uniformly for a.e.* $x \in \Omega$.

If there are numbers $\theta > p$, $s_0 > 0$ *such that*

$$0 < \theta G(x,s) \leq sg(x,s) \tag{3.29}$$

for a.e. $x \in \Omega \setminus \omega$, $\forall s \leq -s_0$ *and for a.e* $x \in \Omega$, $\forall s \geq s_0$, *then problem* (1.6) *has a nontrivial solution.*

Remark 3.1. *Before passing to the proof let us note that by virtue of* (3.29) *the exponent* q *entering in* (1.2) *necessarily lies in the interval* (p, p^*). *Indeed, since*

$$\frac{\theta}{s} \leq \frac{g(x,s)}{G(x,s)} \ \ \text{for a.e. } x \in \Omega, \ \forall s \geq s_0,$$

integrating from s_0 *to* $t > s_0$, *one obtains*

$$G(x,t) \geq \gamma_1(x)t^\theta \ \text{for a.e. } x \in \Omega, \ \forall t > s_0, \tag{3.30}$$

with $\gamma_1(x) := G(x, s_0)s_0^{-\theta} > 0$ *for a.e.* $x \in \Omega$, *which together with* (3.29) *show that* q *in* (1.2) *is forced to be* $> p$.

Proof of Theorem 3.2. By Proposition 3.3 it is clear that Φ_1 satisfies the Palais-Smale condition. We claim that under the assumptions of the theorem there exist a number $\rho > 0$ and an element $\overline{e} \in K \setminus \{0\}$ such that

$$\inf_{\|v\|_{\lambda_1} = \rho} \Phi_1(v) > \Phi_1(0) = 0 \tag{3.31}$$

and

$$\lim_{t \to +\infty} \Phi_1(t\overline{e}) = -\infty. \tag{3.32}$$

Then, obviously (2.3) is accomplished with $e = t\overline{e}$, t sufficiently large, and Theorem 2.2 applies, yielding the conclusion.

By (i) and (ii) one can find numbers $\sigma \in (0, \lambda_1)$ and $\delta_0 > 0$ such that

$$G(x,s) \leq \frac{\lambda_1 - \sigma}{p}|s|^p \quad \text{for a.e. } x \in \Omega \setminus \omega,\ \forall s \in [-\delta_0, 0) \tag{3.33}$$

and

$$G(x,s) \leq \frac{\lambda_1 - \sigma}{p}s^p \quad \text{for a.e. } x \in \Omega,\ \forall s \in (0, \delta_0]. \tag{3.34}$$

From (3.8) there is a constant $k = k(\delta_0)$ with

$$G(x,s) \leq k|s|^q \ \text{ for a.e. } x \in \Omega,\ \forall |s| > \delta_0. \tag{3.35}$$

For an arbitrary $u \in K$ the sets $\Omega_-(u)$ and $\Omega_+(u)$ are defined by (3.12) and recall that $\Omega_-(u) \subset \Omega \setminus \omega$. By (3.33) and (3.35) we have

$$\begin{aligned}\int_{\Omega_-(u)} G(x,u) &= \int_{\Omega_-(u)\cap[-\delta_0 \leq u]} G(x,u) + \int_{[u<-\delta_0]} G(x,u)\\ &\leq \frac{\lambda_1 - \sigma}{p}\int_{\Omega_-(u)} |u|^p + k\int_{\Omega_-(u)} |u|^q.\end{aligned} \tag{3.36}$$

Similarly, (3.34) and (3.35) imply

$$\begin{aligned}\int_{\Omega_+(u)} G(x,u) &= \int_{\Omega_+(u)\cap[u \leq \delta_0]} G(x,u) + \int_{[u>\delta_0]} G(x,u)\\ &\leq \frac{\lambda_1 - \sigma}{p}\int_{\Omega_+(u)} u^p + k\int_{\Omega_+(u)} u^q.\end{aligned} \tag{3.37}$$

From (3.36) and (3.37) we infer

$$\int_\Omega G(x,u) \leq \frac{\lambda_1 - \sigma}{p}\|u\|^p_{L^p} + k\|u\|^q_{L^q}. \tag{3.38}$$

Taking into account the continuity of the embedding $W \subset L^q(\Omega)$, from (3.38), (3.4) and (3.5) we estimate Φ_1 as follows

$$\begin{aligned}\Phi_1(u) = \mathcal{G}(u) + \varphi_\alpha(u) &= -\int_\Omega G(x,u) + \frac{1}{p}\left(\|\nabla u\|^p_{L^p} + \alpha\|u\|^p_{L^p}\right)\\ &\geq \frac{1}{p}\left(\|\nabla u\|^p_{L^p} + \alpha\|u\|^p_{L^p} + (\sigma - \lambda_1)\|u\|^p_{L^p}\right) - \tilde{k}\|u\|^q_{\lambda_1}\\ &\geq \frac{1}{p}\left(\|\nabla u\|^p_{L^p} + \alpha\|u\|^p_{L^p} + (\sigma - \lambda_1)\frac{\|\nabla u\|^p_{L^p} + \alpha\|u\|^p_{L^p}}{\lambda_1}\right) - \tilde{k}\|u\|^q_{\lambda_1}\\ &\geq \frac{\sigma}{2p\lambda_1}\|u\|^p_{\lambda_1} - \tilde{k}\|u\|^q_{\lambda_1}\end{aligned}$$

showing that (3.31) holds true with some $\rho > 0$ sufficiently small (because $q > p$, cf. Remark 3.1).

Next, we deal with (3.32). Since $\lambda_1 < +\infty$, it is easy to see that there are $u_0 \in K \setminus \{0\}$ and $\sigma_0 > 0$ with either $|\{x \in \Omega : u_0(x) < -\sigma_0\}| > 0$ or $|\{x \in \Omega : u_0(x) > \sigma_0\}| > 0$. Then, the cone property of K enables us to find some $\overline{e} \in K \setminus \{0\}$ such that at least one of the sets

$$\Omega_< := \{x \in \Omega : \overline{e}(x) < -s_0\},$$
$$\Omega_> := \{x \in \Omega : \overline{e}(x) > s_0\}$$

has a positive measure. Also, similarly to (3.30) is obtained

$$G(x,t) \geq \gamma_2(x)|t|^\theta \text{ for a.e. } x \in \Omega \setminus \omega,\ \forall t < -s_0, \tag{3.39}$$

with some $\gamma_2 \in L^\infty(\Omega)$, $\gamma_2(x) > 0$ for a.e. $x \in \Omega \setminus \omega$. For $t \geq 1$, taking into account the inclusions

$$\Omega_< \subset [t\overline{e} < -s_0] \subset \Omega \setminus \omega,$$
$$\Omega_> \subset [t\overline{e} > s_0]$$

and using (3.9), (3.30) and (3.39) we estimate $-\mathcal{G}(t\overline{e})$ as follows:

$$\begin{aligned}
-\mathcal{G}(t\overline{e}) &= \int_{[t\overline{e}<-s_0]} G(x, t\overline{e}) + \int_{[t\overline{e}>s_0]} G(x, t\overline{e}) + \int_{[|t\overline{e}|\leq s_0]} G(x, t\overline{e}) \\
&\geq t^\theta \left(\int_{[t\overline{e}<-s_0]} \gamma_2 |\overline{e}|^\theta + \int_{[t\overline{e}>s_0]} \gamma_1 \overline{e}^\theta \right) - k|\Omega| \\
&\geq t^\theta \left(\int_{\Omega_<} \gamma_2 |\overline{e}|^\theta + \int_{\Omega_>} \gamma_1 \overline{e}^\theta \right) - k|\Omega| \\
&= k_0 t^\theta + k_1
\end{aligned}$$

where $k_0 = k_0(\overline{e}) > 0$, $k_1 \in I\!R$ are constants. Therefore, as $\theta > p$, we get

$$\begin{aligned}
\Phi_1(t\overline{e}) &= \mathcal{G}(t\overline{e}) + \varphi_\alpha(t\overline{e}) \\
&\leq -k_0 t^\theta - k_1 + \frac{t^p}{p} \left(\|\nabla \overline{e}\|_{L^p} + \alpha \|\overline{e}\|_{L^p} \right) \to -\infty \ , \text{ as } t \to +\infty \ ,
\end{aligned}$$

i.e. (3.32), and the proof is complete. □

Example 3.1. *For the sake of simplicity let us consider the one dimensional frame, i.e., $N = 1$. We take $\Omega = (-1, 1)$, $\omega = (0, 1)$ and let $S := [\Omega \times (1, \infty)] \cup [(\Omega \setminus \omega) \times (-\infty, -1)]$. Defining $g : \Omega \times I\!R \to I\!R$ by*

$$g(x, s) = \begin{cases} q|s|^{q-2}s, & \textit{if } (x, s) \in S, \\ \\ 0 & \textit{otherwise}, \end{cases}$$

we have

$$G(x, s) = \begin{cases} |s|^q - 1, & \textit{if } (x, s) \in S, \\ \\ 0 & \textit{otherwise}, \end{cases}$$

and it is a simple matter to check that the requirements of Theorem 3.2 are satisfied with $\theta = q > p$.

Remark 3.2. *Theorem 3.1 and Theorem 3.2 respectively are generalizations of Theorem 4.1 and Theorem 4.2 from* [7].

4. Applications and further remarks

As already pointed out in Section 1, solving problem (1.6) we implicitly solve the hemivariational inequality (1.7). This is immediate by a basic result of Clarke (see p. 84 in [3]) yielding:

$$\int_\Omega (-G)^0(x, u; v) \geq \mathcal{G}^0(u; v) \geq (\mathcal{G}\big|_W)^0(u; v), \ \ \forall u, v \in W. \tag{4.1}$$

In this respect Theorem 3.1 and Theorem 3.2 also appear as being existence results for problem (1.7). In the smooth case, meaning g Carathéodory, condition (3.21) is automatically satisfied and clearly these theorems provide sufficient conditions ensuring the existence of solutions for problem (1.8).

Remark 4.1. *In the case when* $\omega = \emptyset$ *we have* $K = W$ *and problem* (1.6) *becomes*

$$\begin{cases} \text{Find } u \in W \text{ such that} \\ T_\alpha(u) \in \partial(-\mathcal{G}|_W)(u). \end{cases} \tag{4.2}$$

Since by the chain rule (see Clarke [3]*, p. 45) one has* $\partial(-\mathcal{G}|_W)(u) \subset \partial(-\mathcal{G})(u)$ *it follows that if* u *solves* (4.2) *then there is some* $w \in \partial(-\mathcal{G})(u) \subset L^{q'}(\Omega)$*, such that*

$$\langle T_\alpha(u), v\rangle = \int_\Omega wv, \quad \forall v \in W.$$

Next, we give applications to existence of weak solutions for some differential inclusions problems with p-Laplacian. Let Γ_0 be a closed subset of Γ having positive surface measure, $\varepsilon > 0$, and let us denote

$$\frac{\partial u}{\partial \nu_p} := |\nabla u|^{p-2}\frac{\partial u}{\partial \nu}$$

where ν stands for the unit outward normal on Γ. We discuss the following discontinuous boundary value problems

$$(D) \qquad \begin{cases} -\Delta_p u \in \left[\underline{g}(x,u), \overline{g}(x,u)\right] \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma \end{cases}$$

$$(N) \qquad \begin{cases} -\Delta_p u + \varepsilon |u|^{p-2}u \in \left[\underline{g}(x,u), \overline{g}(x,u)\right] \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \nu_p} = 0 \text{ on } \Gamma \end{cases}$$

$$(M) \qquad \begin{cases} -\Delta_p u \in \left[\underline{g}(x,u), \overline{g}(x,u)\right] \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma_0, \quad \dfrac{\partial u}{\partial \nu_p} = 0 \text{ on } \Gamma \setminus \Gamma_0 \end{cases}$$

$$(P) \qquad \begin{cases} -\Delta_p u + \varepsilon |u|^{p-2}u \in \left[\underline{g}(x,u), \overline{g}(x,u)\right] \text{ in } \Omega, \\ u = \text{ constant on } \Gamma, \quad \displaystyle\int_\Gamma \frac{\partial u}{\partial \nu_p} \mathrm{d}\Gamma = 0 \end{cases}$$

Associated with the above problems will be the constant

$$a = \begin{cases} 0 & \text{for } (D) \text{ and } (M) \\ \varepsilon & \text{for } (N) \text{ and } (P). \end{cases}$$

We denote

$$W_{\Gamma_0} = \left\{u \in W^{1,p}(\Omega) : u|_{\Gamma_0} = 0\right\},$$
$$W_1 = \left\{u \in W^{1,p}(\Omega) : u = \text{ constant on } \Gamma\right\};$$
$$\lambda_{1,D} = \inf\left\{\frac{\|\nabla u\|_{L^p}}{\|u\|_{L^p}} : u \in W_0^{1,p}(\Omega)\right\},$$
$$\lambda_{1,M} = \inf\left\{\frac{\|\nabla u\|_{L^p}}{\|u\|_{L^p}} : u \in W_{\Gamma_0}^{1,p}(\Omega)\right\},$$
$$\lambda_{1,N} = \lambda_{1,P} = \varepsilon.$$

Note that, by virtue of Poincaré inequality, both of the $\lambda_{1,D}$ and $\lambda_{1,M}$ are > 0. Also, noticing that the subspaces W_{Γ_0} and W_1 contain $W_0^{1,p}(\Omega)$, Green's formula makes natural the following

Definition 4.1. *An element $u \in W_0^{1,p}(\Omega)$ (resp. $W^{1,p}(\Omega)$, W_{Γ_0}, W_1) is said to be a weak solution (in short, solution) of problem (D) (resp. (N), (M), (P)) if there exists a measurable function $w : \Omega \to I\!R$ such that*

$$w(x) \in \left[\underline{g}(x, u(x)), \overline{g}(x, u(x))\right] \text{ for a.e. } x \in \Omega$$

and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + a \int_\Omega |u|^{p-2} uv = \int_\Omega wv, \tag{4.3}$$

for all $v \in W_0^{1,p}(\Omega)$ (resp. $W^{1,p}(\Omega)$, W_{Γ_0}, W_1).

Theorem 4.1. *Assume* (1.2) *and*

$$\limsup_{|s|\to\infty} \frac{pG(x,s)}{|s|^p} < \lambda_{1,D} \text{ (resp. } \lambda_{1,N},\ \lambda_{1,M},\ \lambda_{1,P}\text{) uniformly for a.e. } x \in \Omega\ .$$

Then problem (D) (resp. (N), (M), (P)) has at least one solution.

Proof. Clearly, (4.3) rewrites

$$\langle T_a(u), v\rangle = \int_\Omega wv.$$

Taking into account Remark 4.1 and (3.25) we apply Theorem 3.1 with $W = W_0^{1,p}(\Omega)$ (resp. $W^{1,p}(\Omega)$, W_{Γ_0}, W_1), $\alpha = a$ and $\omega = \emptyset$. □

Theorem 4.2. *Assume* (1.2) *and* (3.21), *together with*

$$\limsup_{|s|\to 0} \frac{pG(x,s)}{|s|^p} < \lambda_{1,D} \text{ (resp. } \lambda_{1,N},\ \lambda_{1,M},\ \lambda_{1,P}\text{) uniformly for a.e. } x \in \Omega\ .$$

If there are numbers $\theta > p$, $s_0 > 0$ such that

$$0 < \theta G(x,s) \le sg(x,s) \quad \text{for a.e. } \ x \in \Omega,\ \forall\ |s| \ge s_0,$$

then problem (D) (resp. (N), (M), (P)) has a nontrivial solution.

Proof. This follows by the argument in the proof of Theorem 4.1 but with Theorem 3.2 instead of Theorem 3.1. □

Remark 4.2. *Since Theorem 3.1 does not ask hypothesis (H_{λ_1}), Theorem 4.1 clearly remains true for $\varepsilon = 0$ in the case of problems (N) and (P). Theorem 4.1 generalizes Theorem 5.1 in* [6] *which concerns with problem (D). An existence result for problem (N) was obtained by Hu, Matzakos and Papageorgiou in* [9]. *This is of a different type and it is based on the nonsmooth variant of the Saddle Point Theorem due to Chang* [2]. *We also mention the recent result obtained by Filippakis and Papageorgiou* [8] *in a resonant case for problem (D).*

Remark 4.3. *Theorem 4.2 extends to the case of problems (N), (M) and (P) e.g. the following results concerning problem (D): Corollary 3.11 in Ambrosetti and Rabinowitz* [1] *and Theorem 2.15 in Rabinowitz* [12] *($p = 2$, g continuous), Theorem 3.6 (resp. Theorem 18) in Dincă, Jebelean and Mawhin* [4] *(resp.* [5]*) (g Carathéodory), Theorem 5.3 in Chang* [2] *($p = 2$) and Theorem 5.2 in Dincă, Jebelean and Motreanu* [6].

Other possible choices for W can be found in [7].

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